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SOME HERMITE-HADAMARD TYPE INEQUALITIES FOR $(\varphi, p, \mu)-$PREINVEX FUNCTIONS

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Abstract. In the present paper, a new class of convex functions isintroduced which is called $(\varphi, p, \mu)-$preinvex functions.With the help of this new class we prove some of Hermite-Hadamard type inequalities for $(\varphi, p, \mu)-$preinvex functions.

1. INTRODUCTION AND PRELIMINARES

Fractional calculus (see [8], [9], [14]) arise in the mathematical modeling of various problems in sciences and engineering such as mathematics, physics, chemistry and biology.

Fractional integral operators are studied by lots of authors (see [1], [15], [18], [21], [26]) with many applications in differents fields. (see [5], [6]). Also, these operators have allow to extended results about integral inequalities of many types (see [1], [12], [13]), for instance, Hermite-Hadamard integral inequalities (see [3], [6], [7]).

Currently, several extensions and generalizations are inroduced by many researchers for classical convexity (see [15], [17], [18], [24]). A significant generalizations of convex functions are invex functions introduced by Hanson (see [7]).

In recent years, generalizations of classic convexity have been considered various study. The theory concept of is one of them, which is introduced by Hanson. This theory has approved special attention by several authors over the years. Preinvex function on the invex sets can be characterized by a class of variational inequalities, called the variational-like inequalities.

The $f : J \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on an interval J of real numbers $a, b \in J$ and $a < b$, if the following inequalities

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x) \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

holds,it is called Hermite-Hadamard type inequality.An important extension of convex functions was the introduction of preinvex function [21].

The following definitions were introduced in [21] .

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Definition 1.1. A set $K_\eta \in \mathbb{R}^n$ is said in order to be invex set with respect to $\eta(.,.)$, if

$$u + \tau\eta(v,u) \in K_\eta,\ \forall u,v \in K_\eta,\quad \tau \in [0,1]\,. \tag{1.2}$$

The invex set K_η is also called $\eta-$connected set. $\eta(v,u) = v - u$, the invex set reduces in order to classical convex set.

Definition 1.2. A function $f : K_\eta \to \mathbb{R}$ is said in order to be preinvex with respect to arbitrary bifunction $\eta\,(.,.)\,$, if

$$f\,(u + \tau\eta\,(v,u)) \leq (1-\tau)\,f\,(u) + \tau f\,(v)\,,\ \forall u,v \in K_\eta,\ \tau \in [0,1]\,. \tag{1.3}$$

The function f is said in order to be preconcave if and only if $-f$ is preinvex .

For $\eta\,(v,u) = v - u$, the preinvex functions becomes convex functions in the classical sense.

In $[21]$, Noor et. al. noticed that every convex function is a preinvex function. Clearly, every convex set is an invex set but the converse is not true.

Definition 1.3. Let $\eta\,(.,.)\ : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is supposed to have the following property:

$$\eta\,(v + \tau_1\eta\,(u,v)\,, v + \tau_2\eta\,(u,v)) = (\tau_1 - \tau_2)\,\eta\,(u,v)\,,$$

$\forall \tau_1, \tau_2 \in [0,1]\,,\ \tau_1 \leq \tau_2,\ u,v \in K_\eta.$

In [21], Noor et al. gave the below interesting Riemann-Liouville integral inequalities of Hermite-Hadamard-Noor type for preinvex function

$$f\left(\tfrac{2a+\eta(b,a)}{2}\right) \leq \tfrac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(u)du \leq \tfrac{f(a)+f(b)}{2}.$$

We now recall some known concepts which will be helpful in obtaining some of our main results.

[21], Beta functions $B\,(.,.)$ are defined as ;

$$B\,(x,y) = \int_0^1 \tau^{x-1}\,(1-\tau)^{y-1}\,d\tau, \tag{1.4}$$

is Euler Beta function with

$$B\,(x,y) = \frac{\Gamma\,(x)\,\Gamma\,(y)}{\Gamma\,(x+y)}. \tag{1.5}$$

In the present paper, we introduced the notion of $(\varphi,p,\mu)-$preinvex Functions. Some new and interesting estimates of the integral

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi\,(a) + \eta\,(\varphi\,(b)\,,\varphi\,(a)) - \varphi\,(u))^{\frac{p}{k}}\,(\varphi\,(u) - \varphi\,(a))^{\frac{\mu}{k}}\,f\,(\varphi\,(u))\,d\varphi\,(u)\,,$$

via $(\varphi,p,\mu)-$preinvex functions are obtained. Some special classical are also deduced from this result.

The $K_\eta = [\varphi\,(u)\,,\varphi\,(u) + \eta\,(\varphi\,(v)\,,\varphi\,(u))]$ be a nonempty closed set in $\mathbb{R}^n$. Let $f : K_\eta \to \mathbb{R}$ be a continuous function and let $\eta(.,.) : K_\eta \times K_\eta \to \mathbb{R}^n$ be a continuous bifunction.

2. Main Results

We introduce notations,.... In this section, in order to obtain main results we will introduce the some new concepts of (φ, p, μ)-preinvex functions and its variant forms.

The following definition is defined by us.

Definition 2.1. Let J be an interval $\varphi(u), \varphi(v) \in J$ with $\varphi(u) < \varphi(u) + \eta(\varphi(v), \varphi(u))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : K_\eta \to \mathbb{R}$ be (φ, p, μ) $-$preinvex function with respect to bifunction $\eta(.,.)$, If

$$f(\varphi(u) + \tau\eta(\varphi(v), \varphi(u))) \leq \tau^p (1-\tau)^\mu [f(\varphi(u)) + f(\varphi(v))], \tag{2.1}$$

where $\varphi(u), \varphi(v) \in K_\eta,\ \tau \in [0,1]$, some fixed $p > 0, \mu > 0$.

Remark 2.2. Let J be an interval $\varphi(u), \varphi(v) \in J$ with $\varphi(u) < \varphi(u) + \eta(\varphi(v), \varphi(u))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : J \subseteq \mathbb{R} \to \mathbb{R}$ be a φ convex function on $J = [\varphi(a), \varphi(b)]$. The present study, function $\eta(.,.) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is supposed to have the following property:
(2.2)
$$\eta(\varphi(v) + \tau_1\eta(\varphi(u), \varphi(v)), \varphi(v) + \tau_2\eta(\varphi(u), \varphi(v))) = (\tau_1 - \tau_2)\eta(\varphi(u), \varphi(v)),$$

where $\tau_1, \tau_2 \in [0,1],\ \varphi(\tau_1) \leq \varphi(\tau_2)$.

1. If $\tau_1 = \tau_2 = 0$, afterwards (1.4) implies that $\eta(\varphi(v), \varphi(v)) = 0$ for all $\varphi(v) \in \mathbb{R}$.

2. If $\tau_1 = 0$ and $\tau_2 = \tau > 0$ afterwards,
$\eta(\varphi(v), \varphi(v) + \tau\eta(\varphi(u), \varphi(v))) = -\tau\eta(\varphi(u), \varphi(v))$ for all $\varphi(u), \varphi(v) \in \mathbb{R}$.

3. If $\eta(\varphi(u), \varphi(v)) > 0$ for some $(\varphi(u), \varphi(v)) \in \mathbb{R}$ afterwards,
$\eta(\varphi(v), \varphi(v) + \tau\eta(\varphi(u), \varphi(v))) \leq 0$ for all $\tau \in [0,1]$, (1.4) implies that η has not constant sign on $\mathbb{R} \times \mathbb{R}$.

Theorem 2.3. *If J be an interval $\varphi(a), \varphi(b) \in J$ with $\varphi(a) < \varphi(a) + \eta(\varphi(b), \varphi(a))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : K_\eta \to \mathbb{R}$ be (φ, p, μ) $-$preinvex function on $J = [\varphi(a), \varphi(b)]$ with respect to bifunction $\eta(.,.)$ with $\eta(\varphi(b), \varphi(a)) > 0$. If $f \in L[\varphi(a) + \eta(\varphi(b), \varphi(a))]$, then the following inequality hold*

$$\begin{aligned} &2^{p+\mu-1} f\left(\tfrac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \\ &\leq \tfrac{1}{\eta(\varphi(b),\varphi(a))} \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\varphi(u)) d\varphi(u) \\ &\leq B(p+1, \mu+1)[f(\varphi(a)) + f(\varphi(b))]. \end{aligned} \tag{2.3}$$

Proof. Since $\eta(.,.)$ satisfies (1.4) and f is (φ, p, μ) $-$preinvex function,afterwards, i.e. with $\varphi(u) = \varphi(a) + \tau\eta(\varphi(b), \varphi(a)),\ \varphi(v) = \varphi(a) + (1-\tau)\eta(\varphi(b), \varphi(a))$ we have,

$$f\left(\tfrac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \leq \tfrac{f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))+f(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))}{2^{p+\mu}}. \tag{2.4}$$

Integrating both sides of the preceding inequality with respect to τ on $[0,1]$, we get,

$$2^{p+\mu-1}\int_0^1 f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\leq \frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\varphi(u))d\varphi(u)$$

$$\leq \int_0^1 \tau^p(1-\tau)^\mu [f(\varphi(a))+f(\varphi(b))]d\tau.$$

i.e.

$$\frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\varphi(u))d\varphi(u)$$

$$\leq B(p+1,\mu+1)[f(\varphi(a))+f(\varphi(b))].$$

Thus the proof is complated.

If we take $\varphi(x)=x$ in Theorem 1, we obtain the results in [21]. □

Theorem 2.4. *Let J be an interval $\varphi(a), \varphi(b) \in J$ with $\varphi(a) < \varphi(a)+\eta(\varphi(b),\varphi(a))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f, g : K_\eta \to \mathbb{R}$ be $(\varphi,p,\mu)-$preinvex functions on $J = [\varphi(a), \varphi(b)]$ with respect to bifunction $\eta(.,.)$, and $\eta(\varphi(b),\varphi(a)) > 0$.If $f, g \in L[\varphi(a)+\eta(\varphi(b),\varphi(a))]$, then the following inequalities hold,*

$$2^{2(p+\mu)-1} f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) g\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\leq \frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} [(\varphi(u)-\varphi(a))(\eta(\varphi(b),\varphi(a))-(\varphi(u)-\varphi(a)))]^{2(p+\mu)}$$

$$\times f(\varphi(u))g(\varphi(u))d\varphi(u)$$

$$\leq B(2p+1,2\mu+1)[M(\varphi(a),\varphi(b))+N(\varphi(a),\varphi(b))],$$

where

$$M(\varphi(a),\varphi(b)) = f(\varphi(a))g(\varphi(a))+f(\varphi(b))g(\varphi(b)),$$

$$N(\varphi(a),\varphi(b)) = f(\varphi(a))g(\varphi(b))+f(\varphi(b))g(\varphi(a)).$$

Proof. Since f and g are $(\varphi,p,\mu)-$preinvex function, afterwards,

$$f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) g\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\leq \left[\frac{f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))+f(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))}{2^{p+\mu}}\right]$$

$$\times \left[\frac{g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))+g(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))}{2^{p+\mu}}\right]$$

$$= \frac{1}{2^{2(p+\mu)}}[f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))$$

$$+f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))g(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))$$

$$+f(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a))g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))$$

$$+f(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a))g(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))].$$

This implies

$$f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)g\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\leq \frac{1}{2^{2(p+\mu)}}\left[f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))\right.$$

$$+f(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a))g(\varphi(a)+(1-\tau)\eta(\varphi(b),\varphi(a)))$$

$$\left.+2\tau 2^p(1-\tau)^{2\mu}[f(\varphi(a))+f(\varphi(b))][g(\varphi(a))+g(\varphi(b))]\right].$$

Integraling the preceding inequality with respect to τ on $[0,1]$, we get,

$$f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)g\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\leq \frac{1}{2^{2(p+\mu)-1}\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}\left[(\varphi(u)-\varphi(a))(\eta(\varphi(b),\varphi(a))-(\varphi(u)-\varphi(a)))\right]^{2(p+\mu)}$$

$$\times f(\varphi(u))g(\varphi(u))d\varphi(u)$$

$$\leq B(2p+1,2\mu+1)[M(\varphi(a),\varphi(b))+N(\varphi(a),\varphi(b))].$$

The proof is completed.

If we take $\varphi(x)=x$ in Theorem 2 , we obtain the results in [21] . □

Theorem 2.5. *If J be an interval $\varphi(a),\varphi(b)\in J$ with $\varphi(a)<\varphi(a)+\eta(\varphi(b),\varphi(a))$ and $\varphi: J\to\mathbb{R}$ a continuous increasing function. Let $f,g: K_\eta\to\mathbb{R}$ be (φ,p,μ) −preinvex functions on $J=[\varphi(a),\varphi(b)]$ with respect to bifunction $\eta(.,.)$, and $\eta(\varphi(b),\varphi(a))>0$.If $f,g\in L[\varphi(a)+\eta(\varphi(b),\varphi(a))]$, following is obtained,*

$$\frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}f(\varphi(u))g(\varphi(u))d\varphi(u)$$

$$\leq B(2p+1,2\mu+1)[M(\varphi(a),\varphi(b))+N(\varphi(a),\varphi(b))],$$

where

$$M(\varphi(a),\varphi(b))=f(\varphi(a))g(\varphi(a))+f(\varphi(b))g(\varphi(b)),$$

$$N(\varphi(a),\varphi(b))=f(\varphi(a))g(\varphi(b))+f(\varphi(b))g(\varphi(a)).$$

Proof. Since f and g are (φ,p,μ) −preinvex functions, afterwards,

$$f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))\leq\tau^p(1-\tau)^\mu[f(\varphi(a))+f(\varphi(b))],$$

and

$$g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))\leq\tau^p(1-\tau)^\mu[g(\varphi(a))+g(\varphi(b))].$$

Multiplying both sides of the preceding inequality and afterwards integrating the resulting inequality with respect to τ over $[0,1]$, we obtain,

$$\int_0^1 f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))g(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))d\tau$$

$$\leq\int_0^1\tau^{2p}(1-\tau)^{2\mu}[f(\varphi(a))+f(\varphi(b))][g(\varphi(a))+g(\varphi(b))]d\tau.$$

i.e,

$$\frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\varphi(u))g(\varphi(u))d\varphi(u)$$

$$\leq B(2p+1,2\mu+1)\left[M(\varphi(a),\varphi(b))+N(\varphi(a),\varphi(b))\right],$$

which complete the proof.

If we take $\varphi(x)=x$ in Theorem 3 , we obtain the results in [21] . □

Lemma 2.6. *Let J be an interval $\varphi(a),\varphi(b)\in J$ with $\varphi(a)<\varphi(a)+\eta(\varphi(b),\varphi(a))$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:K_\eta\to\mathbb{R}$ be a (φ,p,μ) −preinvex function on $J=[\varphi(a),\varphi(b)]$ so that $f\in L[\varphi(a)+\eta(\varphi(b),\varphi(a))]$, afterwards*

$$\int_0^1\tau^{\frac{\alpha}{k}}(1-\tau)^{\frac{\beta}{k}}f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))d\tau$$

$$=\frac{1}{\eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1}(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}(\varphi(u)-\varphi(a))^{\frac{\alpha}{k}}(\varphi(a)+\eta(\varphi(b),\varphi(a))-\varphi(u))^{\frac{\beta}{k}}$$

$$\times f(\varphi(u))d\varphi(u).$$

Proof. Suppose a function $f\in L[\varphi(a)+\eta(\varphi(b),\varphi(a))], k\geq 1$, afterwards,

$$\int_0^1\tau^{\frac{\alpha}{k}}(1-\tau)^{\frac{\beta}{k}}f(\varphi(a)+\tau\eta(\varphi(b),\varphi(a)))d\tau$$

i.e. with $\varphi(u)=\varphi(a)+\tau\eta(\varphi(b),\varphi(a)),\ d\varphi(u)=\eta(\varphi(b),\varphi(a))\,d\tau$ in the above integral, then

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}\left(\frac{\varphi(u)-\varphi(a)}{\eta(\varphi(b),\varphi(a))}\right)^{\frac{\alpha}{k}}\left(1-\frac{\varphi(u)-\varphi(a)}{\eta(\varphi(b),\varphi(a))}\right)^{\frac{\beta}{k}}\frac{d\varphi(u)}{\eta(\varphi(b),\varphi(a))}$$

$$=\frac{1}{\eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1}(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}(\varphi(u)-\varphi(a))^{\frac{\alpha}{k}}(\varphi(a)+\eta(\varphi(b),\varphi(a))-\varphi(u))^{\frac{\beta}{k}}$$

$$\times f(\varphi(u))d\varphi(u),$$

which complete the proof.

If we take $\varphi(x)=x$, and $k=1$ in Lemma 1 , we obtain the results in [21] . □

Theorem 2.7. *LetJ be an interval $\varphi(a),\varphi(b)\in J$ with $\varphi(a)<\varphi(a)+\eta(\varphi(b),\varphi(a))$ and $\varphi:J\to\mathbb{R}$ a continuous increasing function. Let $f:K_\eta\to\mathbb{R}$ be (φ,p,μ) −preinvex function on $J=[\varphi(a),\varphi(b)]$ with respect to bifunction $\eta(.,.)$ with $\eta(\varphi(b),\varphi(a))>0$.If $f\in L[\varphi(a)+\eta(\varphi(b),\varphi(a))]$, afterwards*

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))}(\varphi(u)-\varphi(a))^{\frac{\alpha}{k}}(\varphi(a)+\eta(\varphi(b),\varphi(a))-\varphi(u))^{\frac{\beta}{k}}f(\varphi(u))d\varphi(u)$$

$$\leq\eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1}(\varphi(b),\varphi(a))\,B\left(\frac{\alpha}{k}+p+1,\frac{\beta}{k}+\mu+1\right)[f(\varphi(a))+f(\varphi(b))].$$

Proof. Using Lemma 1, definition of Beta function the fact that f is (φ, p, μ) −preinvex function, for some fixed $k \geq 1$ we have

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi(u) - \varphi(a))^{\frac{\alpha}{k}} (\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(u))^{\frac{\beta}{k}} f(\varphi(u)) d\varphi(u)$$

$$= \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \int_0^1 \tau^{\frac{\alpha}{k}} (1-\tau)^{\frac{\beta}{k}} f(\varphi(a) + \tau\eta(\varphi(b), \varphi(a))) d\tau$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \int_0^1 \tau^{\frac{\alpha}{k}+p} (1-\tau)^{\frac{\beta}{k}+\mu} [f(\varphi(a)) + f(\varphi(b))] d\tau$$

$$= \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) B\left(\frac{\alpha}{k} + p + 1, \frac{\beta}{k} + \mu + 1\right) [f(\varphi(a)) + f(\varphi(b))],$$

which complete the proof.
If we take $\varphi(x) = x$ and $k = 1$ in Theorem 4 , we obtain the results in [21] . □

Theorem 2.8. *If J be an interval $\varphi(a), \varphi(b) \in J$ with $\varphi(a) < \varphi(a) + \eta(\varphi(b), \varphi(a))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : K_\eta \to \mathbb{R}$ be (φ, p, μ) −preinvex function on $J = [\varphi(a), \varphi(b)]$ with respect to bifunction $\eta(.,.)$, and $\eta(\varphi(b), \varphi(a)) > 0$.If $f \in L[\varphi(a) + \eta(\varphi(b), \varphi(a))]$, afterwards*

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi(u) - \varphi(a))^{\frac{\alpha}{k}} (\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(u))^{\frac{\beta}{k}} f(\varphi(u)) d\varphi(u)$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(r\frac{\alpha}{k} + 1, \frac{\beta}{k} r + 1)\right]^{\frac{1}{r}}$$

$$\times \left[B(p+1, \mu+1) \left(|f(\varphi(a))|^{\frac{r}{r-1}} + |f(\varphi(b))|^{\frac{r}{r-1}}\right)\right]^{\frac{r-1}{r}}.$$

Proof. From Lemma 1, Hölder inequality, definition of beta functions and the fact that $|f|^{\frac{r}{r-1}}$ is (φ, p, μ) −preinvex function, for some fixed $r > 1$, we have

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi(u) - \varphi(a))^{\frac{\alpha}{k}} (\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(u))^{\frac{\beta}{k}} f(\varphi(u)) d\varphi(u)$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[\int_0^1 \tau^{r\frac{\alpha}{k}} (1-\tau)^{\frac{\beta}{k}r}\right]^{\frac{1}{r}}$$

$$\times \left[\int_0^1 |f(\varphi(a) + \eta(\varphi(b), \varphi(a)))|^{\frac{r}{r-1}} d\tau\right]^{\frac{r-1}{r}}$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(r\frac{\alpha}{k}+1, \frac{\beta}{k}r+1)\right]^{\frac{1}{r}}$$

$$\times \left[\int_0^1 \tau^p (1-\tau)^\mu |f(\varphi(a) + \eta(\varphi(b), \varphi(a)))|^{\frac{r-1}{r}} d\tau\right]$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(r\frac{\alpha}{k}+1, \frac{\beta}{k}r+1)\right]^{\frac{1}{r}}$$

$$\times \left[\int_0^1 \tau^p (1-\tau)^\mu \left[|f(\varphi(a))|^{\frac{r}{r-1}} + |f(\varphi(b))|^{\frac{r}{r-1}}\right] d\tau\right]^{\frac{r-1}{r}}$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(r\frac{\alpha}{k}+1, \frac{\beta}{k}r+1)\right]^{\frac{1}{r}}$$

$$\times \left[B(p+1, \mu+1) \left(|f(\varphi(a))|^{\frac{r}{r-1}} + |f(\varphi(b))|^{\frac{r}{r-1}}\right)\right]^{\frac{r-1}{r}}.$$

The proof is done.

If we take $\varphi(x) = x$ and $k = 1$ in Theorem 5 , we obtain the results in [21] . □

Theorem 2.9. *If J be an interval $\varphi(a), \varphi(b) \in J$ with $\varphi(a) < \varphi(a)+\eta(\varphi(b), \varphi(a))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : K_\eta \to \mathbb{R}$ be (φ, p, μ) −preinvex function on $J = [\varphi(a), \varphi(b)]$ with respect to bifunction $\eta(.,.)$, and $\eta(\varphi(b), \varphi(a)) > 0$.If $f \in L[\varphi(a) + \eta(\varphi(b), \varphi(a))]$, afterwards*

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi(u) - \varphi(a))^{\frac{\alpha}{k}} (\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(u))^{\frac{\beta}{k}} f(\varphi(u)) d\varphi(u)$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(\frac{\alpha}{k}+1, \frac{\beta}{k}+1)\right]^{\frac{r-1}{r}}$$

$$\times \left[B(\frac{\alpha}{k}+p+1, \frac{\beta}{k}+\mu+1) (|f(\varphi(a))|^r + |f(\varphi(b))|^r)\right]^{\frac{1}{r}}.$$

Proof. Using Lemma 1, Holder's inequality, definition of beta functions and the fact that $|f|^r$ is (φ, p, μ) −preinvex function and for some fixed $r > 1$, then

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} (\varphi(u) - \varphi(a))^{\frac{\alpha}{k}} (\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(u))^{\frac{\beta}{k}} f(\varphi(u)) d\varphi(u)$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[\int_0^1 (1-\tau)^{\frac{\alpha}{k}} \tau^{\frac{\beta}{k}} d\tau\right]^{\frac{r-1}{r}}$$

$$\times \left[\int_0^1 \tau^{\frac{\alpha}{k}} (1-\tau)^{\frac{\beta}{k}} |f(\varphi(a) + \tau\eta(\varphi(b), \varphi(a))|^r d\tau\right]^{\frac{1}{r}}$$

$$\leq \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(\tfrac{\alpha}{k}+1, \tfrac{\beta}{k}+1\right]^{\frac{r-1}{r}}$$

$$\times \left[\int_0^1 \varphi(\tau)^{\alpha} (1-\varphi(\tau))^{\beta} [|f(\varphi(a))|^r + |f(\varphi(b))|^r] d\tau\right]^{\frac{1}{r}}$$

$$= \eta^{\frac{\alpha}{k}+\frac{\beta}{k}+1} (\varphi(b), \varphi(a)) \left[B(\tfrac{\alpha}{k}+1, \tfrac{\beta}{k}+1\right]^{\frac{r-1}{r}}$$

$$\times \left[B(\tfrac{\alpha}{k}+p+1, \tfrac{\beta}{k}+\mu+1) [|f(\varphi(a))|^r + |f(\varphi(b))|^r]\right]^{\frac{1}{r}}.$$

Thus the proof is completed.

If we take $\varphi(x) = x$ and $k = 1$ in Theorem 6 , we obtain the results in [21] . □

Using different technique, we obtain the following theorem.

Theorem 2.10. *If J be an interval $\varphi(a), \varphi(b) \in J$ with $\varphi(a)+\eta(\varphi(b), \varphi(a))$ and $\varphi : J \to \mathbb{R}$ a continuous increasing function. Let $f : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \to (0, \infty)$ be differentiable and (φ, p, μ) −preinvex function on $J = [\varphi(a), \varphi(b)]$ afterwards,*

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(x)\, dx \geq \frac{1}{\eta(\varphi(b),\varphi(a))} f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)$$

$$\times \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} \exp\left[\left\langle \frac{f'\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)}{f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)}, \eta\left(x, \frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right)\right\rangle\right].$$

Proof. Suppose a function $f \in L[\varphi(a) + \eta(\varphi(b), \varphi(a))]$ and f is $(\varphi, p, \mu)-preinvex$ function. afterwards,

$$\log f(x) - \log f(y) \geq \left\langle \tfrac{d}{d\tau}(\log f(y)), \eta(x, y)\right\rangle,$$

namely,

$$\log \frac{f(x)}{f(y)} \geq \left\langle \frac{f'(y)}{f(y)}, \eta(x, y)\right\rangle,$$

$$f(x) \geq f(y) \exp\left[\left\langle \frac{f'(y)}{f(y)}, \eta(x, y)\right\rangle\right].$$

Taking $y = \frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}$,in the preceding inequality, we have

$$f(x) \geq f\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \exp\left[\left\langle \frac{f'(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}{f(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}, \eta(x, \frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})\right\rangle\right],$$

$$x \in [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))]$$

integrating the preceding inequality with respect to x on $[\varphi(a), \varphi(a)+\eta(\varphi(b),\varphi(a))]$, we have,

$$\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(x)\,dx$$

$$\geq \frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})$$

$$\times \exp\left[\left\langle \frac{f'(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}{f(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}, \eta(x, \frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})\right\rangle\right] dx$$

$$\geq \exp\left[\frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} f(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})\right.$$

$$\left.\times \left\langle \frac{f'(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}{f(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})}, \eta(x, \frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2})\right\rangle dx\right].$$

Hence the proof is completed.

If we take $\varphi(x) = x$ in Theorem 7 , we obtain the results in [22] . □

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