On Optimality Conditions for a Convex Optimization Problem with Polyhedral Discrete Inclusions

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Abstract. In this paper, using by the concept of locally dual mapping, the optimality conditions for solution of a convex optimization problem are determined. Specifically, the sets and the inclusions in the considered problem are polyhedral. Furthermore, the problem is given by a discrete inclusions system with discrete time. The polyhedral discrete inclusions in the problem are defined by a set-valued map.

Keywords: Set-valued mapping, discrete inclusions, locally dual mapping, necessary and sufficient conditions, polyhedral mapping.

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1. Introduction

This paper is devoted to an investigation of convex optimization problem described by discrete inclusions. As it is well known that most of optimization problems such as differential games, models of economic dynamics, macroeconomic problems, etc. are now described in terms of set-valued mappings [1] and form a component part of the control theory and mathematical economics [6]. Among these problems, by virtue of their wide applicability (especially in models of economic dynamics), an important role is played by problems with polyhedral mappings [8]. We refer the reader to the monographs by Pschenichny [8], Rockafellar [9], Aubin and Frankowska [1], Mordukhovich [7] and the papers of Mahmudov [3, 4, 5].

2. Necessary Concepts

The required definitions and theorems are given in this section.

A set $M \subseteq \mathbb{R}^n$ is called a polyhedral set if it can be expressed the intersection of a finite number of closed half-space, that is if M is the solution of the inequality system

$$\langle x, b_i \rangle \leqslant \alpha_i, \quad i = 1, 2, .., m$$

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where $b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$ then M is polyhedral. So any polyhedral set can be expressed as follows

$$\{x \in \mathbb{R}^n : Ax \leqslant b\}$$

where A is an $m \times n$ matrices, b is an m-dimensional column vector. Any polyhedral set is closed and convex. See [2] for further information.

Definition 2.1. *If for every $x \in K$ and $\lambda > 0$ we have $\lambda x \in K$ then K is called a cone. If K is convex set then the cone K is called a convex cone.*

Definition 2.2. *Let K be a cone. Dual cone of K is defined as follows*

$$K^* = \{x^* \in X^* : \langle x, x^* \rangle \geqslant 0, \ \forall x \in K\}$$

Theorem 2.1. *Polyhedral cones can be given with the solution of a linear homogen inequality system as below*

$$\langle x, x_k^* \rangle \geqslant 0, \quad k = 1, 2, .., l. \tag{1}$$

Theorem 2.2. *If a polyhedral cone K is given by the linear inequality system as in* (1) *then its dual cone K^* is also polyhedral cone and $K^* = \{x^* : x^* = \sum_{k=1}^{l} \gamma_k x_k^*, \ \gamma_k \geqslant 0\}$.*

Theorem 2.3. *For the polyhedral cones $K_1, K_2, .., K_m$ the following relation is valid:*

$$(K_1 \cap K_2 \cap .. \cap K_m)^* = K_1^* + K_2^* + .. + K_m^* \, .$$

Let X and Y be finite dimensional Euclidean spaces and $Z = X \times Y$. A set-valued mapping $a : X \to Y$ is a mapping that associates with any $x \in X$ a set $a(x) \in Y$. For any set $M \subseteq Z$ the formula $a(x) = \{y : (x, y) \in M\}$ defines a set-valued mapping. Notice that M is the graph of the mapping a. A set-valued mapping a completely characterized by its graph. The graph is denoted $gph\, a$ and defined by $gph\, a = \{(x, y) : y \in a(x)\}$. The set $dom\, a = \{x : a(x) \neq \emptyset\}$ is called the domain of a. The norm of the set $a(x)$ is defined by $\|a(x)\| = \sup_y\{\|y\| : y \in a(x)\}$ and it is assume as specific that $\|\emptyset\| = 0$. For more information about the set-valued mapping see [1].

Definition 2.3. *A set-valued mapping is called convex if the set $gph\, a$ is convex.*

In a clear statement, if for any $x_1, x_2 \in X$ the relation

$$a(\lambda x_1 + (1 - \lambda)x_2) \supseteq \lambda a(x_1) + (1 - \lambda)a(x_2), \ 0 \leqslant \lambda \leqslant 1$$

is satisfied then the mapping a is called a convex set-valued map.

Definition 2.4. *A set-valued mapping is called closed if the set $gph\, a$ is closed.*

Definition 2.5. *A set-valued mapping is polyhedral if the set $gph\, a$ is polyhedral.*

Definition 2.6. *The cone of tangent directions of the set $gph\, a$ at a point $z \in gph\, a$ is denoted by $K_a(z)$ and defined as below*

$$K_a(z) = con(gph\, a - z) = \{\bar{z} : \bar{z} = \lambda(z_1 - z), z_1 \in gph\, a,\ \lambda > 0\}\ . \tag{2}$$

Definition 2.7. *The locally dual mapping of a convex set-valued mapping at a point $z \in gph\, a$ is denoted by $a^*(y^*; z)$ and defined by $a^*(y^*; z) = \{x^* : (-x^*, y^*) \in K_a^*(z)\}$.*

The set $\partial f(x) = \{x^* \in \mathbb{R}^n : f(x) - f(x_0) \geqslant \langle x^*, x - x_0\rangle, \forall x \in \mathbb{R}^n\}$ is called the subdifferential of f at x. Obviously $\partial f(x)$ is a closed convex set. In general, the subdifferential may be empty. If $\partial f(x)$ is not empty then f is said to be subdifferentiable at x.

The cone of tangent directions $K_M(x_0)$ of the set at a point $x_0 \in M$ is called a local tent, if for each $\overline{x}_0 \in ri(K_M(x_0))$ (where riK is relative interior of K) there exists a convex cone Q and a continuous mapping ψ defined in a neighborhood of the origin such that:

a) $\overline{x}_0 \in ri(Q)$, $Lin\, Q = Lin\, K_M(x_0)$ ve $Q \subseteq K_M(x_0)$

b) $\psi(\overline{x}) = \overline{x} + r(\overline{x})$, $\overline{x} \to 0$ için $\|\overline{x}\|^{-1} r(\overline{x}) \to 0$

c) $x_0 + \psi(\overline{x}) \in M$, $\overline{x} \in Q \cap \varepsilon B$.

Notice that if M is a convex set then $K_M(x_0) = con(M - x_0)$ where$x_0 \in M$ is a local tent.

Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be an arbitrary function. Let us denote $dom\, f = \{x : f(x) < +\infty\}$. Take a point $x \in dom\, f$ and consider the following function for $\overline{x} \in X, (\overline{x} \neq 0)$

$$F(\overline{x}, x) = \sup_{r(.)} \limsup_{\lambda \downarrow 0} \frac{f(x + \lambda\overline{x} + r(\lambda)) - f(x)}{\lambda}$$

where $r(\lambda) \in \mathbb{R}^n$, $\lambda^{-1} r(\lambda) \to 0$, $\lambda \downarrow 0$.

The function $h(\cdot, x)$ is called a convex upper approximations (CUA) of a function f at every fixed point $x \in dom\, f$, if

1) $h(\overline{x}, x) \geqslant F(\overline{x}, x)$ for all $\overline{x} \neq 0$

2) $h(\cdot, x)$ is a closed (lower semi-continuous) positively homogeneous convex function

If the function $h(\overline{x}, x)$ is a CUA for f at the point x then the set $\partial h(0,x) = \{x^* \in \mathbb{R}^n : h(\overline{x}, x) \geqslant \langle \overline{x}, x^* \rangle,\ \overline{x} \in \mathbb{R}^n\}$ equals to $\partial f(x)$.

Let us now consider the following problem in nonconvex case

$$\begin{array}{c} f(x) \to \min \\ x \in M \end{array} \tag{3}$$

where $M \subseteq \mathbb{R}^n$ is an any set and f is a function which has an UCA.

Theorem 2.4. [8] *Suppose that in the problem* (3)*,* $x_0 \in M$ *is a point that gives the minimum value for* f*,* $h(\overline{x}, x_0)$ *is an UCA for the function* f *and the cone* $K_M(x_0)$ *is a locally tent for* M *at* x_0*. Then*

$$intdom\, h(., x_0) \cap K_M(x_0) \neq \emptyset$$

if $\partial f(x_0) \cap K_M^*(x_0) \neq \emptyset$.

Theorem 2.5. [8] *Let the point* x_0 *be the minimum point for the function* f *on the set* $M = \bigcap_{i=1}^m M_i$*. Further suppose that* $h(\overline{x}, x_0)$ *is an UCA for the function* f *at the point* x_0*, for all* $i = 1, 2..., m$ *the cones* $K_{M_i}(x_0)$ *are locally tents and the next condition is valid*

$$intdom\, h(., x_0) \cap (\bigcap_{i=1}^m K_{M_i}(x_0)) \neq \emptyset$$

Then there exist a number $\lambda \geqslant 0$ *and vectors* $x_i^* \in K_{M_i}^*(x_0)$ *not equal simultaneously to zero, such that*

$$\lambda x_0^* = \sum_{i=1}^m x_i^*, \quad x_0^* \in \partial f(x_0)\ .$$

Let us now consider the following problem:

$$\sum_{t=0}^{T} g(x_t, t) \to \min \tag{4}$$

$$x_{t+1} \in a(x_t),\ t = 0, 1, ..., T-1 \tag{5}$$

$$x_0 \in N,\ x_T \in M \tag{6}$$

Suppose that a trajectory $\{\widetilde{x}_t\}_{t=0}^T$ is an optimal solution for the problem (4)-(6) and

(1) a is a set-valued mapping such that the tangent directional cones $K_a(\widetilde{x}_t, \widetilde{x}_{t+1})$, $t = 0, 1, ..., T-1$ are locally tents,

(2) the tangent directional cones $K_N(\widetilde{x}_0)$ and $K_M(\widetilde{x}_T)$ are locally tents,

(3) the functions $g(x,t)$, $t=0,1,...,T$ have an UCA $h_t(0,\widetilde{x}_t)$ at the points $\widetilde{x}_t$ such that it is contious according to $\overline{x}$ in other words the subdifferential $\partial g(\widetilde{x}_t,t)=\partial h_t(0,\widetilde{x}_t)$ is defined.

Let w be a vector in the space $\mathbb{R}^{(T+1)n}$ consists of n-dimensional components x_t, $t=0,1,...,T$.

One can show that using the followings

$$\begin{aligned} f(w) &= \sum_{t=0}^{T} g(x_t,t) \\ \widetilde{N} = \{w : x_0 \in N\}\,,\ &\widetilde{M} = \{w : x_T \in M\} \\ \widetilde{M_t} = \{w : (x_t,x_{t+1}) &\in gph\, a\},\ t=0,1,...,T-1 \end{aligned} \tag{7}$$

the problem (4)-(6) is equivalent to the following problem:

$$\begin{gathered} f(w)\rightarrow \min \\ w\in \widetilde{N}\cap(\bigcap_{t=o}^{T-1}\widetilde{M_t})\cap\widetilde{M}\ . \end{gathered} \tag{8}$$

Using by Theorem 2.5 the necessary conditions for the problem (8) is given in the next theorem.

Theorem 2.6. [8] *Let us suppose that the conditions 1-3 above are satisfied. Then in order to the trajectory $\{\widetilde{x}_t\}_t^T$ of the system (5) satisfying the conditions (6) minimizing the sum (4) it is necessary that there exist a number $\lambda\in\{0,1\}$ and vectors x_t^*, $t=0,1,...,T$, x_e^* not equal simultaneously to zero, such that*

$$x_t^*\in a^*(x_{t+1}^*;(\widetilde{x}_t,\widetilde{x}_{t+1}))+\lambda\partial g(\widetilde{x}_t,t),\ t=0,1,...,T-1 \tag{9}$$

$$x_T^*+x_e^*\in\lambda\partial g(\widetilde{x}_T,T), \tag{10}$$

$$x_0^*\in K_N^*(\widetilde{x}_0), \tag{11}$$

$$x_e^*\in K_M^*(\widetilde{x}_T)\ . \tag{12}$$

When $\lambda=1$ then these conditions are also sufficient.

Remark 2.1. *If a set-valued mapping is convex then $K_a(z)$ is a locally tent and the below condition (1) holds. In case the set N is convex then the tangent directional cone $K_N(x)$ is a locally tent and the below condition (2) holds. If the function g is convex then the below condition (3) is valid.*

Corollary 2.7. *If the mapping a, the sets N,M and the function g are convex then from Remark 2.1 the Theorem 2.6 is valid.*

Corollary 2.8. *In case of polyhedral from Theorem 2.3, the number* λ *equals to 1 in the conditions* (9) *and* (10).

3. Necessary and Sufficient Conditions for Polyhedral Discrete Inclusions

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, let $z = (x, y)$ be the pair of elements $x, y \in \mathbb{R}^n$, and let $\langle x, y \rangle$ be their inner product. Let us now consider the problem for polyhedral discrete inclusions with a convex function:

$$\sum_{t=0}^{T} g(x_t, t) \longrightarrow \min \tag{13}$$

$$x_{t+1} \in a(x_t) \ , \quad t = 0, 1, .., T-1 \tag{14}$$

$$x_0 \in N_0 \quad , \quad x_T \in M_T \tag{15}$$

where T is a finite natural number and the functions g are convex. The mapping a, the sets N_0 and M_T are defined as follows

$$a(x) = \{y : Ax - By \leqslant d\} \tag{16}$$

$$N_0 = \{x_0 : Nx_0 \leqslant p\} \tag{17}$$

$$M_T = \{x_T : Mx_T \leqslant q\} \ . \tag{18}$$

where A and B are $m \times n$ matrices, d is m-dimensional column vector, N and M rectangular matrices, while p, q are column vectors of appropriate dimensions.

Lemma 3.1. *Let* A *and* B *be* $m \times n$ *matrices and* d *be an* m*-dimensional column vector. Then, the set-valued mapping* $a(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *defined by* $a(x) = \{y : Ax - By \leqslant d\}$ *is polyhedral, convex and closed.*

Proof. It is known that the graph of the set-valued mapping $a(x) = \{y : Ax - By \leqslant d\}$ is the set $gph\, a = \{(x, y) : Ax - By \leqslant d\} \subset \mathbb{R}^{2n}$. Because the set $gph\, a$ is the solution set of the linear inequality systems $Ax - By \leqslant d$, the set $gph\, a$ is polyhedral. Consequently $gph\, a$ is convex. By the Definition 2.3 and Definition 2.5 the set-valued mapping a is a convex and polyhedral. Lets show that the set $gph\, a$ is closed. Lets consider a sequence (x_k, y_k) such that $(x_k, y_k) \to (x, y)$ and $(x_k, y_k) \in gph\, a$ for all $k \in \mathbb{N}$. The inclusion $(x_k, y_k) \in gph\, a$, $k \in \mathbb{N}$ implies that the inequality systems $Ax_k - By_k \leqslant d$ for all $k \in \mathbb{N}$. By passing to limit for $k \to \infty$ we have $Ax - By \leqslant d$. Hence we have the inclusion $(x, y) \in gph\, a$. Consequently, the set $gph\, a$ is closed in the space $\mathbb{R}^{2n}$ and by the Definition 2.4 the mapping a is closed.

In the following lemma it is calculated the locally dual mapping for a mapping whose graph is polyhedral set.

Lemma 3.2. *Assume that a polyhedral mapping $a(x) = \{y : Ax - By \leqslant d\}$ is given, where A and B are $r \times n$ matrices and d is an r-dimensional column vector. Then the locally dual mapping for the mapping a at the point $z_0 = (x_0, y_0)$ is the following form:*

$$a^*(y^*; z_0) = \{x^* = A^*\lambda : y^* = B^*\lambda, \langle Ax_0 - By_0 - d, \lambda\rangle = 0, \lambda \geqslant 0\}, \tag{19}$$

where A^*, B^* are the transposes of matrices A and B, respectively.

Proof. It is known that $gph\, a = \{(x, y) : Ax - By \leqslant d\}$. Now, for a point $z_0 = (x_0, y_0) \in gph\, a$, let us now define the index set

$$I_0 = \{i : A_i x_0 - B_i y_0 = d_i, i = 1, .., r\}$$

where A_i, B_i are the ith row of the matrices A and B, respectively, and d_i is ith entry of the vector d. Let us now determine the cone $K_a(z_0)$ using by the Definition 2.6. To determine it we should research that for which points $\overline{z} = (\overline{x}, \overline{y})$ the inclusion $z_0 + \lambda\overline{z} \in gph\, a$ is satisfied for sufficiently small $\lambda > 0$.

If $i \in I_0$ then $A_i x_0 - B_i y_0 = d_i$. Hence the inequality

$$A_i(x_0 + \lambda\overline{x}) - B_i(y_0 + \lambda\overline{y}) = A_i x_0 - B_i y_0 + \lambda(A_i\overline{x} - B_i\overline{y}) = d_i + \lambda(A_i\overline{x} - B_i\overline{y}) \leqslant d_i$$

is satisfied, but this situation is possible only for the points $\overline{z} = (\overline{x}, \overline{y})$ which satisfy

$$A_i\overline{x} - B_i\overline{y} \leqslant 0, \quad i \in I_0\ . \tag{20}$$

If $i \notin I_0$ then the following inequality

$$A_i(x_0 + \lambda\overline{x}) - B_i(y_0 + \lambda\overline{y}) = (A_i x_0 - B_i y_0) + \lambda(A_i\overline{x} - B_i\overline{y}) < d_i$$

is satisfied independently from the point $\overline{z}$ for sufficiently small numbers $\lambda > 0$. Consequently, the points $\overline{z}$ is satisfied the inequality (20) if and only if the inclusion $z_0 + \lambda\overline{z} \in gph\, a$ is satisfied for sufficiently small $\lambda > 0$, shortly the Definition 2.6 implies that

$$K_a(z_0) = \{(\overline{x}, \overline{y}) : A_i\overline{x} - B_i\overline{y} \leqslant 0,\ i \in I_0\}\ .$$

Because of the inequality system in (20) equals to the following system

$$\langle\overline{x}, -A_i\rangle + \langle\overline{y}, B_i\rangle \geqslant 0, \quad i \in I_0 \tag{21}$$

from Theorem 2.1 it can be deduced that $K_a(z_0)$ is a polyhedral cone and if we apply the Theorem 2.2 to the system (21) we have $(x^*, y^*) \in K_a^*(z_0)$ if and only if

$$x^* = -\sum_{i \in I_0} A_i^* \lambda_i, \quad y^* = \sum_{i \in I_0} B_i^* \lambda_i, \qquad \lambda_i \geqslant 0\ . \tag{22}$$

If we take λ_i as λ and $\lambda_i = 0$ for $i \notin I_0$ then the dual cone $K_a^*(z_0)$ is obtained in the following form

$$K_a^*(z_0) = \{(x^*, y^*) : x^* = -A^*\lambda, y^* = B^*\lambda, \lambda \geqslant 0, \langle Ax_0 - By_0 - d, \lambda\rangle = 0\}\ . \tag{23}$$

Finaly using by (23) and Definition 2.7 we have the equality (19). ∎

We will now find the dual cone for the tanget directions cone at a point on a polyhedral set by the next lemma.

Lemma 3.3. *Let $N = \{x : Ax \leqslant p\}$ be a polyhedral set such that A is an $r \times n$ matrix and p is an r-dimensional column vector. Then the dual cone to the tanget cone $K_N(x_0)$ at a point $x_0 \in N$ must be the following form:*

$$K_N^*(x_0) = \{x_0^* = -A^*\lambda : \langle Ax_0 - p, \lambda\rangle = 0, \lambda \geqslant 0\}. \tag{24}$$

where A^* is the tranpose of A.

Proof. Because it is the solution set of the inequality system $Ax \leqslant p$, the set $\{x : Ax \leqslant p\}$ is polyhedral and consequently convex. Lets define an index set $I_0 = \{i : A_i x_0 = p_i, i = 1, .., r\}$ for $x_0 \in N$ similarly in Lemma 3.2. For $i \in I_0$ and sufficiently small $\lambda \geqslant 0$ the inequality $A_i(x_0 + \lambda\overline{x}) = p_i + \lambda A_i \overline{x} \leqslant p_i$ is valid only in case $A_i\overline{x} \leqslant 0$, $i \in I_0$ and so $\langle \overline{x}, -A_i\rangle \geqslant 0, \quad i \in I_0$. If $i \notin I_0$ then the inequality $A_i(x_0 + \lambda\overline{x}) = A_i x_0 + \lambda A_i \overline{x} < p_i$ is independently valid from the point $\overline{x}$ for sufficiently small $\lambda \geqslant 0$. Consequently $K_N(x_0) = \{\overline{x} : \langle \overline{x}, -A_i\rangle \geqslant 0, i \in I_0\}$. From Theorem 2.1 that cone is polyhedral. Because of Theorem 2.2 and the definition of dual cone one can has $x^* \in K_N^*(x_0)$ if and only if $x^* = -\sum_{i \in I_0} A_i^* \lambda_i, \quad \lambda_i \geqslant 0$. If we take λ as the column vector that its components are λ_i and $\lambda_i = 0$ for $i \notin I_0$ then the desired dual cone is obtained in format (24). ∎

Theorem 3.4. *In order that the trajectory $\{\widetilde{x}_t\}_{t=0}^T$ of the discrete polyhedral inclusion* (14) *be a solution of the polyhedral optimization problem with the boundary conditions* (13)-(15), *it is necessary and sufficient that there exist vectors $x_t^*, t = 0, 1, .., T$ and x^*, satisfying the relations*

$$\begin{gathered} x_t^* = A^*\lambda_t + u_t^*, \quad u_t^* \in \partial_x g(\widetilde{x}_t, t), \quad x_{t+1}^* = B^*\lambda_t,\ \lambda_t \geqslant 0, \\ \langle A\widetilde{x}_t - B\widetilde{x}_{t+1} - d, \lambda_t\rangle = 0, \quad t = 0, 1, .., T-1, \end{gathered} \tag{25}$$

$$x_T^* + x^* \in \partial_x g(\widetilde{x}_T, T), \tag{26}$$

$$x_0^* = -N^*\gamma_0, \quad \langle N\widetilde{x}_0 - p, \gamma_0\rangle = 0, \quad \gamma_0 \geqslant 0, \tag{27}$$

$$x^* = -M^*\gamma_T, \quad \langle M\widetilde{x}_T - q, \gamma_T\rangle = 0, \quad \gamma_T \geqslant 0\ . \tag{28}$$

Proof. From Lemma 3.2 we have

$$a^*(y^*; z) = \{x^* = A^*\lambda : y^* = B^*\lambda, \langle Ax - By - d, \lambda\rangle = 0, \lambda \geqslant 0\}\ . \tag{29}$$

On the other hand due to the Lemma 3.3 for $\widetilde{x}_0 \in N_0$ and $\widetilde{x}_T \in M_T$ we have

$$\begin{aligned} K^*_{N_0}(\widetilde{x}_0) &= \{x_0^* = -N^*\gamma_0 : \langle N\widetilde{x}_0 - p, \gamma_0\rangle = 0, \gamma_0 \geqslant 0\}, \\ K^*_{M_T}(\widetilde{x}_T) &= \{x_T^* = -M^*\gamma_T : \langle M\widetilde{x}_T - q, \gamma_T\rangle = 0, \gamma_T \geqslant 0\}\ . \end{aligned} \tag{30}$$

Lets now use these results with Result 2.7 and Theorem 2.6. According to Result 2.8, λ in (9) equals to 1. In (9) if we take $x^* = x_t^*$, $y^* = x_{t+1}^*$, $x = \widetilde{x}_t$, $y = \widetilde{x}_{t+1}$ and consider (29) then we have the conditions $x_t^* = A^*\lambda_t + u_t^*$, $u_t^* \in \partial_x g(\widetilde{x}_t, t)$, $x_{t+1}^* = B^*\lambda_t$, $\langle A\widetilde{x}_t - B\widetilde{x}_{t+1} - d, \lambda_t\rangle = 0$, $\lambda \geqslant 0$ so the conditions (25). Again according to Result 2.8 λ in (10) equals to 1. If we take $x_e^* = x^*$ we have the condition (26). In (11) if we take $N = N_0$ and consider (30) then we have the condition $x_0^* = -N^*\gamma_0$, $\langle N\widetilde{x}_0 - p, \gamma_0\rangle = 0$, $\gamma_0 \geqslant 0$ so the condition (27). Similarly in (12) if we take $M = M_T$ and consider (30) then we have the condition $x^* = -M^*\gamma_T$, $\langle M\widetilde{x}_T - q, \gamma_T\rangle = 0$, $\gamma_T \geqslant 0$ so the condition (28). So all conditions are obtained and since $\lambda = 1 > 0$ according to Theorem 2.6 these conditions are sufficient in the same time. ∎

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