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Existence and Uniqueness Results for a Computer Virus Spreading Model with Atangana-Baleanu Derivative

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Abstract

A virus is programs that are often hidden inside a file that appears to be harmless and cause unexpected and unwanted events on the computer when the file is run. Viruses, unlike others, are software designed to maliciously damage systems. Viruses have the ability to copy themselves, jump to the files you work with, and delete files, change the file content and make it unavailable. They can do all this without the knowledge of the user and without the need for any command. Therefore, in order to better understanding of computer virus, there are many mathematical models in the literature. In the present study, we consider computer virus spreading model benefiting from Atangana-Baleanu derivative in Caputo sense with non-local and non-singular kernels. The solution properties of our fractional model are established benefiting from Arzelo-Ascoli theorem.

Keywords: Arzelo-Ascoli theorem, Atangana-Baleanu derivative, existence and uniqueness results.

1. Introduction

Computer viruses are some computer programs that can damage your computer in different ways when it works. If these programs (or virus codes) are run, they will start damaging your computer according to the way they are programmed. In addition, after viruses are activated in a system, they have the ability to reproduce, spread to other files on your computer, infect other computers over the network and many more. Many mathematical models are being studied to cope with the spread of computer viruses and for better understanding their structure such as SIR model, SEIRS model, SIRS model [1-8].

Due to memory properties of fractional derivatives, fractional operators gain increasing interest from various directions in the modeling of biological process, neural networks, engineering, physics, etc [9-20]. It is clear that Riemann Liouville and Caputo fractional derivatives have singular kernels. To cope with this problem, Atangana and Baleanu have proposed a new fractional derivative named as Atangana-Baleanu (AB) fractional derivative with Mittag-Leffler kernel. There are many extensive treatment and several applications of this recently defined AB derivatives in the literature [21-28].

Taking these motivations and AB derivative into consideration, in this article we examine the existence and uniqueness conditions for the solutions of the below computer virus spreading model presented by [29]:

$$\begin{aligned} \frac{dS}{dt} &= b - \beta SI - \mu S, \\ \frac{dE}{dt} &= (1-p)\beta SI - \gamma E + \varepsilon I - \mu E, \\ \frac{dI}{dt} &= p\beta SI + \gamma E - \varepsilon I - \alpha I - \mu I. \end{aligned} \qquad \mathbf{(1.1)}$$

We choose this model because it defines a novel virus spreading system considering the possibility of a virus outbreak on a network with restricted antivirus ability. Here, the total number of computers attached the internet are separated into four class: The number of susceptible computers $S(t)$ is the set of external not infected computers which are attached to the network, the number of exposed computers $E(t)$ is the set of all latent computers at which viruses are latent, the number of infected computers $I(t)$ is the set of infected computers at which viruses are explode. The model parameters are: b represents the ratio where external computers are linked with the network, β stands for the ratio where having a link to one latent computer, γ shows the ratio where one latent computer explodes, μ displays the ratio where one computer is out from the network, α displays the recovery ratio of infected computers and connect to the the ability of the anti-virus software. Owing to a possible link with infected computers, susceptible computer is latent with possibility $(1-p)\beta I(t)$, or explodes with possibility $p\beta I(t)$ where $p>0$ is a fixed parameter. Since the network's anti-virus ability is restricted, the virus is momentarily suppressed with possibility ε where $\varepsilon>0$ is a fixed parameter.

In this paper, inspired by the above model, we give properties of the solution of our fractional model with AB derivative by Arzelo-Ascoli theorem.

2. Preliminaries

Now, we will give basic definitions related to AB fractional derivative.

Definition 2.1. Let $f\in H^1(a,b)$ be a function, $a<b$ and $r\in[0,1]$. The AB derivative in Caputo sense of f is defined as [30]

$$ {}^{ABC}_{\ 0}D^r_t f(t)=\frac{K(r)}{1-r}\int_a^t f'(x)E_r\left[-r\frac{(t-x)^r}{1-r}\right]dx \quad \textbf{(2.1)}$$

with $K(r)$ is a normalization function with $K(0)=K(1)=1$.

Definition 2.3. The fractional integral relevant to AB derivative is given by [30]

$$ {}^{AB}_{\ 0}I^r_t f(t)=\frac{1-r}{K(r)}f(t)+\frac{r}{K(r)\Gamma(r)}\int_a^t f(y)(t-y)^{r-1}dy \quad \textbf{(2.2)}$$

3. Existence and Uniqueness of the Solutions

Now, we remodel the system (1.1) using AB derivative in Caputo sense.

$$ {}^{ABC}_{\ 0}D^r_t S=b-\beta SI-\mu S,$$
$$ {}^{ABC}_{\ 0}D^r_t E=(1-p)\beta SI-\gamma E+\varepsilon I-\mu E,$$
$$ {}^{ABC}_{\ 0}D^r_t I=p\beta SI+\gamma E-\varepsilon I-\alpha I-\mu I. \quad \textbf{(3.1)}$$

In this part, the existence and uniqueness of the solutions for the model (3.1) is considered and proved by using AB derivative. For this aim, we apply AB integral to the system (3.1) and we get

$$S(t)-S(0)=K\frac{1-r}{K(r)}\{b-\beta SI-\mu S\}+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}\{b-\beta SI-\mu S\}dy,$$
$$E(t)-E(0)=\frac{1-r}{K(r)}\{(1-p)\beta SI-\gamma E+\varepsilon I-\mu E\}+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}\{(1-p)\beta SI-\gamma E+\varepsilon I-\mu E\}dy,$$
$$I(t)-I(0)=\frac{1-r}{K(r)}\{p\beta SI+\gamma E-\varepsilon I-\alpha I-\mu I\}+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}\{p\beta SI+\gamma E-\varepsilon I-\alpha I-\mu I\}dy. \quad \textbf{(3.2)}$$

For the simplification of equations in (3.2), we define our kernels below:

$$s(t,S(t))=b-\beta SI-\mu S,$$
$$s(t,E(t))=(1-p)\beta SI-\gamma E+\varepsilon I-\mu E,$$
$$s(t,I(t))=p\beta SI+\gamma E-\varepsilon I-\alpha I-\mu I. \quad \textbf{(3.3)}$$

Now, we will consider the operator $G:H\to H$ defined as

$$GS(t)=\frac{1-r}{K(r)}s(t,S(t))+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}s(y,S(y))dy,$$
$$GE(t)=\frac{1-r}{K(r)}s(t,E(t))+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}s(y,E(y))dy,$$
$$GI(t)=\frac{1-r}{K(r)}s(t,I(t))+\frac{r}{K(r)\Gamma(r)}\int_0^t(t-y)^{r-1}s(y,I(y))dy. \quad \textbf{(3.4)}$$

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Lemma 3.1. Let $M \subset H$ be a bounded set, assume that S,E and I satisfy Lipschitz condition
$\|S(t_2)-S(t_1)\| \le p\|t_2-t_1\|$,
$\|E(t_2)-E(t_1)\| \le r\|t_2-t_1\|$,
$\|I(t_2)-I(t_1)\| \le q\|t_2-t_1\|$.

For some positive constants *p*, *q* ,*r*. Then $\overline{G(M)}$ is compact.

Proof 3.1 Let $P = \max\left\{\frac{1-r}{K(r)} + s(t,S(t))\right\}$, $0 \le S(t) \le K_1$ for some positive constant K_1. For $S(t) \in M$, then we get

$$\|GS(t)\| = \frac{1-r}{K(r)}\|s(t,S(t))\| + \frac{r}{K(r)\Gamma(r)}\int_0^t (t-y)^{r-1}\|s(y,S(y))\|dy \le \frac{1-r}{K(r)}P + \frac{rPt^r}{K(r)\Gamma(r+1)}. \quad \textbf{(3.5)}$$

Let $R = \max\left\{\frac{1-r}{K(r)} + s(t,E(t))\right\}$, $0 \le E(t) \le K_2$ for some positive constant K_2. For $E(t) \in M$, then we get

$$\|GE(t)\| = \frac{1-r}{K(r)}\|s(t,E(t))\| + \frac{r}{K(r)\Gamma(r)}\int_0^t (t-y)^{r-1}\|s(y,E(y))\|dy \le \frac{1-r}{K(r)}R + \frac{rRt^r}{K(r)\Gamma(r+1)} \quad \textbf{(3.6)}$$

And similarly, we consider third equation, let $Q = \max\left\{\frac{1-r}{K(r)} + s(t,I(t))\right\}$, $0 \le I(t) \le K_3$ for some positive constant K_3. For $I(t) \in M$

$$\|GI(t)\| = \frac{1-r}{K(r)}\|s(t,I(t))\| + \frac{r}{K(r)\Gamma(r)}\int_0^t (t-y)^{r-1}\|s(y,I(y))\|dy \le \frac{1-r}{K(r)}Q + \frac{rQt^r}{K(r)\Gamma(r+1)}. \quad \textbf{(3.7)}$$

Thus from equations (3.5)-(3.7) we can conclude that the operator G is bounded.

Now, we assume $t_1 < t_2$ and $S(t) \in M$. Taking into account that *M* is bounded, for $\varepsilon > 0$, if $|t_2 - t_1| < \delta$ we have

$$\begin{aligned}\|GS(t_2) - GS(t_1)\| &= \Bigg\| \frac{1-r}{K(r)} s(t_2, S(t_2)) + \frac{r}{K(r)\Gamma(r)}\int_0^{t_2}(t_2-y)^{r-1} s(y,S(y))dy \\ &\quad - \frac{1-r}{K(r)} s(t_1,S(t_1)) - \frac{r}{K(r)\Gamma(r)}\int_0^{t_1}(t_1-y)^{r-1} s(y,S(y))dy \Bigg\| \\ &\le \frac{1-r}{K(r)}\|s(t_2,S(t_2)) - s(t_1,S(t_1))\| \\ &\quad + \Bigg\|\frac{r}{K(r)\Gamma(r)}\int_0^{t_2}(t_2-y)^{r-1} s(y,S(y))dy - \frac{r}{K(r)\Gamma(r)}\int_0^{t_1}(t_1-y)^{r-1} s(y,S(y))dy\Bigg\| \\ &\le \frac{1-r}{K(r)}\|s(t_2,S(t_2)) - s(t_1,S(t_1))\| \\ &\quad + \frac{rK_1}{K(r)\Gamma(r)}\left\{\int_0^{t_2}(t_2-y)^{r-1}dy - \int_0^{t_1}(t_1-y)^{r-1}dy\right\}.\end{aligned} \quad \textbf{(3.8)}$$

We have that

$$\int_0^{t_2}(t_2-y)^{r-1}dy - \int_0^{t_1}(t_1-y)^{r-1}dy = \frac{(t_2-t_1)^r}{r}. \quad \textbf{(3.9)}$$

We study the following

$$\|s(t_2,S(t_2)) - s(t_1,S(t_1))\| \le (\beta(a+b)+\mu)\|S(t_2)-S(t_1)\| \le J_1\|t_2-t_1\|. \quad \textbf{(3.10)}$$

Then, putting Eqs. (3.9)-(3.10) in (3.8), we obtain

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$$\left\|GS(t_2)-GS(t_1)\right\| \le \frac{1-r}{K(r)} J_1 \left\|t_2-t_1\right\|$$

$$+\frac{rK}{K(r)\Gamma(r)} \frac{\left\|t_2-t_1\right\|^r}{r}$$

Let $\delta_1 = \dfrac{\varepsilon}{\frac{1-r}{K(r)} J_1 + \frac{rK_1}{K(r)\Gamma(r+1)}}$ then we get

$$\left\|GS(t_2)-GS(t_1)\right\| \le \varepsilon.$$

In the same manner, we can get the following results for the other two functions: If we get $\varepsilon > 0$, we have

$$\delta_2 = \frac{\varepsilon}{\frac{1-r}{K(r)} J_2 + \frac{rK_2}{K(r)\Gamma(r+1)}},$$

$$\delta_3 = \frac{\varepsilon}{\frac{1-r}{K(r)} J_3 + \frac{rK_3}{K(r)\Gamma(r+1)}},$$

then we obtain

$$\left\|GE(t_2)-GE(t_1)\right\| \le \varepsilon,$$

$$\left\|GI(t_2)-GI(t_1)\right\| \le \varepsilon.$$

Thus $\overline{G(M)}$ is equicontinuous and so $\overline{G(M)}$ is compact by Arzelo-Ascoli Theorem. □

Theorem 3.2. Let $S:[a,b]\times[0,\infty)\to[0,\infty)$ be increasing for each t in $[a,b]$ and be a continuous function. Suppose that one can find u,v satisfying $M(D)u \le S(t,u)$, $M(D)v \ge S(t,v)$ for $0 \le u(t) \le v(t)$ and $a \le t \le b$. Thus, system (1.1) has a positive solution.

Proof 3.2. Now we handle the fixed point operator G. We know that the operator $G:H\to H$ is completely continuous. Let $S_1 \le S_2, E_1 \le E_2, I_1 \le I_2$ then we get

$$GS_1(t) \le \frac{1-r}{K(r)} s(t,S_1(t)) + \frac{r}{K(r)\Gamma(r)} \int_0^t (t-y)^{r-1} \left\|s(y,S_1(y))\right\| dy \le GS_2(t). \quad \textbf{(3.11)}$$

Following similar steps, we obtain

$$GE_1(t) \le \frac{1-r}{K(r)} s(t,E_1(t)) + \frac{r}{K(r)\Gamma(r)} \int_0^t (t-y)^{r-1} \left\|s(y,E_1(y))\right\| dy \le GE_2(t) \quad \textbf{(3.12)}$$

and

$$GI_1(t) \le \frac{1-r}{K(r)} s(t,I_1(t)) + \frac{r}{K(r)\Gamma(r)} \int_0^t (t-y)^{r-1} \left\|s(y,I_1(y))\right\| dy \le GI_2(t) \quad \textbf{(3.13)}$$

Thus, G is increasing operator. From the conjecture, $Gm \ge m$, $Gn \le n$. So $G:\langle m,n\rangle \to \langle m,n\rangle$ is compact and continuous from Lemma 3.1. Thus, H is a normal cone. □

Now, we will investigate the uniqueness of solutions. To manage this, we study the followings:

$$\left\|GS_1(t)-GS_2(t)\right\| \le \frac{1-r}{K(r)} \left\|s(t,S_1(t))-s(t,S_2(t))\right\|$$

$$+\frac{r}{K(r)\Gamma(r)} \int_0^t (t-y)^{r-1} \left\|s(y,S_1(y))-s(y,S_2(y))\right\| dy$$

$$\le \frac{1-r}{K(r)} F_1 \left\|S_1(t)-S_2(t)\right\|$$

$$+\frac{r}{K(r)\Gamma(r)} F_1 \int_0^t (t-y)^{r-1} \left\|S_1(y)-S_2(y)\right\| dy$$

which gives

$$\left\|GS_1(t)-GS_2(t)\right\| \le \left\{\frac{1-r}{K(r)} F_1 + \frac{rF_1 b^r}{K(r)\Gamma(r+1)}\right\} \times \left\|S_1(t)-S_2(t)\right\| \quad \textbf{(3.14)}$$

By a similar method, we obtain

$$\left\|GE_1(t)-GE_2(t)\right\| \le \left\{\frac{1-r}{K(r)} F_2 + \frac{rF_2 b^r}{K(r)\Gamma(r+1)}\right\} \times \left\|E_1(t)-E_2(t)\right\|. \quad \textbf{(3.15)}$$

$$\left\|GI_1(t)-GI_2(t)\right\| \le \left\{\frac{1-r}{K(r)} F_3 + \frac{rF_3 b^r}{K(r)\Gamma(r+1)}\right\} \times \left\|I_1(t)-I_2(t)\right\|. \quad \textbf{(3.16)}$$

Therefore, if the following conditions satisfy

$$\frac{1-r}{K(r)}F_1+\frac{rF_1b^r}{K(r)\Gamma(r+1)}<1,$$

$$\frac{1-r}{K(r)}F_2+\frac{rF_2b^r}{K(r)\Gamma(r+1)}<1,$$

$$\frac{1-r}{K(r)}F_3+\frac{rF_3b^r}{K(r)\Gamma(r+1)}<1,$$

the mapping G is a contraction, so it has a fixed point by Banach fixed-point theorem. Thus the new model has a unique positive solution.

4. Numerical Results

Here, in order to observe how fractional order *r* affects behavior of the fractional model (1.1), some numerical simulations of this model are depict benefiting from the numerical method presented by Toufik and Atangana in the paper [31]. We select the parameters *b=5*, $\beta = 0.007$, $\mu = 0.6$, $\gamma = 0.25$, $\varepsilon = 0.3$, $p = 0.7$ with the initial conditions *S(0)=3*, *E(0)=1*, *I(0)=1* as given in [29]. From Figure1, we see the behavior of susceptible, exposed and infected computers. From Figure 2, it is visible that as *r* goes up, the number of susceptible computers *S(t)* increases while the number of exposed computers *E(t)* and infected computers *I(t)* decreases.

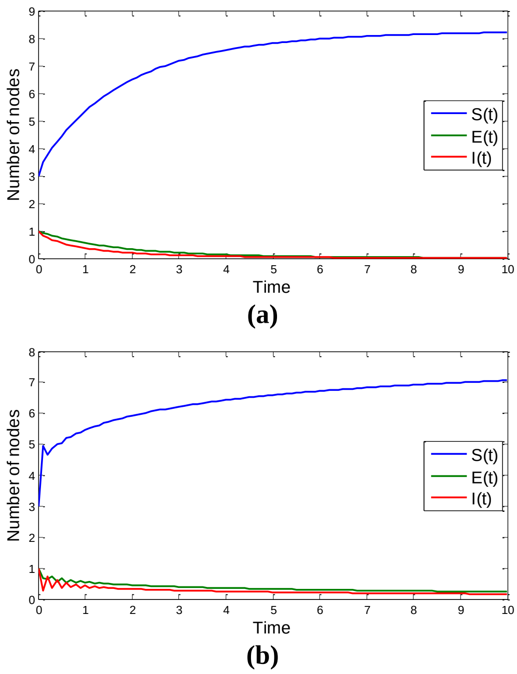

Figure 1. Numerical simulations for the model (1.1) at *r=0.95* and *r=0.6*, respectively.

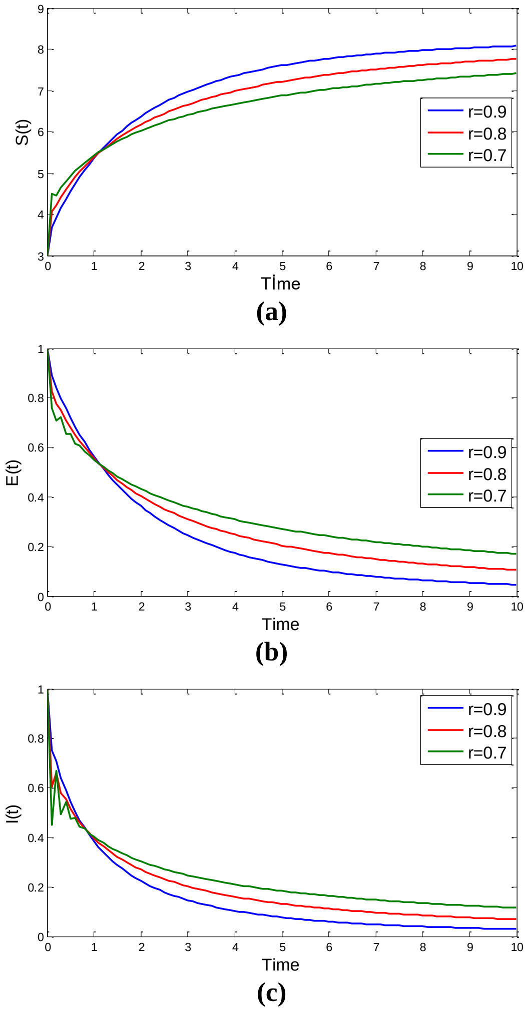

Figure 2. The behavior of the fractional computer virus spreading model components for distinct values *r*.

5. Conclusion

In this study, Atangana–Beleanu derivative with Mittag-Leffler kernel has been applied in reformulating the computer virus spreading model presented by Xu and Ren [29]. Then, Arzelo-Ascoli theorem is used to prove the existence and uniqueness properties of the considered model. Benefiting from Toufik-Atangana method [31], some numerical simulations are depicted with several values *r* and briefly interpreted. We expect that the present study will be more helpful to construct computer virus spreading model with fractional derivatives.

Author's Contributions

Sümeyra Uçar drafted the manuscript, compiled information from the literature, and designed the figures and tables.

Ethics

There are no ethical issues after the publication of this manuscript.

References

1. Piqueira, JRC, Araujo, VO. 2009. A modified epidemiological model for computer viruses, Applied Mathematics and Computation; 213(2): 355–360.

2. Wierman, JC, Marchette, DJ. 2004. Modeling computer virus prevalence with a susceptible-infected susceptible model with reintroduction, Computational Statistics Data Analysis; 45(1): 3–23.

3. Li, XZ, Zhou, LL. 2009. Global stability of an SEIR epidemic model with vertical transmission and saturating contact rate, Chaos Solitons & Fractals; 40(2): 874–884.

4. Li, G, Zhen, J. 2005. Global stability of an SEI epidemic model with general contact rate, Chaos, Solitons Fractals; 23(3): 997–1004.

5. Jin, Y, Wang, W, Xiao, S. 2007. An SIRS model with a nonlinear incidence rate, Chaos, Solitons Fractals; 34(5): 1482–1497.

6. Yang, LX, Yang, XF, Zhu, QY, Wen, LS. 2013. A computer virus model with graded cure rates, Nonlinear Analysis: Real World Applications; 14(1): 414-442.

7. Yang, LX, Yang, XF. 2012. The spread of computer viruses under the influence of removable storage devices, Applied Mathematics and Computation; 219(8): 3419-3422.

8. Wang, FG, Zhang, YK, Wang, CG, Ma, JF, Moon, SJ. 2010. Stability analysis of a SEIQV epidemic model for rapid spreading worms, Computers & Security; 29(4): 410-418.

9. Özdemir, N, Karadeniz, D, İskender, BB. 2009. Fractional optimal control problem of a distributed system in cylindrical coordinates, Physics Letters A; 373(2): 221-226.

10. Oldham, KB, Spanier, J. The Fractional Calculus; New York, Academic Press, 1974.

11. Kilbas, AA, Srivastava, HM, Trujillo, JJ. Theory and applications of fractional differential equations; Amsterdam, Elsevier, 2006.

12. Özdemir, N, Agrawal, OP, İskender, BB, Karadeniz, D. 2009. Fractional optimal control of a 2-dimensional distributed system using eigenfunctions. Nonlinear Dynamics; 55:251-260.

13. Yavuz, M, Özdemir, N. 2018. European vanilla option pricing model of fractional order without singular kernel. Fractal Fractional; 2(1): 3.

14. Evirgen, F. 2011. Multistage adomian decomposition method for solving NLP problems over a nonlinear fractional dynamical system, Journal of Computational and Nonlinear Dynamics; 6.

15. Evirgen, F. 2016. Analyze the optimal solutions of optimization problems by means of fractional gradient based system using VIM, An International Journal of Optimization and Control: Theories & Applications (IJOCTA); 6(2): 75-83.

16. Atangana, A, Koca, İ. 2016. On the new fractional derivative and application to nonlinear Baggs andFreedman model, Journal of Nonlinear Sciences and Applications; 9(5): 2467-2480.

17. Alkahtani, BSTA, Atangana A, Koca İ. 2016. A new nonlinear triadic model of predator prey based on derivative with non-local and non-singular kernel, Advances in Mechanical Engineering; 8(11).

18. Mekkaoui, T, Atangana, A. 2017. New numerical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, European Physical Journal Plus; 132(10).

19. Morales-Delgado VF, Gomez-Aguilar JF, Taneco-Hernandez MA, Escobar-Jimenez RF, Olivares-Peregrino VH. 2018. Mathematical modeling of the smoking dynamics using fractional differential equations with local and nonlocal kernel, Journal of Nonlinear Sciences Applications; 11(8): 994-1014.

20. Özdemir, N, Yavuz M. 2017. Numerical Solution of Fractional Black-Scholes Equation by Using the Multivariate Pade Approximation, Acta Physica Polonica A; 132: 1050-1053.

21. Yavuz, M, Özdemir, N, Başkonuş, HM. 2017. Solutions of partial differential equations using the fractional operator involving Mittag-Leffler kernel, The European Physical Journal Plus; 133(6).

22. Koca, İ. 2018. Analysis of rubella disease model with non-local and non-singular fractional derivatives, An International Journal of Optimization and Control: Theories & Applications (IJOCTA); 8(1): 17-25.

23. Uçar, S, Uçar E, Özdemir, N, Hammouch Z. 2019. Mathematical analysis and numerical simulation for a smoking model with Atangana Baleanu derivative, Chaos, Solitons Fractals; 118: 300-306.

24. Özdemir, N, Uçar E. 2020. Investigating of an immune system-cancer mathematical model with Mittag-Leffler kernel, AIMS Mathematics, 5(2):1519-1531.

25. Baleanu, D, Fernandez, A. 2018. On some new properties of fractional derivatives with Mittag-Leffler kernel, Communications Nonlinear Science and Numerical Simulation; 59: 444–462.

26. Fernandez, A, Baleanu D, Srivastava, HM. 2019. Series representations for fractional-calculus operators involving generalised Mittag-Leffler functions, Communications Nonlinear Science and Numerical Simulation; 67: 517-527.

27. Avcı, D, Yetim A. 2018. Analytical solutions to the advection-diffusion equation with the Atangana-Baleanu derivative over a finite domain, Journal of Balıkesir University Institute Science and Technology; 20(2): 382–395.

28. Yavuz, M, Bonyah E. 2019. New approaches to the fractional Dynamics of schistosomiasis disease model, Physica A: Statistical Mechanics and its Applications; 525: 373-393.

29. Xu, Y, Ren J. 2016, Propagation Effect of a Virus Outbreak on a Network with Limited Anti-Virus Ability; Plos One, Article ID: e0164415

30. Atangana, A, Baleanu, D. 2016. New fractional derivatives with non-local and non-singular kernel: theory and applications to heat transfer model, Thermal Science; 20(2): 763-769.

31. Toufik, M, Atangana, A. 2017. New numeriical approximation of fractional derivative with non-local and non-singular kernel: Application to chaotic models, The European Physical Journal Plus; 132.