Local Group-Groupoids

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Abstract. It is known that if X is a topological group, then the fundamental groupoid $\pi_1(X)$ is a group-groupoid, i,e, a group object in the category of groupoids. The group structure of a group-groupoid lifts to a covering groupoid. Further if G is a group-groupoid, then the category $GpGdAct(G)$ of group-groupoid operations and the category $GpGdCov/G$ of group-groupoid coverings of G are equivalent.

In this paper we prove the corresponding results for local topological groups and local group objects in the category of groupoids.

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1. Introduction

The theory of covering groupoids plays an important role in the applications of groupoids (cf. [1], [6]) and covering spaces are not only important for algebraic topology, but also they have important applications in many other branches of mathematics including differential topology, the theory of topological groups and the theory of Riemann surfaces.
There are two important results about group-groupoids given in [2]. One is that the group structure of a group-groupoid lifts to a covering groupoid, i.e., if G is a group-groupoid and $p\colon \widetilde{G} \to G$ is covering morphism of groupoids, then $\widetilde{G}$ becomes a group-groupoid such that p is group-groupoid morphism. The other is that if G is a group-groupoid, then the category $GpGdCov/G$ of covering morphisms over G is equivalent to the category $GpGdAct(G)$ of group-groupoid actions of G on groups.
In this paper we introduce the notion of a local group-groupoid as a local group object in the category of groupoids and prove local group-groupoid versions of these results. For the first result we prove that if G is a local group-groupoid and $p\colon \widetilde{G} \to G$ is covering morphism of groupoids, then $\widetilde{G}$ becomes a group-groupoid such that p is a local group-groupoid morphism. For the second result we prove that if G is a local group-groupoid, then the category $LGpGdCov/G$ of local group-groupoid covers is equivalent to the category $LGpGdAct(G)$ of local group-groupoid actions of G on local groups.

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2. Covering maps and covering morphisms of groupoids

We assume the usual theory of covering maps. All spaces X are assumed to be locally path connected and semilocally 1-connected, so that each path component of X admits a simply connected cover. Recall that a covering map $p\colon \widetilde{X} \to X$ of connected spaces is called *universal* if it covers every covering of X in the sense that if $q\colon \widetilde{Y} \to X$ is another covering of X then there exists a map $r\colon \widetilde{X} \to \widetilde{Y}$ such that $p = qr$ (hence r becomes a covering). A covering map $p\colon \widetilde{X} \to X$ is called *simply connected* if $\widetilde{X}$ is simply connected. So a simply connected covering is a universal covering.

Definition 2.1. We call a subset A of X *liftable* if it is open, path connected and A lifts to each covering of X, that is, if $p\colon \widetilde{X} \to X$ is a covering map, $\imath\colon A \to X$ is the inclusion map, and $\widetilde{x} \in \widetilde{X}$ satisfies $p(\widetilde{x}) = x \in A$, then there exists a map (necessarily unique) $\widetilde{\imath}\colon A \to \widetilde{X}$ such that $p\widetilde{\imath} = \imath$ and $\widetilde{\imath}(x) = \widetilde{x}$.

It is easy to see that A is liftable if and only if it is open, path connected and for all $x \in A$, the fundamental group $\pi_1(A, x)$ is mapped to singleton by the morphism induced by the inclusion map $\imath\colon A \to X$ and e is the identity element of the fundamental group $\pi_1(X, x)$. Remark that if X is a semilocally simply connected topological space then each point $x \in X$ has a liftable neighbourhood.

A *groupoid* is a small category in which each morphism is an isomorphism, that is, a groupoid G has a set of morphisms, which we call just *elements* of G, a set O_G of *objects* together with maps $s, t\colon G \to O_G$ and $\epsilon\colon O_G \to G$ such that $s\epsilon = t\epsilon = 1_{O_G}$, the identity map. The maps s, t are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion*. If $a, b \in G$ and $t(a) = s(b)$, then the *composite* ab exists such that $s(ab) = s(a)$ and $t(ab) = t(b)$. So there exists a partial composite defined by the map $G_t \times_s G \to G, (a, b) \mapsto ab$, where $G_t \times_s G$ is the pullback of t and s. Further, this partial composite is associative, for $x \in O_G$ the element $\epsilon(x)$ denoted by 1_x acts as the identity, and each element a has an inverse a^{-1} such that $s(a^{-1}) = t(a)$, $t(a^{-1}) = s(a)$, $aa^{-1} = (\epsilon s)(a)$, $a^{-1}a = (\epsilon t)(a)$. The map $G \to G$, $a \mapsto a^{-1}$ is called the *inversion*.

So a group can be thought as a groupoid with only one object and a groupoid can be thought as a group with many objects.

In a groupoid G for $x, y \in O_G$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. We say G is *connected* if for all $x, y \in O_G$, $G(x, y)$ is not empty.

For $x \in O_G$ we denote the star $\{a \in G \mid s(a) = x\}$ of x by G_x. The *object group* at x is $G(x) = G(x,x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f\colon H \to G$ and $O_f\colon O_H \to O_G$ such that $s \circ f = O_f \circ s$, $t \circ f = O_f \circ t$ and $f(ab) = f(a)f(b)$ for all $(a,b) \in H_t \times_s H$. For such a morphism we simply write $f\colon H \to G$.

Covering morphisms and universal covering groupoids of groupoids are defined in [1] as follows:

Definition 2.2. Let $p\colon \widetilde{G} \to G$ be a morphism of groupoids. Then p is called a *covering morphism* and $\widetilde{G}$ *a covering groupoid* of G if for each $\widetilde{x} \in O_{\widetilde{G}}$ the restriction of p

$$p_x\colon (\widetilde{G})_{\widetilde{x}} \to G_{p(\widetilde{x})}$$

is bijective.

As an example a group homomorphism $p\colon \widetilde{G} \to G$ is a covering morphism if and only if it is an isomorphism.

A covering morphism $p\colon \widetilde{G} \to G$ is called *connected* if both $\widetilde{G}$ and G are connected.

A connected covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e. if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\widetilde{p}\colon \widetilde{G} \to \widetilde{H}$ such that $q\widetilde{p} = p$ (and hence $\widetilde{p}$ is also a covering morphism), this is equivalent to that for $\widetilde{x}, \widetilde{y} \in O_{\widetilde{G}}$ the set $\widetilde{G}(\widetilde{x}, \widetilde{y})$ has not more than one element.

A pointed morphism $p\colon \widetilde{G}, \widetilde{x} \to G, x$ is called a *covering morphism* if the morphism $p\colon \widetilde{G} \to G$ is a covering morphism.

Definition 2.3. Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids and $f\colon (H, z) \to (G, x)$ a morphism of groupoids. We say f lifts to p if there exists a unique morphism $\widetilde{f}\colon (H, z) \to (\widetilde{G}, \widetilde{x})$ such that $f = p\widetilde{f}$.

For any groupoid morphism $p\colon \widetilde{G} \to G$ and an object $\widetilde{x}$ of $\widetilde{G}$ we call the subgroup $p(\widetilde{G}(\widetilde{x}))$ of $G(p\widetilde{x})$ the *characteristic group* of p at $\widetilde{x}$. The following result gives a criterion on the liftings of morphisms ([1]).

Theorem 2.1. *Let $p\colon (\widetilde{G}, \widetilde{x}) \to (G, x)$ be a covering morphism of groupoids and $f\colon (H, z) \to (G, x)$ a morphism such that H is connected. Then the morphism $f\colon (H, z) \to (G, x)$ lifts uniquely to a morphism $\widetilde{f}\colon (H, z) \to (\widetilde{G}, \widetilde{x})$ if and only if the characteristic group of f is contained in that of p.*

As a result of this theorem we have the following corollary

Corollary 2.2. *Let* $p\colon (\widetilde{G},\widetilde{x}) \to (G,x)$ *and* $q\colon (\widetilde{H},\widetilde{z}) \to (G,x)$ *be connected covering morphisms with characteristic groups* C *and* D *respectively. If* $C \subseteq D$*, then there is a unique covering morphism* $r\colon (\widetilde{G},\widetilde{x}) \longrightarrow (\widetilde{H},\widetilde{z})$ *such that* $p = qr$*. If* $C = D$*, then* r *is an isomorphism.*

Let X be a topological space. Then we have a category $TCov/X$ of covering maps $p\colon \widetilde{X} \to X$ of topological spaces. Further we have a category $GdCov/\pi_1 X$ of covering morphisms $p\colon \widetilde{G} \to \pi_1 X$.
The following result was given in [1].

Proposition 2.1. Let X be a connected topological space which has a universal cover. Then the categories $TCov/X$ and $GdCov/\pi_1 X$ are equivalent.

Let G be a groupoid. An *action of* G *on a set* consists of a set X, a function $\theta\colon X \to O_G$ and a partial function $X_\theta \times_s G \to X, (x,a) \mapsto xa$ defined on the pullback $X_\theta \times_s G$ of θ and s such that
(i) $\theta(xa) = t(a)$
(ii) $x(ab) = (xa)b$
(iii) $x1_{\theta(x)} = x$
As an example if $p\colon \widetilde{G} \to G$ is a covering morphism of groupoids and $X = O_{\widetilde{G}}$, $\theta = O_p$, then we obtain an action of G on X via θ by assigning to $\widetilde{x} \in X$ and $a \in G_{p(\widetilde{x})}$ the target of the unique lift $\widetilde{a}$ of a with source $\widetilde{x}$.
Given such an action, *action groupoid* $G \ltimes X$ is defined to be the groupoid with object set X and elements of $(G \ltimes X)(x,y)$ the pairs (a,x) such that $a \in G(\theta(x),\theta(y))$ and $xa = y$. The groupoid composite is defined to be

$$(a,x)(b,y) = (ab,x))$$

when $y = xa$

Theorem 2.3 ([1]). *Let* x *be an object of a connected groupoid* G *and let* C *be a subgroup of the object group* $G(x)$*. Then there exists a covering morphism* $q\colon (\widetilde{G}_C,\widetilde{x}) \longrightarrow (G,x)$ *with characteristic group* C*.*

Proof. We give the sketch proof for the technical methods: Let X be the set of cosets $Ca = \{ca \mid c \in C\}$ for a in G_x. Let $\theta\colon X \to O_G$ send Ca to the final point of a. Then G

acts on X by

$$X_\theta \times_s G \to X, (Ca, g) \mapsto C(ag).$$

The required groupoid $\widetilde{G}_C$ is taken to be the action groupoid $G \ltimes X$. Then the projection $q\colon \widetilde{G}_C \to G$ given on objects by $\theta\colon X \to O_G$ and on elements by $(g, Ca) \mapsto g$, is a covering morphism of groupoids and has the characteristic group C. So the groupoid composite is defined by

$$(g, Ca)(h, Cb) = (gh, Ca).$$

The required object $\widetilde{x} \in \widetilde{G}_C$ is the coset C. $\square$

3. Local topological groups

Local Lie group is defined in [5]. We deal with the topological case of this definition. For this we first state local group notion algebraically as follows.

Definition 3.1. A set L is called a *local group* if there exists

a) a distinguish element $e \in L$, the identity element,

b) a multiplication $\mu\colon \mathcal{U} \longrightarrow L, (x, y) \longmapsto x \circ y$ defined on a subset $\mathcal{U}$ of $L \times L$ such that $(\{e\} \times L) \cup (L \times \{e\}) \subseteq \mathcal{U}$,

c) an inversion map $\imath\colon V \longrightarrow L, x \longmapsto \overline{x}$ defined on a subset $e \in V \subseteq L$ such that $V \times \imath(V) \subseteq \mathcal{U}$ and $\imath(V) \times V \subseteq \mathcal{U}$,

all satisfying the following properties:

(i) Identity: $e \circ x = x = x \circ e$ for all $x \in L$

(ii) Inverse: $\imath(x) \circ x = e = x \circ \imath(x)$ for all $x \in V$

(iii) Associativity: If $(x, y), (y, z), (x \circ y, z)$ and $(x, y \circ z)$ all belong to $\mathcal{U}$, then

$$x \circ (y \circ z) = (x \circ y) \circ z).$$

We denote such a local group by $(L, \mu, \mathcal{U}, \imath, V)$ or only by L. Here we note that if $\mathcal{U} = L \times L$ and $V = L$, then a local group becomes a group. So the concept of local group generalizes that of group.

As the topological version of Definition 3.1 a local topological group is defined as follows:

Definition 3.2. In Definition 3.1, if L is a topological space such that $\mathcal{U}$ is open in $L \times L$, V is open in L, the maps μ and $\imath$ are continuous, then $(L, \mu, \mathcal{U}, \imath, V)$ is called a *local topological group.*

If $\mathcal{U} = L \times L$ and $V = L$, then a local topological groups becomes a topological group.

Example 3.1. ([5]) Let G be a topological group with the group multiplication $\mu\colon G\times G\longrightarrow G, (x,y)\mapsto x\circ y$ and the inversion map $\imath\colon G\longrightarrow G, x\mapsto x^{-1}$. Let L be an open neighbourhood of the identity element e in G. Then we can choose a local topological group $(L,\mu,\mathcal{U},\imath,V)$ taking $\mathcal{U}=(L\times L)\cap\mu^{-1}(L)$ and $V=L\cap L^{-1}$. Here the group multiplication μ and the inversion $\imath$ on G are restricted to define a local group multiplication and an inversion maps on L.
More generally if $\mathcal{U}$ and V are chosen such that

$$(\{e\}\times L)\cup(L\times\{e\})\subseteq\mathcal{U}\subseteq(L\times L)\cap\mu^{-1}(L)$$

$$\{e\}\subseteq V\subseteq L\cap\imath^{-1}(L)$$

and

$$(V\times\imath(V))\cup(\imath(V)\times V)\subseteq\mathcal{U}$$

then we have a local topological groupoid.

Definition 3.3. Let $(L,\mu,\mathcal{U},\imath,V)$ and $(\widetilde{L},\widetilde{\mu},\widetilde{\mathcal{U}},\widetilde{\imath},\widetilde{V})$ be local topological groups. A continuous map $f\colon L\longrightarrow\widetilde{L}$ is called a local topological group morphism if
(i) $(f\times f)(\mathcal{U})\subseteq\widetilde{\mathcal{U}}$, $f(V)\subseteq\widetilde{V}$, $f(e)=\widetilde{e}$,
(ii) $f(x\circ y)=f(x)\circ f(y)$ for $(x,y)\in\mathcal{U}$, and
(iii) $f(\imath(x))=\widetilde{\imath}(f(x))$ for $x\in V$.

A local topological group morphism is called a *homeomorphism* if it is one-to one and onto , with continuous inverse.
Note that the composition of local topological group homomorphisms is again a local group homomorphism. So local topological groups and morphisms between them form a category. We write *LTGp* for this category.

Proposition 3.1. *Let $(L,\mu,\mathcal{U},\imath,V)$ be a local topological group and A an open neighbourhood of the identity element e in L. Then there is an open neighbourhood B of e in L such that $B^2\subseteq A$. Further if A is liftable then B can be chosen to be liftable.*

Proof. Since $(L,\mu,\mathcal{U},\imath,V)$ is a local topological group, the multiplication $\mu\colon\mathcal{U}\to L$, $(a,b)\mapsto a\circ b$ is continuous. So there is an open neighbourhood $\mathcal{B}$ of (e,e) in $\mathcal{U}$ such that $\mu(\mathcal{B})\subseteq A$. Therefore there are open neighbourhoods B_1 and B_2 of e in L such that $B_1\circ B_2\subseteq A$. Hence for $B=B_1\cap B_2$ we have $B^2\subseteq A$ as required.
Further if A is liftable then B can be chosen as liftable. For if A is liftable then for each $x\in B$, the fundamental group $\pi_1(B,x)$ is mapped to the singleton by the morphism

induced by the inclusion map $\imath\colon B \to L$. But B is not necessarily path connected and hence not necessarily liftable. Since the path component $C_e(B)$ of e in B is liftable and satisfies these conditions, if necessary, we can replace B by $C_e(B)$ and suppose that B is liftable. □

Definition 3.4. Let L and $\widetilde{L}$ be local topological groups. A morphism $f\colon \widetilde{L} \longrightarrow L$ of local topological groups is called a *cover* if it is a covering map on the underlying spaces.

4. Local group-groupoids

We know from [3] that if X is a topological group, then the fundamental groupoid $\pi_1 X$ is a group object in the category of groupoids. This is called a *group-groupoid*. If L is a local topological group rather than a topological group, then similarly we obtain a local group object in the category of groupoids. As a local group object in the category of groupoids we define a *local group-groupoid* as follows:

Definition 4.1. A *local group-groupoid* G is a groupoid in which O_G and G both have local group structures such that the following maps are the local morphisms of groupoids:
(i) $\mu\colon \mathcal{U} \longrightarrow G, (a,b) \mapsto a \circ b$
(ii) $\imath\colon V \to G, a \mapsto \overline{a}$
(iii) $e\colon (\star) \to G$, where $(\star)$ is singleton.

Remark 4.1. Let G be a local group-groupoid. Then from Definition 4.1, we have the following:
(i) If $(a,b) \in \mathcal{U} \subseteq G \times G$, then $s(a \circ b) = s(a) \circ s(b)$ and $t(a \circ b) = t(a) \circ t(b)$
(ii) If $a \in V \subseteq G$, then $s(\overline{a}) = \overline{s(a)}$ and $t(\overline{a}) = \overline{t(a)}$
(iii) If e is the identity of local group O_G, then 1_e is the identity of local group G.

In a local group-groupoid G for $a, b \in G$ the groupoid composite is denoted by ab when $s(b) = t(a)$ and the local group multiplication by $a \circ b$ when $(a,b) \in \mathcal{U}$. The local group inverse of $a \in V$ is denoted by $\overline{a}$ while the groupoid inverse is written a^{-1}.

Proposition 4.1. *In a local group-groupoid G we have*

$$(ac) \circ (bd) = (a \circ b)(c \circ d)$$

for $a, b, c, d \in G$ such that the necessary compositions and multiplications are defined

Proof. Since μ preserves the groupoid composite for $a, b, c, d \in G$ such that $((a, b), (c, d) \in \mathcal{U}$, $(ac, bd) \in \mathcal{U}$ and ac, bd are defined we have that

$$\begin{aligned}(ac) \circ (bd) &= \mu[ac, bd] \\ &= \mu[(a, b)(c, d)] \\ &= \mu(a, b)\mu(c, d) \\ &= (a \circ b)(c \circ d).\end{aligned}$$

□

Proposition 4.2. *If L is a local topological group, then the fundamental groupoid $\pi_1 L$ is a local group-groupoid.*

Proof. Let L be a local topological group, with the continuous multiplication

$$\mu\colon \mathcal{U} \to L, (x, y) \mapsto x \circ y$$

and the continuous inversion

$$\imath\colon V \to L, x \mapsto \overline{x}.$$

Then these maps induce the following morphisms

$$\pi_1\mu\colon \pi_1\mathcal{U} \to \pi_1 L, [(a, b)] \mapsto [a \circ b]$$

$$\pi_1\imath\colon \pi_1 V \to \pi_1 L, [a] \mapsto \overline{[a]} = [\overline{a}].$$

Here the path $a \circ b$ is defined by $(a \circ b)(t) = a(t) \circ b(t)$ for $0 \leq t \leq 1$. Note that since (a, b) is a path in $\mathcal{U}$, the path $a \circ b$ is defined. The path in L denoted by $\overline{a}$ is defined by $\overline{a}(t) = \overline{a(t)}$. So $\pi_1 L$ becomes a local group. To prove that $\pi_1 L$ is a local group-groupoid we have to check the conditions of Definition 4.1.

Let (a, b) and (c, d) be the paths in $\mathcal{U}$ such that the multiplications of paths ac and bd are defined such that (ac, bd) is a path in $\mathcal{U}$. Then evaluating at $t \in [0, 1]$ gives the interchange law

$$(ac) \circ (bd) = (a \circ b)(c \circ d)$$

i.e, the map $\mu\colon \mathcal{U} \longrightarrow G, (a, b) \mapsto a \circ b$ is a morphism of groupoids. The other conditions are trivial. □

Example 4.1. If X is a local group, then $G = X \times X$ becomes a local group-groupoid on X. Here a pair (x, y) is a morphism from x to y and the groupoid composite is defined by $(x,y)(z,u)=(x,u)$ whenever $y=z$. The local group multiplication is defined by

$$(x,y)\circ(z,u)=(x\circ z,y\circ u)$$

whenever $x\circ z$ and $y\circ u$ are defined.

Proposition 4.3. *Let G be a local group-groupoid, e the identity of O_G. Then the connected component $C(G)_e$ of e in groupoid is a local group-groupoid.*

Proof. Recall that the connected component $C(G)_e$ of e is a full subgroupoid of G on those objects x such that $G(x,e)$ is nonempty. Since G is a local group-groupoid there are local morphisms $\mu\colon\mathcal{U}\longrightarrow G$, $\imath\colon V\to G$ and $e\colon(\star)\to G$, where $(\star)$ is singleton. Then restrictions of these local morphisms yield the following local morphisms
(i) $\mu\colon\mathcal{U}\cap(C(G)_e\times C(G)_e)\longrightarrow C(G)_e$
(ii) $\imath\colon V\cap C(G)_e\to C(G)_e$
(iii) $e\colon(\star)\to C(G)_e)$. □

Proposition 4.4. Let G be a local group-groupoid and e the identity of O_G. Then the star $G_e=\{a\in G\colon s(a)=e\}$ of e becomes a local group.

Proof. Let G be a local group-groupoid with local morphism $\mu\colon\mathcal{U}\longrightarrow G$. We have $s(a\circ b)=s(a)\circ s(a)$ for $(a,b)\in\mathcal{U}$. So the restriction of local morphism $\mu\colon\mathcal{U}\longrightarrow G$ yields a local morphism $\mu\colon\mathcal{U}\cap(G_e\times G_e)\longrightarrow G_e$. Similarly, since for $a\in V$, $s(\overline{a})=\overline{s(a)}$, the restriction of local morphism $\imath\colon V\longrightarrow G$ yields $\imath\colon V\cap G_e\longrightarrow G_e$. The rest of the proof are straightforward. □

5. Covering morphisms of local group-groupoids

Definition 5.1. Let H and G be two local group-groupoids. A *morphism* $f\colon H\to G$ from H to G is a morphism of underlying groupoids preserving local group structure, i.e, $f(a\circ b)=f(a)\circ f(b)$ for $(a,b)\in\mathcal{U}\subseteq H\times H$.

A morphism $f\colon H\to G$ of local group-groupoids is called *covering* (resp. *universal covering*) if it is a covering morphism (resp. universal covering) on underlying groupoids.

Example 5.1. If $p\colon\widetilde{L}\to L$ is a covering morphism of local topological groups, then the induced morphism $\pi_1 p\colon\pi_1\widetilde{L}\to\pi_1 L$ is a covering morphism of local group-groupoids.

Definition 5.2. Suppose that G is a local group-groupoid and e is the identity of O_G. Let $\widetilde{G}$ be a groupoid, $p\colon \widetilde{G} \to G$ a covering morphism of groupoids and $\tilde{e} \in O_{\widetilde{G}}$ such that $p(\widetilde{e}) = e$. We say that the local group structure of G *lifts* to $\widetilde{G}$ if there exists a local group structure on $\widetilde{G}$ with the identity element $\tilde{e} \in O_{\widetilde{G}}$ such that $p\colon \widetilde{G} \to G$ is a morphism of local group-groupoids.

We now use Theorem 2.3 to prove that the local group structure of a group-groupoid lifts to a covering groupoid.

Theorem 5.1. *Let $\widetilde{G}$ be a local groupoid and G a group-groupoid whose underlying groupoid is connected. Suppose that $p\colon \widetilde{G} \to G$ is a covering morphism of groupoids, e is the identity element of the group O_G and $\widetilde{e}$ is an element of $O_{\widetilde{G}}$ such that $p(\widetilde{e}) = e$. Then the local group structure of G lifts to $\widetilde{G}$ with identity $\widetilde{e}$.*

Proof. Let C be the characteristic group of $p\colon (\widetilde{G}, \widetilde{e}) \to (G, e)$. Then by Theorem 2.3 we have a covering morphism $q\colon (\widetilde{G}_C, \widetilde{e}) \to (G, e)$ with characteristic group C. So by Corollary 2.2 the covering morphisms p and q are equivalent. Therefore it is sufficient to prove that the group structure of G lifts to $\widetilde{G}_C$ by the covering morphism $q\colon (\widetilde{G}_C, \widetilde{x}) \to (G, e)$.

Let $m\colon \mathcal{U} \to G, (g, h) \mapsto g \circ h$ be the local group multiplication of the local group-groupoid G. Now define a local group multiplication on $X = O_{\widetilde{G}_C}$ by

$$(Ca) \circ (Cb) = C(a \circ b)$$

when $a \circ b$ is defined and a local group multiplication on $\widetilde{G}_C$ by

$$(g, Ca) \circ (h, C \circ b) \mapsto (g \circ h, C(a \circ b)).$$

when $g \circ h$ and $a \circ b$ are defined.

Here note that if $a, b \in G_e$ and $a \circ b$ is defined, then $a \circ b \in G_e$ and so $C(a \circ b)$ is defined. It is straightforward to see that $\widetilde{G}_C$ is a local group-groupoid. When the necessary groupoid composites and local group multiplications are possible we have

$$\begin{aligned} (g, Ca)(k, Cc) \circ (h, Cb)(t, Cd) &= (gk, Ca) \circ (ht, Cb) \\ &= (gk \circ ht, C(a \circ b)). \end{aligned}$$

$$\begin{aligned} ((g, Ca) \circ (h, Cb))((k, Cc) \circ (t, Cd)) &= (g \circ h, C(a \circ b))(k \circ t, C(c \circ d)) \\ &= ((g \circ h)(k \circ t), C(a \circ b)). \end{aligned}$$

Since G is a local group-groupoid, when the necessary groupoid composites and local group multiplications are possible we have that $gk \circ ht = (g \circ h)(k \circ t)$, so

$$(g, Ca)(k, Cc) \circ (h, Cb)(t, Cd) = ((g, Ca) \circ (h, Cb))((k, Cc) \circ (t, Cd))$$

i.e, the interchange low is satisfied.

Further the morphism q preserves the local group structure as follows: If the necessary groupoid composites and local group multiplications are possible then we have that

$$\begin{aligned} q((g, Ca) \circ (h, C \circ b)) &= q(g \circ h, C \circ (a \circ b)) \\ &= g \circ h \\ &= q(g, Ca) \circ q(h, Cb). \end{aligned}$$

□

Definition 5.3. Let G be a local group-groupoid and L a local group. An *action* or *operation* of the local group-groupoid G on L consists of a local group morphism $\theta\colon L \to O_G$ and an operation of the underlying groupoid of G on the underlying set of L via $\theta\colon L \to O_G$ such that the following interchange law holds

$$(xa) \circ (yb) = (a \circ b)(x \circ y)$$

whenever both sides are defined.

We write (L, θ) for such an action.

As an example if G is a local group-groupoid, then G acts on $L = O_G$ via the identity map $L \to O_G$.

Example 5.2. If $f\colon H \to G$ is a covering morphism of local group-groupoids, then the local group-groupoid G acts on O_H.

Example 5.3. Let G be a local group-groupoid which acts on a local group L. Then the action groupoid $G \ltimes L$ is a local group-groupoid with local group operation defined by

$$(a, x) \circ (b, y) = (a \circ b, x \circ y)$$

whenever both sides are defined and the projection $p\colon G \ltimes L \to G, (a, x) \mapsto a$ is a covering morphism of local group-groupoids.

Let G be a local group-groupoid. We obtain a category denoted by $LGpGdCov/G$ whose objects are the covering morphisms of groupoids $f\colon H \to G$ which are also local group morphisms and a morphism from $f\colon H \to G$ to $g\colon K \to G$ is a morphism $h\colon H \to K$ of local group-groupoids which becomes also a covering morphism.

A morphism of local group-groupoid operations $f\colon (L,\theta) \to (L',\theta')$ is a morphism of local groups $f\colon L \to L'$ such that $\theta = \theta' f$ and $f(xa) = (fx)a$ whenever xa is defined. So we have a category $LGpGdAct(G)$.

Theorem 5.2. *For a local group-groupoid G, the categories $LGpGdAct(G)$ and $LGpGdCov(G)$ are equivalent.*

Proof. For an action (L,θ) of G, we have the action groupoid $G \ltimes L$ and a local group structure on it defined by

$$(a,x) \circ (b,y) = (a \circ b, x \circ y)$$

whenever both sides are defined. Then $G \ltimes L$ becomes a local group-groupoid and the projection $p\colon G \ltimes L \to G, (a,x) \to a$ is a covering morphisms of local group-groupoid. Therefore we have an object of the category $LGpGdCov/G$. This gives a functor

$$\Gamma\colon LGpGdAct(G) \to LGpGdCov(G).$$

Conversely if $f\colon H \to G$ is a covering morphism of local group-groupoids, then we have a bijection $O_H \; {}_f{\times}_s \; G \to H$. Then the composition of this bijection with the final point map $t\colon H \to O_H$ gives an action $O_H \; {}_f{\times}_s \; G \to O_H$ via $O_f\colon O_H \to O_G$. So we obtain a functor

$$\Phi\colon LGpGdCov(G) \to LGpGdAct(G).$$

The natural equivalences $\Gamma\Phi \simeq 1$ and $\Phi\Gamma \simeq 1$ follow. $\square$

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