MATHEMATICAL SCIENCES AND APPLICATIONS E-NOTES

8 (1) 69-77 (2020) ©MSAEN

HTTPS://DOI.ORG/10.36753/MATHENOT.661351

An Extended Family of Slant Curves in $S-$manifolds

Şaban Güvenç

Abstract
In this paper, we define an extended family of slant curves (i.e. $\theta_\alpha-$slant curves) in $S-$manifolds. We give two examples of such curves in $\mathbb{R}^{2n+s}(-3s)$, where we choose $n = 1$, $s = 2$. Finally, we study biharmonicity of these curves in $S-$space forms.

Keywords: *$\theta_\alpha-$slant curve; $S-$manifold; biharmonic curve.*

AMS Subject Classification (2020): *Primary: 53C25; Secondary: 53C40; 53A04.*

1. Introduction

In [6], J. Eells and L. Maire studied selected topics in harmonic maps. In this paper, they suggested k-harmonic maps. G. Y. Jiang dealed with the case $k = 2$ in [11]. He derived the first and second variational formulas for 2-harmonic maps. On the other hand, in [4], B. Y. Chen published a survey article, which is divided into 25 sections. In one of these sections, he considered a biharmonic submanifold of Euclidean space as $\Delta H = 0$, where Δ denotes the Laplace operator and H denotes the mean curvature vector field. If the ambient space is considered as Euclidean, then Jiang's and Chen's results match.

In [5], J. T. Cho, J. Inoguchi and J. E. Lee defined and studied slant curves in Sasakian manifolds. They proved a theorem, which is similar to the classical theorem of Lancret for curves in Euclidean 3-space. They showed that a non-geodesic curve in a Sasakian 3-manifold is a slant curve if and only if the ratio of $(\tau \pm 1)$ and κ is constant, where κ and τ denotes the geodesic curvature and torsion of the curve, respectively. They gave some interesting examples. Notably, in the Heisenberg group with an appropriate metric, they exhibited slant helices and a slant curve which is not a helix.

In [8], D. Fetcu and C. Oniciuc obtained a method of producing biharmonic submanifolds in a Sasakian space form using the flow of characteristic vector field ξ. They showed that under the flow action of ξ a biharmonic integral submanifold is carried to a biharmonic anti-invariant submanifold. Following their idea, the present author and C. Özgür considered biharmonic slant curves in S-space forms [9].

It is a natural motivation to generalize the results of slant curves to $\theta_\alpha-$slant curves in $S-$manifolds. In Section 2, we give the fundamental definitions and theorems of S-space forms, biharmonic maps and Frenet curves. In Section 3, we define an extented family of slant curves, namely $\theta_\alpha-$slant curves, in $S-$manifolds and give two examples. In Section 4, we obtain the necessary and sufficient conditions for $\theta_\alpha-$slant curves in $S-$space forms to be proper biharmonic.

Received : *18–12–2019,* ***Accepted :*** *14–02–2020*

2. Preliminaries

Let (M,g) be a $(2n+s)$-dimensional Riemann manifold. M is called a *framed metric manifold* with a *framed metric structure* $(f,\xi_\alpha,\eta^\alpha,g)$, $\alpha\in\{1,...,s\}$, if it satisfies:

$$\begin{gathered} f^2X=-X+\sum_{\alpha=1}^{s}\eta^\alpha(X)\xi_\alpha,\quad \eta^\alpha(f(X))=0,\quad \eta^\alpha(\xi_\beta)=\delta_{\alpha\beta},\quad f(\xi_\alpha)=0,\\ g(X,Y)=g(fX,fY)+\sum_{\alpha=1}^{s}\eta^\alpha(X)\eta^\alpha(Y),\\ \eta^\alpha(X)=g(X,\xi),\quad d\eta^\alpha(X,Y)=-d\eta^\alpha(Y,X)=g(X,fY), \end{gathered} \tag{2.1}$$

where f is a $(1,1)-$type tensor field of rank $2n$; $\xi_1,...,\xi_s$ are vector fields; $\eta^1,...,\eta^s$ are 1-forms and g is a Riemannian metric on M; $X,Y\in TM$ and $\alpha,\beta\in\{1,...,s\}$ (see [13], [15]). $(f,\xi_\alpha,\eta^\alpha,g)$ is said to be an *S-structure*, if the Nijenhuis tensor of f is equal to $-2d\eta^\alpha\otimes\xi_\alpha$, for all $\alpha\in\{1,...,s\}$ [1].

If $s=1$, a framed metric structure is the same as an almost contact metric structure and an S-structure is the same as a Sasakian structure. For an S-structure, we have the following equations [1]:

$$(\nabla_X f)Y=\sum_{\alpha=1}^{s}\left\{g(fX,fY)\xi_\alpha+\eta^\alpha(Y)f^2X\right\}, \tag{2.2}$$

and

$$\nabla\xi_\alpha=-f\ , \tag{2.3}$$

for all $\alpha=1,...,s$. In case of $s=1$, (2.3) can be calculated from (2.2).

Let $X\in T_pM$ be orthogonal to $\xi_1,...,\xi_s$. The plane section spanned by $\{X,fX\}$ is called an *f-section* in T_pM and its sectional curvature is called an *f-sectional curvature*. Let $(M,f,\xi_\alpha,\eta^\alpha,g)$ be an S-manifold. If M has constant f-sectional curvature, its curvature tensor R *is* given by

$$\begin{aligned} R(X,Y)Z=\sum_{\alpha,\beta}&\left\{\eta^\alpha(X)\eta^\beta(Z)f^2Y-\eta^\alpha(Y)\eta^\beta(Z)f^2X\right.\\ &\left.-g(fX,fZ)\eta^\alpha(Y)\xi_\beta+g(fY,fZ)\eta^\alpha(X)\xi_\beta\right\}\\ &+\tfrac{c+3s}{4}\left\{-g(fY,fZ)f^2X+g(fX,fZ)f^2Y\right\}\\ &\tfrac{c-s}{4}\left\{g(X,fZ)fY-g(Y,fZ)fX+2g(X,fY)fZ\right\}, \end{aligned} \tag{2.4}$$

for $X,Y,Z\in TM$ [3]. In this case, M is called an *S-space form* and it is denoted by $M(c)$. In case of $s=1$, an S-space form is the same as a Sasakian space form [2].

Let (M,g) and (N,h) be Riemannian manifolds and $\varphi:M\to N$ a differentiable map. A *harmonic map* is a critical point of the energy functional of φ, which is defined as

$$E(\varphi)=\frac{1}{2}\int_M|d\varphi|^2\,\upsilon_g,$$

(see [7]). Furthermore, a *biharmonic map* is a critical point of the bienergy functional

$$E_2(\varphi)=\frac{1}{2}\int_M|\tau(\varphi)|^2\,\upsilon_g,$$

where $\tau(\varphi)=trace\nabla d\varphi$ and it is called *the first tension field* of φ. Jiang derived the biharmonic map equation [11]

$$\tau_2(\varphi)=-J^\varphi(\tau(\varphi))=-\Delta\tau(\varphi)-traceR^N(d\varphi,\tau(\varphi))d\varphi=0,$$

where J^φ denotes the Jacobi operator of φ. It is obvious that harmonic maps are biharmonic. So, non-harmonic biharmonic maps are called *proper biharmonic*.

Let $\gamma:I\to M$ be a unit-speed curve in an n-dimensional Riemannian manifold (M,g). The curve γ is called a *Frenet curve of osculating order* r $(1\leq r\leq n)$, if there exists orthonormal vector fields $T,E_2,...,E_r$ along the curve validating the Frenet equations

$$\begin{aligned} T &= \gamma', \\ \nabla_T T &= \kappa_1 E_2, \\ \nabla_T E_2 &= -\kappa_1 T + \kappa_2 E_3, \\ &\dots \\ \nabla_T E_r &= -\kappa_{r-1} E_{r-1}, \end{aligned} \tag{2.5}$$

where $\kappa_1, ..., \kappa_{r-1}$ are positive functions called the curvatures of γ. If $\kappa_1 = 0$, then γ is called a *geodesic*. If κ_1 is a non-zero positive constant and $r = 2$, γ is called a *circle*. If $\kappa_1, ..., \kappa_{r-1}$ are non-zero positive constants, then γ is called a *helix of order r* $(r \geq 3)$. If $r = 3$, it is shortly called a *helix*.

A submanifold of an S-manifold is said to be an *integral submanifold* if $\eta^\alpha(X) = 0$, $\alpha \in \{1, ..., s\}$, where X is tangent to the submanifold [12]. A *Legendre curve* is a 1-dimensional integral submanifold of an S-manifold $(M^{2n+s}, f, \xi_\alpha, \eta^\alpha, g)$. More precisely, a unit-speed curve $\gamma : I \to M$ is a Legendre curve if T is g-orthogonal to all ξ_α $(\alpha = 1, ...s)$, where $T = \gamma'$ [14].

3. $\theta_\alpha-$Slant Curves in $S-$manifolds

In this section, we define an extension of slant curves in S-manifolds. Firstly, we give the following definition:

Definition 3.1. Let $M = (M^{2n+s}, f, \xi_\alpha, \eta^\alpha, g)$ be an S-manifold and $\gamma : I \to M$ a unit-speed curve. γ is called a $\theta_\alpha-$*slant curve*, if there exist constant angles θ_α $(\alpha = 1, ..., s)$ such that $\eta^\alpha(T) = \cos\theta_\alpha$. Here, we call θ_α the *contact angles of* γ.

One can easily see that Definition 3.1 extends the family of slant curves to $\theta_\alpha-$slant curves. In particular, a $\theta_\alpha-$slant curve is called *slant* if its all contact angles are equal (see [9]).

For a a $\theta_\alpha-$slant curve, if we differentiate $\eta^\alpha(T) = \cos\theta_\alpha$ along the curve γ, we obtain

$$\eta^\alpha(E_2) = 0, \tag{3.1}$$

for all $\alpha = 1, ..., s$. From now on, we use the following notations:

$$A = \sum_{\alpha=1}^{s} \cos^2\theta_\alpha, \ B = \sum_{\alpha=1}^{s} \cos\theta_\alpha, \ V = \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha.$$

The following corollary is directly obtained:

Corollary 3.1. *If γ is slant, then*

$$A = s\cos^2\theta, B = s\cos\theta, V = \cos\theta \sum_{\alpha=1}^{s} \xi_\alpha,$$

where θ denotes the equal contact angles of γ.

Let γ be a non-geodesic unit-speed $\theta_\alpha-$slant curve. Using equation 2.1, we find

$$g(fT, fT) = 1 - A \geq 0.$$

If $A = 1$, then we have $fT = 0$, that is, $T = V$. Hence, we get

$$\nabla_T T = \nabla_V V = 0,$$

which means γ is a geodesic. As a result, we can give the following proposition:

Proposition 3.1. *For a non-geodesic unit-speed $\theta_\alpha-$slant curve in an S-manifold,*

$$A = \sum_{\alpha=1}^{s} \cos^2\theta_\alpha < 1.$$

Note that, if γ is slant, we obtain Proposition 3.1 in [9].

From equations 2.1 and 2.5, we obtain the following statement:

Proposition 3.2. *For a non-geodesic unit-speed $\theta_\alpha-$slant curve in an S-manifold $(M, f, \xi_\alpha, \eta^\alpha, g)$, we have*

$$\nabla_T fT = (1-A)\sum_{\alpha=1}^{s}\xi_\alpha + B(-T+V) + \kappa_1 f E_2. \tag{3.2}$$

Now we give the following examples of non-trivial $\theta_\alpha-$slant curves in $\mathbb{R}^{2n+s}(-3s)$, choosing $n=1$, $s=2$. For detailed information on $\mathbb{R}^{2n+s}(-3s)$, see [10].

Example 3.1. $\gamma : I \to \mathbb{R}^4(-6)$, $\gamma(t) = \left(t, 0, t, \sqrt{2}t\right)$ is a $\theta_\alpha-$slant curve with the contact angles $\theta_1 = \frac{\pi}{3}$, $\theta_2 = \frac{\pi}{4}$. In fact, γ is a $\theta_\alpha-$slant circle with $\kappa_1 = \frac{\sqrt{2}+1}{2}$.

Example 3.2. Let c_i be arbitrary constants $(i = 1, ..., 4)$, $t_0 \in I$, θ_1 and θ_2 constants such that $A = \cos^2\theta_1 + \cos^2\theta_2 < 1$. Let us consider a smooth function $u : I \to \mathbb{R}$ and define $\gamma_i : I \to \mathbb{R}$ $(i = 1, ..., 4)$ as

$$\gamma_1(t) = c_1 + 2\sqrt{1-A}\int_{t_0}^{t}\cos(u(p))dp,$$

$$\gamma_2(t) = c_2 + 2\sqrt{1-A}\int_{t_0}^{t}\sin(u(p))dp,$$

$$\gamma_3(t) = c_3 + 2t\cos\theta_1$$
$$+2\sqrt{1-A}\int_{t_0}^{t}\cos(u(q))\left(c_2 + 2\sqrt{1-A}\int_{t_0}^{q}\sin(u(p))dp\right)dq,$$

$$\gamma_4(t) = c_4 + 2t\cos\theta_2$$
$$+2\sqrt{1-A}\int_{t_0}^{t}\cos(u(q))\left(c_2 + 2\sqrt{1-A}\int_{t_0}^{q}\sin(u(p))dp\right)dq.$$

Then $\gamma : I \to \mathbb{R}^4(-6)$, $\gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t), \gamma_4(t))$ is a $\theta_\alpha-$slant curve with the contact angles θ_1 and θ_2.

4. Biharmonic $\theta_\alpha-$Slant Curves in $S-$Space Forms

In this section, we consider proper biharmonic $\theta_\alpha-$slant curves in S-space forms. Let γ be a unit-speed $\theta_\alpha-$slant curve in an S-space form $(M, f, \xi_\alpha, \eta^\alpha, g)$. Then, we have

$$R(T, \nabla_T T)T = -\kappa_1\left[B^2 + \frac{c+3s}{4}(1-A)\right]E_2 - 3\kappa_1\frac{c-s}{4}g(fT, E_2)fT,$$

$$\begin{aligned}
\tau_2(\gamma) &= \nabla_T\nabla_T\nabla_T T - R(T, \nabla_T T)T \\
&= -3\kappa_1\kappa_1' T \\
&\quad + \left(\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + \kappa_1\left[B^2 + \frac{c+3s}{4}(1-A)\right]\right)E_2 \\
&\quad + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')E_3 + \kappa_1\kappa_2\kappa_3 E_4 \\
&\quad + 3\kappa_1\frac{c-s}{4}g(fT, E_2)fT.
\end{aligned} \tag{4.1}$$

As a result, we can state the following theorem:

Theorem 4.1. *γ is a proper-biharmonic $\theta_\alpha-$slant curve in an S-space form $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if $\kappa_1 = constant > 0$ and*

$$3\frac{c-s}{4}g\left(fT, E_2\right)fT = \left[\kappa_1^2 + \kappa_2^2 - B^2 - \frac{c+3s}{4}\left(1-A\right)\right]E_2 - \kappa_2' E_3 - \kappa_2\kappa_3 E_4. \tag{4.2}$$

Proof. Let γ be a proper-biharmonic $\theta_\alpha-$slant curve. Then $\kappa_1 > 0$ and $\tau_2(\gamma) = 0$. If we take the inner-product of both sides with T, we find $\kappa_1 = constant > 0$. Hence, from equation (4.1), we obtain equation (4.2). Conversely, if $\kappa_1 = constant > 0$ and equation (4.2) is satisfied, we find $\tau_2(\gamma) = 0$, which completes the proof. □

We will consider equation (4.2) from all points of view. Our discussions are based on the question: "When do the coefficients of fT vanish?". First discussion is for the absence of the term with fT in equation (4.2). Second discussion is for the non-vanishing coefficients.

First Discussion: The term with fT vanishes.

i) $c = s$.

In this case, equation (4.2) becomes

$$0 = \left[\kappa_1^2 + \kappa_2^2 - B^2 - s\left(1-A\right)\right]E_2 - \kappa_2' E_3 - \kappa_2\kappa_3 E_4. \tag{4.3}$$

As a result, we give the following Theorem:

Theorem 4.2. *Under the assumption $c = s$; γ is a proper-biharmonic $\theta_\alpha-$slant curve in $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if either γ is a circle with $\kappa_1 = \sqrt{B^2 + s(1-A)}$ or a helix with $\kappa_1^2 + \kappa_2^2 = B^2 + s(1-A)$.*

Proof. From equation (4.3), since $\{E_2, E_3, E_4\}$ is $g-$orthonormal, the proof is clear. □

ii) $c \neq s$ and $fT \perp E_2$.

Under these assumptions, equation (4.2) gives us

$$0 = \left[\kappa_1^2 + \kappa_2^2 - B^2 - \frac{c+3s}{4}\left(1-A\right)\right]E_2 - \kappa_2' E_3 - \kappa_2\kappa_3 E_4. \tag{4.4}$$

Firstly, we need to prove the following Lemma:

Lemma 4.1. *Let γ be a $\theta_\alpha-$slant curve of order $r = 3$ in an S-space form $(M, f, \xi_\alpha, \eta^\alpha, g)$ and $fT \perp E_2$. Then, $\{T, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent.*

Proof. Let $r = 3$ and $fT \perp E_2$. Let us denote $S_1 = \{T, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$. In view of equations (2.5), (3.1) and (3.2), we have

$$\begin{aligned} g(E_2, T) &= g(E_2, E_3) = g(E_2, fT) = g(E_2, \nabla_T fT) \\ &= g(E_2, \xi_\alpha) = 0, \end{aligned}$$

for all $\alpha = 1, ..., s$. Thus, S_1 is linearly independent if and only if $S_2 = \{T, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. From the assumption, we have $fT \perp E_2$. If we differentiate $g(fT, E_2) = 0$, we find $g(fT, E_3) = 0$. Since $g(fT, fT) = 1 - A > 0$ is a constant, we obtain $g(fT, \nabla_T fT) = 0$. f is skew-symmetric, so $g(fT, T) = 0$. From equation (2.1), we also have $g(fT, \xi_\alpha) = 0$, for all $\alpha = 1, ..., s$. Then, omitting fT, we get that S_2 is linearly independent if and only if $S_3 = \{T, E_3, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. Now, let us investigate whether T is linearly dependent with other vector fields in S_3. From Frenet equations, $g(T, E_3) = 0$. Equation (3.2) gives us $g(T, \nabla_T fT) = 0$. Assume that $T \in sp\{\xi_1, ..., \xi_s\}$. If we differentiate

$$T = \sum_{\alpha=1}^{s} \cos\theta_\alpha \xi_\alpha,$$

along the curve γ, we get $\kappa_1 = 0$, which is a contradiction. As a result, $T \notin sp\{\xi_1, ..., \xi_s\}$. Hence, S_3 is linearly independent if and only if $S_4 = \{E_3, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. If we differentiate $g(fT, E_3) = 0$, we find $g(\nabla_T fT, E_3) = 0$. Now, let us assume $E_3 \in sp\{\xi_1, ..., \xi_s\}$. If we differentiate

$$E_3 = \sum_{\alpha=1}^{s} \eta^\alpha(E_3)\xi_\alpha,$$

we obtain

$$-\kappa_2 E_2 = \sum_{\alpha=1}^{s} \left\{ \nabla_T \left[\eta^\alpha(E_3) \right] .\xi_\alpha - \eta^\alpha(E_3) fT \right\}.$$

If we take the inner-product of both sides with E_2, we find $\kappa_2 = 0$, which contradicts $r = 3$. Then, $E_3 \notin sp\{\xi_1, ..., \xi_s\}$. So, S_4 is linearly independent if and only if $S_5 = \{\nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent. Equation (3.2) can be rewritten as

$$\nabla_T fT = \sum_{\alpha=1}^{s} \left[(1-A) + B\cos\theta_\alpha \right] \xi_\alpha - BT + \kappa_1 f E_2.$$

Since $fT \perp E_2$ and f is skew-symmetric, we have $fE_2 \perp T$. As a result, the term $(-BT + \kappa_1 fE_2)$ does not vanish, that is, $\nabla_T fT \notin sp\{\xi_1, ..., \xi_s\}$. Consequently, S_5 is linearly independent and the proof is complete. □

In view of Lemma 4.1, we can state the following theorem:

Theorem 4.3. *Under the assumptions $c \neq s$ and $fT \perp E_2$; γ is a proper-biharmonic $\theta_\alpha-$slant curve in $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if either*

a) $dim(M) \geq 4 + s$ and γ is a circle with $\kappa_1 = \frac{1}{2}\sqrt{4B^2 + (c+3s)(1-A)}$, where $\{T, E_2, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent; or

b) $dim(M) \geq 5 + s$ and γ is a helix with $\kappa_1^2 + \kappa_2^2 = B^2 + \frac{c+3s}{4}(1-A)$, where $\{T, E_2, E_3, fT, \nabla_T fT, \xi_1, ..., \xi_s\}$ is linearly independent.

Proof. If we consider Lemma 4.1 and equation (4.4) together, the proof is directly obtained. □

Second Discussion: The term with fT does not vanish.

i) $c \neq s$ and $fT \parallel E_2$.

In this case, since $g(fT, fT) = 1 - A$ and $fT \parallel E_2$, we can write

$$fT = \varepsilon\sqrt{1-A}E_2, \tag{4.5}$$

where $\varepsilon = sgn(g(fT, E_2))$. Then, equation (4.2) becomes

$$3\frac{c-s}{4}(1-A)E_2 = \left[\kappa_1^2 + \kappa_2^2 - B^2 - \frac{c+3s}{4}(1-A) \right] E_2 - \kappa_2' E_3 - \kappa_2\kappa_3 E_4. \tag{4.6}$$

Firstly, we can state the following Lemma:

Lemma 4.2. *Let γ be a non-geodesic $\theta_\alpha-$slant curve in an S-space form $(M, f, \xi_\alpha, \eta^\alpha, g)$ and $fT \parallel E_2$. If κ_1 is a constant, then γ is either a circle or a helix.*

Proof. Let $\kappa_1 = constant > 0$. From equations (2.5), (3.2) and (4.5), after some calculations, we get

$$\kappa_2\varepsilon\sqrt{1-A}E_3 = (1-A)\sum_{\alpha=1}^{s}\xi_\alpha - (B + \varepsilon AD)T + (B + \varepsilon D)V, \tag{4.7}$$

where we denote $D = \kappa_1/\sqrt{1-A}$. Note that

$$g(T,T) = 1,\ g(T, \sum_{\alpha=1}^{s}\xi_\alpha) = B,\ g(T,V) = A,$$

$$g(\sum_{\alpha=1}^{s}\xi_\alpha, \sum_{\alpha=1}^{s}\xi_\alpha) = s,$$

$$g(\sum_{\alpha=1}^{s}\xi_\alpha, V) = B,\ g(V,V) = A.$$

As a result, if we denote the norm of the right-hand side of equation (4.7) by C, we have

$$C = \sqrt{1-A}\sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s},$$

which gives us

$$\kappa_2 = \sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s}.$$

So, κ_2 is a constant. If $\kappa_2 = 0$, then γ is a circle. If $\kappa_2 \neq 0$, equation (4.7) gives us

$$E_3 = a_0 T + a_1 \xi_1 + ... + a_s \xi_s,$$

for some constants $a_0, ..., a_s$. If we differentiate this last equation, we obtain

$$-\kappa_2 E_2 + \kappa_3 E_4 = a_0 \kappa_1 E_2 - a_1 fT - ... - a_s fT. \tag{4.8}$$

If we take the inner-product of equation (4.8) with E_4, considering the fact that $fT \parallel E_2$, we find $\kappa_3 = 0$. In this case, γ is a helix. □

In view of Lemma 4.2, we have the following result:

Theorem 4.4. *Under the assumptions $c \neq s$ and $fT \parallel E_2$; γ is a proper-biharmonic $\theta_\alpha-$slant curve in $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if either*

$a)$ it is a circle with $\kappa_1 = \sqrt{B^2 + c(1-A)}$ with the Frenet frame field

$$\left\{T, \frac{\varepsilon fT}{\sqrt{1-A}}\right\},$$

where $B^2 + c(1-A) > 0$; or

$b)$ it is a helix with $\kappa_1 = \sqrt{1-A}D$, $\kappa_2 = \sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s}$ with the Frenet frame field

$$\left\{T, \frac{\varepsilon fT}{\sqrt{1-A}}, \frac{\varepsilon}{\sqrt{1-A}\sqrt{AD^2 - As + B^2 + 2\varepsilon BD + s}} W\right\},$$

where $AD^2 - As + B^2 + 2\varepsilon BD + s > 0$, $D > 0$ is a constant satisfying

$$D(2\varepsilon B + D) = (1-A)(c-s) \tag{4.9}$$

and W denotes

$$W = (1-A)\sum_{\alpha=1}^{s} \xi_\alpha - (B + \varepsilon AD)T + (B + \varepsilon D)V.$$

Proof. Let γ be proper-biharmonic. Then, $\kappa_1 = constant > 0$ and equation (4.6) must be satisfied. If we take the inner-product of equation (4.6) with E_2, E_3 and E_4, we get

$$\kappa_1^2 + \kappa_2^2 = B^2 + c(1-A), \tag{4.10}$$

$$\kappa_2 = constant, \ \kappa_3 = 0,$$

respectively. From the proof of Lemma 4.2, using equation (4.10), we obtain the curvatures and the Frenet frame field of γ. Furthermore, if γ is a helix, if we replace κ_1 and κ_2 in equation (4.10), we find equation (4.9).

Conversely, let γ be a one of the curves given in $a)$ or $b)$. Then, one can easily show that equation (4.4) is verified. So, γ is proper-biharmonic. □

ii) $c \neq s$ and $g(fT, E_2) \neq 0, 1, -1$.

Since the equality cases are previously investigated, we complete our discussions under the assumptions $c \neq s$ and $g(fT, E_2) \neq 0, 1, -1$. Let us consider a smooth function $m(t)$ such that

$$g(fT, E_2) = \sqrt{1-A}\cos m(t). \tag{4.11}$$

Differentiating this equation, we have

$$\kappa_2 g(fT, E_3) = -\sqrt{1-A}\, m'(t)\sin m(t). \tag{4.12}$$

If we take the inner-product of equation (4.2) with E_2, E_3 and E_4, we find

$$\kappa_1^2 + \kappa_2^2 = B^2 + \frac{c+3s}{4}(1-A) + \frac{3(c-s)}{4} g(fT, E_2)^2, \tag{4.13}$$

$$\kappa_2' + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_3) = 0, \tag{4.14}$$

$$\kappa_2 \kappa_3 + \frac{3(c-s)}{4} g(fT, E_2) g(fT, E_4) = 0, \tag{4.15}$$

respectively. If we multiply equation (4.14) with $2\kappa_2$, equations (4.11) and (4.12), we have

$$2\kappa_2 \kappa_2' + (1-A) \frac{3(c-s)}{4} \left[-2m'(t) \sin m(t) \cos m(t)\right] = 0.$$

If we integrate the last equation, we get

$$\kappa_2^2 = -(1-A) \frac{3(c-s)}{4} \cos^2 m(t) + h_0, \tag{4.16}$$

where h_0 is an arbitrary constant. If we write equation (4.16) in (4.13), we obtain $m(t)$ is constant. As a result, we can write

$$fT = \sqrt{1-A} \left(\cos m E_2 + \sin m E_4\right),$$

where $m \in (0, 2\pi) - \left\{\frac{\pi}{2}, 0, \frac{3\pi}{2}\right\}$. Now, we can give the following theorem:

Theorem 4.5. *Under the assumptions $c \neq s$ and $g(fT, E_2) \neq 0, 1, -1$; γ is a proper-biharmonic $\theta_\alpha-$slant curve in $(M, f, \xi_\alpha, \eta^\alpha, g)$ if and only if κ_1, κ_2 and κ_3 are constants such that*

$$\kappa_1^2 + \kappa_2^2 = B^2 + \frac{c+3s}{4}(1-A) + \frac{3(c-s)}{4} \cos^2 m,$$

$$\kappa_2 \kappa_3 + \frac{3(c-s)}{8}(1-A) \sin 2m = 0,$$

where $fT = \sqrt{1-A} \left(\cos m E_2 + \sin m E_4\right)$ and $m \in (0, 2\pi) - \left\{\frac{\pi}{2}, 0, \frac{3\pi}{2}\right\}$.

References

[1] Blair, D. E.: *Geometry of manifolds with structural group $U(n) \times O(s)$*. J. Differential Geometry. **4**, 155-167 (1970).

[2] Baikoussis, C., Blair, D. E.: *On Legendre curves in contact 3-manifolds*. Geom. Dedicata. **49**, 135-142 (1994).

[3] Cabrerizo, J. L., Fernandez, L. M., Fernandez M.: *The curvature of submanifolds of an S-space form*. Acta Math. Hungar. **62,** 373-383 (1993).

[4] Chen, B.Y.: *A report on submanifolds of finite type*. Soochow J. Math. **22**, 117-337 (1996).

[5] Cho, J. T., Inoguchi, J., Lee, J. E.: *On slant curves in Sasakian 3-manifolds*. Bull. Austral. Math. Soc. **74**, 359-367 (2006).

[6] Eells, Jr. J., Lemaire, L.: Selected topics in harmonic maps. Amer. Math. Soc. Providence, R.I. (1983).

[7] Eells, Jr. J., Sampson, J. H.: *Harmonic mappings of Riemannian manifolds*. Amer. J. Math. **86**, 109-160 (1964).

[8] Fetcu, D., Oniciuc, C.: *Explicit formulas for biharmonic submanifolds in Sasakian space forms*. Pacific J. Math. **240**, 85-107 (2009).

[9] Güvenç, Ş., Özgür, C.: *On slant curves in S-manifolds*. Commun. Korean Math. Soc. **33** (1), 293-303 (2018).

[10] Hasegawa, I., Okuyama, Y., Abe, T.: *On p-th Sasakian manifolds*. J. Hokkaido Univ. Ed. Sect. II A. **37** (1), 1-16 (1986).

[11] Jiang, G. Y.: *2-harmonic maps and their first and second variational formulas*. Chinese Ann. Math. Ser. A. **7**, 389-402 (1986).

[12] Kim, J. S., Dwivedi, M. K., Tripathi, M. M.: *Ricci curvature of integral submanifolds of an S-space form*. Bull. Korean Math. Soc. **44**, 395-406 (2007).

[13] Nakagawa, H.: *On framed f-manifolds*. Kodai Math. Sem. Rep. **18,** 293-306 (1966).

[14] Özgür, C., Güvenç, Ş.: *On biharmonic Legendre curves in S-space forms*. Turkish J. Math. **38** (3), 454-461 (2014).

[15] Yano, K., Kon, M.: Structures on Manifolds. Series in Pure Mathematics, **3**. World Scientific Publishing Co. Singapore (1984).

Affiliations

ŞABAN GÜVENÇ
ADDRESS: Balikesir University, Dept. of Mathematics, 10145, Balikesir-Turkey.
E-MAIL: sguvenc@balikesir.edu.tr
ORCID ID: 0000-0001-6254-4693