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Improved Jacobi Matrix Method for Solving Multi-Functional Integro-Differential Equations with Mixed Delays

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Abstract

In this study, we suggested a novel approach for solving multi-functional integro-differential equations with mixed delays, by using orthogonal Jacobi polynomials. These equations include various classes of differential equations, integro-differential equations and delay differential equations. This new algorithm proposes solutions for each class of these equations and combinations of equation classes, such as Volterra integro-differential equation, Fredholm integro-differential equation, pantograph-delay differential equations. Since the present method is based on fundamental matrix relations and collocation points, numerical solutions can be obtained easily by means of symbolic computation programs. We developed an error estimation algorithm based on the present method for the verification of solutions. Application of the method is illustrated by four numerical examples.

Keywords: Jacobi matrix method, functional integro-differential equation, error estimation algorithm.

1. Introduction

In recent years, many researchers have studied on the numerical solution of different classes of the integro differential equations because integro-differential equations have been one of the principal tools in various areas of applied mathematics, biological models, physics and engineering [1] and also functional integro-differential equations are often used to model some problems with aftereffect in mechanics and the related scientific fields [2]. Our study is aimed to introduce a new numerical solution method of the multi-functional integro-differential equations with mixed delays (mf-IDD Eqs.) which include Volterra integro-differential equations, Fredholm integro-differential equation, pantograph-delay differential equations and also all sub-classes or combinations of these. For this reason, we generate a procedure to obtain a numerical solution for the following mf-IDD equations

$$\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)y^{(k)}(\alpha_{kl}x+\beta_{kl}) = f(x) + \sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs} \int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)y^{(r)}(\mu_{rs}t+\gamma_{rs})dt\,, \quad a \le x,t \le b \tag{1.1}$$

under the initial and boundary conditions

$$\sum_{k=0}^{m-1}\left[a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)\right] = \eta_i, \quad m=\max(m_1,m_3), i=0,1,2,\dots,m-1 \tag{1.2}$$

where $P_{kj}(x)$ and $f(x)$ are known functions defined on the interval, $a \le x \le b$; β_{kj}, λ_{rs}, γ_{rs} a_{ik}, b_{ik} and η_i are real or complex constants, $0 < \alpha_{kj}, \mu_{rs} < 1$, $K_{rs}(x,t)$ are kernel functions and, $y(x)$ is the unknown function to be determined.

In very recent years, several mathematicians have been interested in mf-IDD Equations and their sub-classes for obtaining numerical solutions. Oğuz and Sezer [3] developed Chelyshkov collocation method for the numerical solutions of mixed functional integro-differential equations. Mirzaee et al. [4] gave the numerical solution method based on Euler polynomial for the solutions of Volterra differential equations pantograph-delay type. Kürkçü et al. [5] used Dickson polynomials to enhance matrix based collocation method for mf-IDD equations. Yüzbaşı [6] presented a new numerical approach by using Laguerre polynomials for solving linear pantograph-type Volterra integro-differential equations. Reutskiy [7] studied linear Volterra-Fredholm integro differential equations with linear functional arguments.

We use the orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in order to develop a new approach to obtain the numerical solutions of the mf-IDD equation. The orthogonal Jacobi polynomials are defined with respect to $\omega^{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha > -1, \beta > -1)$ (weight function) on $(-1,1)$ and it is proved that the Jacobi polynomials satisfy the following relation [8, 9];

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} B_n^{(\alpha,\beta,n)} (x-1)^k; \ \ \alpha,\beta > -1$$
$$B_n^{(\alpha,\beta,n)} = 2^{-k} \binom{n+\alpha+\beta+k}{k} \binom{n+\alpha}{n-k}; \quad k = 0,1,2,\dots,n$$

Note that α and β are special parameters of the Jacobi polynomials. The Jacobi polynomials transform to some known orthogonal polynomials with respect to α and β. Some of the most important are Legendre ($\alpha = \beta = 0$), Chebyshev ($\alpha = \beta = -1/2$) and Gegenbauer ($\alpha = \beta$) polynomials. This versatility of Jacobi polynomials are provided that more than one solution unlike the most of others.

By using the definition of the Jacobi polynomials, we assume a solution expressed as the truncated series of orthogonal Jacobi polynomials defined by

$$y(x) \cong y_N^{(\alpha,\beta)}(x) = \sum_{n=0}^{N} a_n P_n^{(\alpha,\beta)}(x)$$

where $P_n^{(\alpha,\beta)}(x), n = 0,1,2,\dots,N$ denote the orthogonal Jacobi polynomials defined above; N is chosen any positive integer such that $N \geq n$ and $a_n, n = 0,1,2,\dots,N$ are unknown coefficients.

Jacobi collocation method is a collocation method based on matrix operations. This matrix operations method named "matrix method" was introduced by Sezer [8] and developed over time by applying to different problems like differential equations [9] and integro-differential equations [10]. The matrix method have also been adapted to many different problems based on various polynomials such as Bessel polynomials [11] and Laguerre polynomials [12] for various classes of the integral equations. Researchers applied the method to various engineering problems. Baykuş and Çevik [13] solved the single-degree-of-freedom (SDOF) system by using Taylor polynomials. Çevik et al. [14] solved the delayed SDOF problem by using exponential functions. Deniz and Sezer [15] solved the nonlinear heat transfer equations using Rational Chebyshev polynomials. In addition, Deniz et al. [16] published a study in which stability analysis of the Taylor Collocation Method is performed.

2. Matrix Representation of Each Term of the Problem

In this section, we transform each term of Eq. (1.1) to matrix form. First, we obtain $P_n^{(\alpha,\beta)}(x)$ orthogonal Jacobi polynomials of matrix form as follows:

$$\mathbf{P}^{(\alpha,\beta)}(x) = \mathbf{X}(x)\mathbf{M}^{(\alpha,\beta)} \qquad \mathbf{(2.1)}$$

where

$$\mathbf{P}^{(\alpha,\beta)}(x) = \left[P_0^{(\alpha,\beta)}(x) \quad P_1^{(\alpha,\beta)}(x) \quad \dots \quad P_N^{(\alpha,\beta)}(x)\right]$$
$$\mathbf{X}(x) = [1 \quad (x-1) \quad (x-1)^2 \quad \dots \quad (x-1)^N]$$

and $\mathbf{M}^{(\alpha,\beta)} = \left[m_{ij}^{(\alpha,\beta)}\right]_{(N+1)\times(N+1)}$ such that

$$m_{ij}^{(\alpha,\beta)} = \begin{cases} 2^{1-i}\binom{i+j-2+\alpha+\beta}{i-1}\binom{j-1+\alpha}{j-i}, & i \leq j \\ 0 \qquad , & i > j \end{cases}$$

We assume the desired solution $y(x)$ of Eq. (1.1) to be defined by the truncated orthogonal Jacobi series in matrix form as follows:

$$\left[y_N^{(\alpha,\beta)}(x)\right] = \mathbf{P}^{(\alpha,\beta)}(x)\mathbf{A} \qquad \mathbf{(2.2)}$$

where $\mathbf{A} = [a_{ij}]_{(N+1)\times 1}$ such that

$$a_{ij} = a_{i-1} \qquad \mathbf{(2.3)}$$

By substituting the matrix form of Jacobi polynomials (2.1) into (2.2), we can obtain the fundamental matrix equation of the approximate solution for the unknown function as [17, 18]

$$\left[y_N^{(\alpha,\beta)}(x)\right] = \mathbf{X}(x)\mathbf{M}^{(\alpha,\beta)}\mathbf{A}. \qquad \mathbf{(2.4)}$$

First, in order to explain the relation between the matrix form of the unknown function $y(x)$ and the matrix form of its derivatives $y^{(k)}(x)$, we introduce the relation between the matrix $\mathbf{X}(x)$ and its derivatives $\mathbf{X}^{(k)}(x)$ which can be expressed as

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\mathbf{B}^k \qquad \mathbf{(2.5)}$$

where $\mathbf{B} = [b_{ij}]_{(N+1)\times(N+1)}$ such that

$$b_{ij} = \begin{cases} i \quad , & j-i=1 \\ 0 \quad , & others \end{cases}$$

Then, using (2.4) and (2.5), one may write

$$\left[y^{(k)}(x)\right] \cong \left[\left(y_N^{(\alpha,\beta)}\right)^{(k)}(x)\right] = \mathbf{X}^{(k)}(x)\mathbf{M}^{(\alpha,\beta)}\mathbf{A} = \mathbf{X}(x)\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \qquad \mathbf{(2.6)}$$

Similarly, the relation between the matrix form of $y(x)$ and the matrix form of its delay forms' derivatives $y^{(k)}(\alpha_{kl}x+\beta_{kl})$ can be expressed as

$$\begin{aligned} y^{(k)}(\alpha_{kl}x+\beta_{kl}) &= \mathbf{X}^{(k)}(\alpha_{kl}x+\beta_{kl})\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \\ &= \mathbf{X}(\alpha_{kl}x+\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \\ &= \mathbf{X}(x)\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \end{aligned} \tag{2.7}$$

where $\mathbf{B}(\alpha_{kj},\beta_{kj}) = [b_{ij}(\alpha_{kl},\beta_{kl})]_{(N+1)\times(N+1)}$ such that

$$b_{ij}(\alpha_{kl},\beta_{kl}) = \begin{cases} \binom{j-1}{i-1}(\alpha_{kj})^{i-1}(\beta_{kj})^{j-i} & , \quad i \le j \\ 0 & , \quad i > j \end{cases}$$

Using (2.7), the matrix form of the differential part of Eq. (1.1) becomes

$$\begin{aligned} &\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)y^{(k)}(\alpha_{kl}x+\beta_{kl}) \\ &= \sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \end{aligned} \tag{2.8}$$

Finally, the matrix form of the integral part of Eq. (1.1) is obtained as follows

$$\begin{aligned} &\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)y^{(r)}(\mu_{rs}t+\gamma_{rs})dt \\ &= \int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\mathbf{X}(t)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A}dt \\ &= \left(\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\mathbf{X}(t)dt\right)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \\ &= \mathbf{Q}_{rs}(x)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \end{aligned}$$

where $\mathbf{Q}_{rs}(x) = [q_{ij}(x)]_{1\times(N+1)}$ such that

$$q_{ij}(x) = \int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)(t-1)^{j-1}dt$$

We can write briefly as follows:

$$\begin{aligned} &\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)y^{(r)}(\mu_{rs}t+\gamma_{rs})dt \\ &= \mathbf{Q}_{rs}(x)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we obtain the matrix form of Eq. (1.1) as follows:

$$\begin{aligned} &\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)\mathbf{X}(x)\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \\ &= f(x) \\ &+\sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs}\mathbf{Q}_{rs}(x)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \end{aligned} \tag{2.10}$$

We can obtain the corresponding matrix forms for the conditions (1.2), by means of the relation (2.4), as

$$\begin{aligned} &\sum_{k=0}^{m-1}\left[a_{ik}y^{(k)}(a)+b_{ik}y^{(k)}(b)\right] \\ &= \sum_{k=0}^{m-1}\left[a_{ik}\mathbf{X}(a)\mathbf{B}^k\mathbf{M}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\mathbf{A} + b_{ik}\mathbf{X}(b)\mathbf{B}^k\mathbf{M}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}\mathbf{A}\right] \\ &= [\eta_i], \\ &i = 0,1,2,\dots,m-1. \end{aligned} \tag{2.11}$$

3. Method of Solution

For constructing the matrix equation, the matrix relations (2.8) and (2.9) are first substituted into Eq. (1.1); then, by using collocation points defined by

$$x_\tau = a + \frac{b-a}{N}\tau, \qquad \tau = 0,1,2,\dots,N$$

the system of matrix equations is obtained as

$$\begin{aligned} &\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x_\tau)\mathbf{X}(x_\tau)\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)}\mathbf{A} \\ &= f(x_\tau) + \sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs}\mathbf{Q}_{rs}(x_\tau)\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\mathbf{A}, \\ &\tau = 0,1,2,\dots,N \end{aligned}$$

Therefore, the fundamental matrix equation becomes

$$\left\{\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} \mathbf{P}_{kl}\mathbf{X}\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k\mathbf{M}^{(\alpha,\beta)} - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4} \lambda_{rs}\overline{\mathbf{Q}_{rs}}\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r\mathbf{M}^{(\alpha,\beta)}\right\}\mathbf{A} = \mathbf{F} \tag{3.1}$$

where $\mathbf{P}_{kl} = [p]_{(N+1)\times(N+1)}$

$$\mathbf{P}_{kl} = \begin{bmatrix} P_{kl}(x_0) & 0 & \cdots & 0 \\ 0 & P_{kl}(x_1) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & P_{kl}(x_N) \end{bmatrix},$$

$$\mathbf{F} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_N) \end{bmatrix},$$

$$\overline{\mathbf{Q}_{rs}} = \begin{bmatrix} \mathbf{Q}_{rs}(x_0) \\ \mathbf{Q}_{rs}(x_1) \\ \vdots \\ \mathbf{Q}_{rs}(x_N) \end{bmatrix} \text{ and } \mathbf{X} = \begin{bmatrix} \mathbf{X}(x_0) \\ \mathbf{X}(x_1) \\ \vdots \\ \mathbf{X}(x_N) \end{bmatrix}.$$

The fundamental matrix equation (2.10) of Eq. (1.1) corresponds to a system of $N+1$ algebraic equations for the $N+1$ unknown coefficients $a_0, a_1, a_2, \cdots, a_N$. Briefly, if we determine

Celal Bayar University Journal of Science
Volume 16, Issue 4, 2020, p 393-401
Doi: 10.18466/cbayarfbe.716634

$$\mathbf{W} = \left(\sum_{k=0}^{m_1} \sum_{i=0}^{m_2} \mathbf{P}_{kl}\mathbf{X}\mathbf{B}(\alpha_{kl},\beta_{kl})\mathbf{B}^k - \sum_{r=0}^{m_3} \sum_{s=0}^{m_4} \lambda_{rs}\overline{\mathbf{Q}_{rs}}\mathbf{B}(\mu_{rs},\gamma_{rs})\mathbf{B}^r \right)\mathbf{M}^{(\alpha,\beta)}$$

then, we can write Eq. (3.1) in the form

$$\mathbf{W}\mathbf{A} = \mathbf{F} \text{ or } [\mathbf{W};\mathbf{F}]. \qquad \textbf{(3.2)}$$

On the other hand, from (2.11), we can obtain the matrix form of conditions briefly as

$$\mathbf{U}_i\mathbf{A} = \eta_i \text{ or } [\mathbf{U}_\mathbf{j};\eta_i], \quad i = 0,1,2,\dots,m-1 \qquad \textbf{(3.3)}$$

such that

$$\mathbf{U}_i = \sum_{k=0}^{m-1} [a_{ik}\mathbf{X}(a) + b_{ik}\mathbf{X}(b)]\,\mathbf{B}^k\mathbf{M}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}$$

Consequently, in order to obtain the solution of Eq. (1.1) under the mixed conditions (1.2), we replace the row matrix (3.3), by last n rows of the augmented matrix (3.2), which yields the required augmented matrix

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}] \qquad \textbf{(3.4)}$$

If $\text{rank}\widetilde{\mathbf{W}} = \text{rank}[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}] = N+1$, then we can write $\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{F}}$. Thus the matrix $\mathbf{A}$ (thereby the coefficients $a_0, a_1, a_2, \cdots, a_N$) is uniquely determined. Eq. (1.1) has also a unique solution under the conditions (1.2). This solution is given by the truncated orthogonal Jacobi series. Thus we get the Jacobi polynomial solution for the arbitrary parameters α and β:

$$y(x) \cong y_N^{(\alpha,\beta)}(x) = \sum_{n=0}^{N} a_n P_n^{(\alpha,\beta)}(x)$$

4. Error Estimation Algorithm and Improved Solution

One of the ultimate objectives in this study is to develop an error estimation algorithm for mf-IDD equations and to obtain an improved solution of the Jacobi polynomial solution. Valid here only that the improved Jacobi polynomial solution is found by using error estimation algorithm. To obtain the error estimation function of the initial solution, we define the residual function of Eq. (1.1) [19] as

$$R_N(x) = \sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)(y_N^{(\alpha,\beta)})^{(k)}(\alpha_{kl}x+\beta_{kl}) - f(x) - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)(y_N^{(\alpha,\beta)})^{(r)}(\mu_{rs}t + \gamma_{rs})dt \qquad \textbf{(4.1)}$$

where $y_N^{(\alpha,\beta)}(x)$ is the approximate solution of the problem (1.1)-(1.2) for special Jacobi parameters α and β. On the other hand, $y_N^{(\alpha,\beta)}(x)$ satisfies the Eq. (1.1)-(1.2):

$$\begin{gathered}\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)(y_N^{(\alpha,\beta)})^{(k)}(\alpha_{kl}x+\beta_{kl}) - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\left(y_N^{(\alpha,\beta)}\right)^{(r)}(\mu_{rs}t + \gamma_{rs})dt = R_N(x) - f(x) \\ \sum_{k=0}^{m-1}\left[a_{ik}(y_N^{(\alpha,\beta)})^{(k)}(a) + b_{ik}(y_N^{(\alpha,\beta)})^{(k)}(b)\right] = \eta_i, \\ i = 0,1,2,\dots,m-1\end{gathered} \qquad \textbf{(4.2)}$$

The error function $e_N{}^{(\alpha,\beta)}(x)$ can be defined as

$$e_N{}^{(\alpha,\beta)}(x) = y(x) - y_N^{(\alpha,\beta)}(x) \qquad \textbf{(4.3)}$$

where $y(x)$ is the exact solution of the problem (1.1)-(1.2). Substituting (4.3) into (1.1)-(1.2) and using (4.1) and (4.2), we derive the error differential equation with homogenous conditions:

$$\begin{gathered}\sum_{k=0}^{m_1}\sum_{l=0}^{m_2} P_{kl}(x)(e_N^{(\alpha,\beta)})^{(k)}(\alpha_{kl}x+\beta_{kl}) - \sum_{r=0}^{m_3}\sum_{s=0}^{m_4}\lambda_{rs}\int_{u_{rs}(x)}^{v_{rs}(x)} K_{rs}(x,t)\left(e_N^{(\alpha,\beta)}\right)^{(r)}(\mu_{rs}t + \gamma_{rs})dt = -R_N(x) \\ \sum_{k=0}^{m-1}\left[a_{jk}\left(e_N^{(\alpha,\beta)}\right)^{(k)}(a) + b_{jk}\left(e_N^{(\alpha,\beta)}\right)^{(k)}(b)\right] = 0\end{gathered} \qquad \textbf{(4.4)}$$

Solving the problem (4.4) in the same way as in Section 3, we get the approximation $e_{N,M}{}^{(\alpha,\beta)}(x)$ to $e_N{}^{(\alpha,\beta)}(x)$, $M > N$ which is the error function based on the residual function $R_N(x)$. Consequently, by means of the orthogonal Jacobi polynomials $y_N^{(\alpha,\beta)}(x)$ and $e_{N,M}{}^{(\alpha,\beta)}(x)$, we obtain the improved Jacobi solution as

$$y_{N,M}^{(\alpha,\beta)}(x) = y_N^{(\alpha,\beta)}(x) + e_{N,M}{}^{(\alpha,\beta)}(x) \qquad \textbf{(4.5)}$$

where, $e_{N,M}{}^{(\alpha,\beta)}(x)$ is the estimated error function.

Celal Bayar University Journal of Science
Volume 16, Issue 4, 2020, p 393-401
Doi: 10.18466/cbayarfbe.716634

5. Illustrative Examples

In this section, we apply the new Jacobi matrix method to four examples by using symbolic computational programing MAPLE. In these examples, the terms $\left|e_N^{(\alpha,\beta)}(x)\right|$ and $\left|E_{N,M}^{(\alpha,\beta)}(x)\right|$ represent the absolute error function for Jacobi polynomial solution and the absolute error function of the improved Jacobi polynomial solution, respectively.

5.1. Example 1

We consider the third order integro-differential equations

$$y'''(x) - xy'(x-1) + 4y\left(\frac{1}{2}x - 1\right) = f(x) - 9\int_{x^2-5}^{x^4+1} xty'^{\left(\frac{1}{3}t-1\right)}dt, \qquad \textbf{(5.1)}$$

$$0 \le x, t \le 1$$
$$y(0) = -2, y'(0) = 0, y''(0) = 12$$

where $f(x) = 3x^{17} - 21x^9 + 72x^7 - 654x^5 + 2390x^3 + 6x^2 - 3384x + 24$ and the exact solution of the problem is $y(x) = 4x^3 + 6x^2 - 2$.

We assume that, for $N = 4$ and $(\alpha, \beta) = (1/2, 1/3)$ which are chosen arbitrarily, the problem has a Jacobi polynomial solution in the form,

$$y(\mathrm{x}) = \mathbf{P}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x)\mathbf{A}$$

such that

$$\mathbf{A} = [a_0 \quad a_1 \quad \cdots \quad a_N]^{\mathbf{T}}$$

and

$$\mathbf{P}^{(\boldsymbol{\alpha},\boldsymbol{\beta})}(x) = \left[P_0^{(\alpha,\beta)}(x) \quad P_1^{(\alpha,\beta)}(x) \quad P_2^{(\alpha,\beta)}(x) \quad P_3^{(\alpha,\beta)}(x) \quad P_4^{(\alpha,\beta)}(x)\right]$$

$$= \begin{bmatrix} 1 \\ \frac{1}{12} + \frac{17x}{12} \\ -\frac{35}{12} + \frac{115}{24}x + \frac{667(x-1)^2}{288} \\ -\frac{805}{96} + \frac{1015}{96}x + \frac{7105(x-1)^2}{576} + \frac{41615(x-1)^3}{10368} \\ -\frac{2135}{128} + \frac{1225}{64}x + \frac{10045(x-1)^2}{256} + \frac{67445(x-1)^3}{2304} + \frac{3574585(x-1)^4}{497664} \end{bmatrix}^{\mathbf{T}}$$

The collocation points are determined as

$$\left\{x_0 = 0,\ \ x_1 = {}^1/_4, x_2 = {}^2/_4\ , x_3 = {}^3/_4\ \ x_4 = 1\right\}$$

and from Eq. (3.1), the fundamental matrix equation of problem is

$$\mathbf{WA} = \mathbf{F}$$

where

$$\mathbf{W} = \begin{bmatrix} 4 & -\frac{16}{3} & \frac{56}{9} & \frac{89005}{5184} & \frac{668255}{62208} \\ 4 & -\frac{265668199}{6291456} & \frac{2542557122971}{9663676416} & -\frac{329287935484420465}{237494511599616} & -\frac{394241050949187375159}{547187354725515264} \\ 4 & -\frac{298781}{4096} & \frac{554514379}{1179648} & -\frac{13410744877075}{5435817984} & \frac{361276085811263}{28991029248} \\ 4 & -\frac{5670779253}{6291456} & \frac{5644627506113}{9663676416} & -\frac{2131519714347125305}{712483534798848} & \frac{7955921113759039089781}{547187354725515264} \\ 4 & -\frac{965}{12} & \frac{165437}{288} & -\frac{19998965}{6912} & \frac{8850194563}{663552} \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 24 & -\frac{13484747456509}{17179869184} & -\frac{181884157}{131072} & -\frac{28337926270647}{17179869184} & -1564 \end{bmatrix}^{\mathbf{T}}$$

Using (3.3), we can write the matrix form of the initial conditions of the problem as follows

Celal Bayar University Journal of Science
Volume 16, Issue 4, 2020, p 393-401
Doi: 10.18466/cbayarfbe.716634

$$[\mathbf{U}\;\;;\;\;\boldsymbol{\gamma}] = \begin{bmatrix} 1 & \frac{1}{12} & -\frac{173}{288} & -\frac{665}{10368} & \frac{233065}{497664} & ; & -2 \\ 0 & \frac{17}{12} & \frac{23}{144} & -\frac{7105}{3456} & -\frac{30835}{124416} & ; & 0 \\ 0 & 0 & \frac{667}{144} & \frac{1015}{1728} & -\frac{454895}{41472} & ; & 12 \end{bmatrix}$$

Consequently, to obtain the solution of Eq. (5.1) under the initial conditions, by replacing the row matrix $[\mathbf{U};\boldsymbol{\gamma}]$, by third and fourth rows of augmented matrix $[\mathbf{W};\mathbf{F}]$, we obtain the required augmented matrix $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$.

By solving the augmented matrix $[\widetilde{\mathbf{W}};\widetilde{\mathbf{F}}]$, we obtain the Jacobi polynomial coefficient matrix

$$\mathbf{A} = \begin{bmatrix} -\frac{6272}{111339} & \frac{576}{493} & \frac{67392}{27347} & \frac{41472}{41615} & 0 \end{bmatrix}^{\mathrm{T}}.$$

From Eq. (3.4), the Jacobi polynomial solution of the problem is

$y_4^{(1/2,1/3)}(x) = 8 + 24(x-1) + 18(x-1)^2 + 4(x-1)^3$ which is the exact solution of the problem. Likewise, we can extract the exact solution of the problem for $N = 5$ and different values of α and β.

5.2. Example 2

We consider the first order integro-differential equations [3, 6]

$$\begin{gathered} y'(x) = y(x) - 2y'\left(x - \frac{1}{2}\right) \\ + (x - x^2)y\left(\frac{1}{2}x - 1\right) \\ + \int_0^x xe^{-t}y(t)dt + \int_0^{x/2} xe^{-t}y'(t)dt + f(x), \\ 0 \le x,t \le 1 \\ y(0) = y'(0) = 1. \end{gathered} \tag{5.2}$$

In Refs. [3, 6], $f(x) = -(x - x^2)e^{\left(\frac{x}{2}-1\right)} + 2e^{\left(x-\frac{1}{2}\right)} - x^2e^{\left(\frac{x}{2}\right)} + xe^{\left(\frac{x}{2}\right)}$, and the exact solution of the problem is $y(x) = e^x$.

Table 1 shows that both Jacobi polynomial solutions and improved Jacobi polynomial solutions are more effective than other methods.

In Table 2, we have presented a comparison of the absolute error values with the estimated absolute errors values, and it is seen that the error estimation is corrected gradually as M increases.

In Table 1 and Table 2 Jacobi polynomial solutions obtained for the Chebyshev base are then $\alpha = \beta = -1/2$. Chebyshev solutions are nearly the same with respect to the arbitrary α and β such that $(\alpha,\beta) = (1/2,1/2)$ and $(\alpha,\beta) = (0,1/3)$.

Table 1. Comparison of the absolute errors of the improved Jacobi method with some other numerical methods for Example 2.

	Present Method			Chelyshkov collocation method [3]	Laguerre collocation method [6]
x_i	$\lvert e_4^{(\alpha,\beta)}(x_i)\rvert$	$\lvert E_{4,5}^{(\alpha,\beta)}(x_i)\rvert$	$\lvert E_{4,6}^{(\alpha,\beta)}(x_i)\rvert$	$N = 4$	$N = 4$
0	0	0	0	0	$1.776e-15$
0.2	$1.646e-4$	$2.714e-5$	$2.717e-6$	$2.926e-4$	$3.460e-4$
0.4	$3.767e-5$	$6.942e-5$	$2.335e-6$	$2.704e-4$	$5.863e-4$
0.6	$4.714e-4$	$5.886e-5$	$1.289e-5$	$4.497e-4$	$2.228e-4$
0.8	$7.898e-4$	$4.248e-5$	$1.425e-5$	$1.740e-3$	$8.626e-4$
1	$8.507e-4$	$6.759e-5$	$5.301e-7$	$2.432e-3$	$1.876e-4$
x_i	$\lvert e_7^{(\alpha,\beta)}(x_i)\rvert$	$\lvert E_{7,8}^{(\alpha,\beta)}(x_i)\rvert$	$\lvert E_{7,9}^{(\alpha,\beta)}(x_i)\rvert$	$N = 7$	$N = 7$
0	0	0	0	0	$1.998e-15$
0.2	$7.603e-7$	$1.672e-8$	$2.079e-8$	$7.378e-6$	$6.022e-7$
0.4	$1.493e-6$	$1.527e-7$	$2.836e-8$	$1.773e-6$	$1.519e-6$
0.6	$6.317e-7$	$3.601e-7$	$6.893e-9$	$1.851e-5$	$1.157e-6$
0.8	$1.797e-6$	$2.488e-7$	$6.128e-8$	$2.969e-5$	$1.133e-6$
1	$3.073e-6$	$2.774e-7$	$5.559e-8$	$2.117e-6$	-

Celal Bayar University Journal of Science
Volume 16, Issue 4, 2020, p 393-401
Doi: 10.18466/cbayarfbe.716634

Table 2. Comparison of the absolute error with the estimated absolute errors for Example 2.

	Actual Absolute Error	Estimated Absolute Errors			
x_i	$\left\|e_4^{(\alpha,\beta)}(x)\right\|$	$\left\|e_{4,7}^{(\alpha,\beta)}(x)\right\|$	$\left\|e_{4,8}^{(\alpha,\beta)}(x)\right\|$	$\left\|e_{4,9}^{(\alpha,\beta)}(x)\right\|$	$\left\|e_{4,10}^{(\alpha,\beta)}(x)\right\|$
0	0	0	0	0	0
0.2	$1.646e-4$	$1.653e-4$	$1.645e-4$	$1.645e-4$	$1.646e-4$
0.4	$3.767e-5$	$3.917e-5$	$3.783e-5$	$3.765e-5$	$3.767e-5$
0.6	$4.714e-4$	$4.708e-4$	$4.710e-4$	$4.714e-4$	$4.714e-4$
0.8	$7.898e-4$	$7.916e-4$	$7.896e-4$	$7.898e-4$	$7.898e-4$
1	$8.507e-4$	$8.476e-4$	$8.504e-4$	$8.507e-4$	$8.507e-4$

5.3. Example 3

Let us consider the Volterra type integro differential equation

$$y'(x) = \int_{x-1}^{x} (\cos(x+t+1)+2)y(t)dt + g(x), \qquad 0 \le x \le 3 \tag{5.3}$$

with the initial condition is $y(0)=1$ and $f(x) = 3cos(x) - \frac{1}{4}\cos(3x+1) - 2 + \frac{1}{2}\sin(x+1) + \sin(2x) - 2\cos(x-1) + \frac{1}{4}\cos(3x+1) - \sin(2x+1)$. The exact solution of the problem is $y(x) = \sin(x) + 1$ [6].

Table 3 is shown the absolute error values of Jacobi polynomial solutions both direct solution for $N = 10$ and improved solutions for $M = 12$ and $M = 14$ in domain interval of the problem. And also, Table 3 is given comparison of the absolute errors values of the present method with Taylor collocation and Laguerre collocation methods.

Figure 1 is shown the comparison of exact solution and Jacobi polynomial solutions ($N = 4$, 5 and 6) and it is clearly seen form the figure that the numerical solutions close the exact solution when N increases in [0,3] domain.

Jacobi polynomial solutions are obtained for the $\alpha = 0$ and $\beta = -1/2$. And also Legendre and Chebyshev solutions are obtained. For N=10 the absolute errors of the Legendre base solution is $7.254e-9$ and the absolute Chebyshev base solution is $8.226e-9$ while the Jacobi polynomial solution is $7.095e-9$.

5.4. Example 4

Consider the first order integro-differential equation

$$y'(x) = 2e^{1-x} - 3y(x) - 3\int_{x-1}^{x} y(t)dt - \int_{x-1}^{x} y'(t)dt\,, 0 \le x,t \le 2 \tag{5.4}$$

under the initial condition

$$y(0) = 1.$$

with the exact solution $y(x) = e^{-x}$ [6, 20, 21].

Table 3. Comparison of the absolute errors of the improved Jacobi method with some other numerical methods for $N = 10$ and for $(N, M) = (10,12), (10,14)$ for Example 3.

	Present Method			Taylor collocation method [20]		Laguerre collocation method [6]
x_i	$\left\|e_{10}^{(\alpha,\beta)}(x)\right\|$	$\left\|E_{10,12}^{(\alpha,\beta)}(x)\right\|$	$\left\|E_{10,14}^{(\alpha,\beta)}(x)\right\|$	16 collocation points	25 collocation points	$N = 10$
0.5	$7.095e-9$	$2.494e-9$	$3.564e-11$	$1.04e-6$	$1.0e-9$	$1.130e-08$
1.0	$7.034e-9$	$3.428e-9$	$5.276e-11$	$2.13e-6$	$7.0e-9$	$2.152e-08$
1.5	$5.704e-9$	$4.999e-9$	$7.585e-11$	$3.03e-6$	$1.3e-8$	$7.639e-09$
2.0	$1.010e-8$	$8.162e-9$	$1.240e-10$	$4.12e-6$	$2.6e-9$	$4.280e-10$
2.5	$1.957e-8$	$1.628e-8$	$2.474e-10$	$6.19e-6$	$4.7e-7$	$5.046e-07$
3	$1.476e-7$	$3.304e-8$	$5.484e-10$	$1.0e-5$	$8.9e-5$	$2.108e-05$

Celal Bayar University Journal of Science
Volume 16, Issue 4, 2020, p 393-401
Doi: 10.18466/cbayarfbe.716634

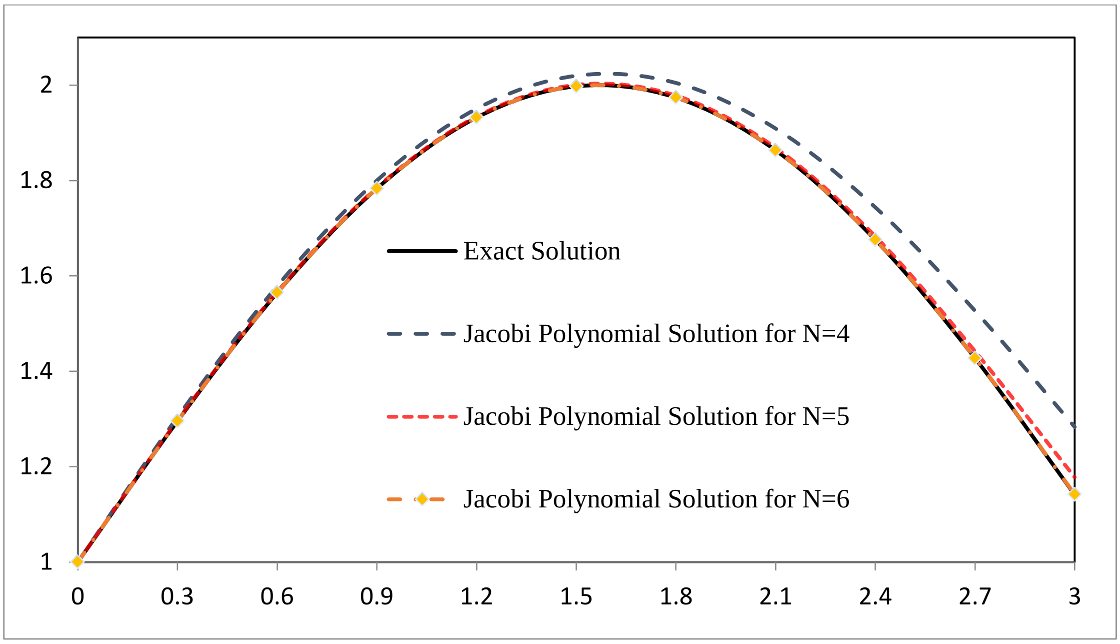

Figure 1. Comparison of exact solution and Jacobi polynomial solutions ($N = 4$, 5 and 6) for Example 3.

Table 4. Comparison of the maximum absolute error values with Jacobi collocation method and Spline collocation method (via various collocation parameters) and Laguerre collocation method for Example 4.

Maximum absolute error values			
Spline collocation method - Gauss I (by using collocation points 30) [6, 21]	$8.33e-9$	Spline collocation method – Gauss I (by using collocation points 60) [6, 21]	$1.30e-10$
Spline collocation method – Radau II (by using collocation points 30) [6, 21]	$1.43e-7$	Spline collocation method – Radau II (by using collocation points 60) [6, 21]	$4.67e-9$
Spline collocation method – Lobatto (by using collocation points 30) [6, 21]	$1.81e-6$	Spline collocation method - Lobatto (by using collocation points 60) [6, 21]	$1.13e-7$
Spline collocation method – Gauss I (by using collocation points 20) [6, 21]	$9.10e-7$	Spline collocation method – Gauss I (by using collocation points 40) [6, 21]	$6.57e-8$
Spline collocation method – Other (by using collocation points 30) [6, 21]	$4.25e-6$	Spline collocation method – Other (by using collocation points 60) [6, 21]	$5.85e-7$
Laguerre collocation method (by using collocation points 10) [6]	$4.80e-8$	Laguerre collocation method (by using collocation points 15) [6]	$2.93e-10$
Present Method (by using collocation points 10)	$2.93e-9$	Present Method (by using collocation points 15)	$1.31e-11$

It is seen clearly from Table 4 that the Jacobi polynomial solution gives better results with respect to Laguerre collocation method [6] and Spline collocation method for the collocation parameters for Gauss, Radau II, Lobatto, Gauss I and named as Other [21].

6. Conclusion

This study presented a new numerical method for solving multi-functional integro-differential equations with mixed delays. This method is based on Jacobi polynomials and matrix operations. From the obtained numerical results, it was concluded that the obtained results are excellent in terms of accuracy and corrections for all four tested problems. The main advantage of the Jacobi collocation method is that it is easily programmable by using symbolic codes.

Therefore, the results of the problems are obtained quickly. Another advantage of the method is that the absolute error value of the obtained solutions decreases when N is increased. Estimated absolute error functions are obtained for the Jacobi Polynomial solutions obtained by the error estimation algorithm given Section 4.

Author's Contributions

M. Mustafa BAHŞI: Drafted and wrote the manuscript, performed the experiment and result analysis.

Mehmet ÇEVİK: Assisted in analytical analysis on the structure, supervised the experiment's progress, result interpretation and helped in manuscript preparation.

Ethics

There are no ethical issues after the publication of this manuscript.

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