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Research Article

Generic $\xi^{\perp}$-Riemannian submersions

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Abstract

As a generalization of semi-invariant $\xi^{\perp}$-Riemannian submersions, we introduce the generic $\xi^{\perp}$- Riemannian submersions. We focus on the generic $\xi^{\perp}$-Riemannian submersions for the Sasakian manifolds with examples and investigate the geometry of foliations. Also, necessary and sufficient conditions for the base manifold to be a local product manifold are obtained and new conditions for totally geodesicity are established. Furthermore, curvature properties of distributions for a generic $\xi^{\perp}$-Riemannian submersion from Sasakian space forms are obtained and we prove that if the distributions, which define a generic $\xi^{\perp}$-Riemannian submersion are totally geodesic, then they are Einstein.

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1. Introduction

Let M be a submanifold of almost contact manifold $\overline{M}$ with almost contact structure φ, then, M is invariant submanifold if $\varphi(T_xM) \subset T_xM$, $\forall\ x \in M$, where T_xM denotes the tangent space to M at point x and $\xi \in T_xM$, and ξ is a characteristic vector. M is called an anti-invariant submanifold if $\varphi(T_xM) \subset T_x^{\perp}M$, $\forall\ x \in M$, where $T_x^{\perp}M$ denotes the normal space to M at the point x. However, a semi-invariant submanifold is a generalization of both the invariant and anti-invariant submanifolds. Semi-invariant submanifold is defined by two orthogonal distributions; one invariant and the other an anti-invariant distribution. Generic submanifolds are generalized semi-invariant submanifolds, although defined by different distributions. A submanifold M of almost contact manifold $\overline{M}$ is called a generic submanifold of $\overline{M}$ if $D_x = (T_xM) \cap \varphi(T_xM)$ defines a smooth distribution $\forall\ x \in M$.

A conventional way to compare two manifolds is by defining smooth maps from one manifold to another. One such map is submersion, whose rank equals to the dimension of the target manifold. An isometric submersion is called a Riemannian submersion.

Riemannian submersion between Riemannian manifolds was first studied by O' Neill and Gray [17,23]. These studies were extended to manifolds with differentiable structures. Several authors have studied different geometric properties of the Riemannian submersions, anti-invariant submersions [8, 18, 21, 29], semi-invariant submersions [2, 4, 6, 24–26, 30].

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Riemannian submersions have applications in physics and mathematics, such as in supergravity and superstring theories [20, 22], Kaluza–Klein theory [12, 19], and the Yang-Mills theory [11, 32]. Also, Frejlich and Dunn et al. [13, 16] obtained submersions of Lie algebra.

Ali and Fatima [7] introduced a generic Riemannian submersion from almost Hermitian manifold onto Riemannian manifold. Several authors have studied submersions of generic submanifolds of a Kaehler manifold [15]. Şahin studied generic Riemannian maps [31]. Akyol introduced generic Riemannian submersions and conformal generic Riemannian submersions from almost product Riemannian submanifolds and almost Hermitian manifold, respectively [1, 3]. Sayar et al. introduced generic submersion from Kaehler manifold [28].

Recently, Akyol et al. defined and studied semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact metric manifolds and investigated the geometry of the new submersions on almost contact manifolds [5].

This work introduces a generic $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds as a generalization of semi-invariant $\xi^{\perp}$-Riemannian submersions from almost contact manifold.

The paper is organized as follows: Section 2, outlines the basic properties of Sasakian manifold and Riemannian submersion; section 3, defines the generic $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemanian manifolds; section 4, investigates the geometry of distributions and show that there are certain product structures on total space of a generic $\xi^{\perp}$-Riemannian submersions from Sasakian manifolds onto Riemannian manifolds; section 5, discuss new conditions for generic $\xi^{\perp}$-Riemannian submersions to be totally geodesic and totally umbilical; finally, section 6, discuss the curvature properties and Einstein conditions of distributions for a generic $\xi^{\perp}$-Riemannian submersion from Sasakian space forms onto Riemannian manifolds.

2. Preliminaries

Let $(N, \varphi, \xi, \eta, g)$ be an almost contact manifold, where φ is a tensor field of type $(1, 1)$, η is a 1-form, ξ is a characteristic vector field, g is the Riemannian metric, such that

$$\varphi^2 = -I + \eta \otimes \xi \quad \text{and } \eta(\xi) = 1, \tag{2.1}$$

$$g(\varphi U, \varphi V) = g(U, V) - \eta(U)\eta(V) \tag{2.2}$$

where $\varphi\xi = 0,\ \eta o\varphi = 0,\ g(\varphi U, V) = -g(U, \varphi V)$ and $\forall\ U, V \in \Gamma(TN)$ [10].

Similarly, an almost contact manifold is normal if

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0$$

where $[\varphi, \varphi]$ is Nijenhuis tensor of φ.

A normal almost contact manifold is Sasakian manifold if and only if

$$(\nabla_U \varphi)V = g(U, V)\xi - \eta(V)U \quad \text{and} \quad \nabla_U \xi = -\varphi U, \tag{2.3}$$

where ∇ is the connection of Levi-Civita covariant differentiation [27].

Let (N, g_N) and (B, g_B) be Riemannian manifolds, where $\dim(N) = k$, $\dim(B) = l$ and $k > l$. A Riemannian submersion $\pi : N \to B$ is a map of N onto B satisfying the following axioms:

(i) π has maximal rank.
(ii)The differential π_* preserves the lenghts of horizontal vectors.

Let $\pi^{-1}(q)$ be a $(k - l)$ dimensional submanifold of N, for any $q \in B$. Then, the submanifolds $\pi^{-1}(q)$ are called fibers.

A vector field on N is referred to as vertical if the fibers are tangent and referred to as horizontal if the fibers are orthogonal. A vector field U on N is called basic if U is horizontal and $\pi-$related to a vector field U on B, i.e., $\pi_* U_p = U_{\pi_*(p)}$ for all $p \in N$. We denote the projection morphisms on the distributions $ker\pi_*$ and $(ker\pi_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively.

We recall that the sections of $\mathcal{V}$ and $\mathcal{H}$ are called vertical vector fields and horizontal vector fields, respectively. A Riemannian submersion $\pi : N \to B$ determines two $(1,2)$ tensor fields $\mathcal{T}$ and $\mathcal{A}$ on N, by the formulas:

$$\mathcal{T}(E,F) = \mathcal{T}_E F = \mathcal{H}\overset{N}{\nabla}_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\overset{N}{\nabla}_{\mathcal{V}E}\mathcal{H}F \tag{2.4}$$

and

$$\mathcal{A}(E,F) = \mathcal{A}_E F = \mathcal{V}\overset{N}{\nabla}_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\overset{N}{\nabla}_{\mathcal{H}E}\mathcal{V}F \tag{2.5}$$

for any $E, F \in \Gamma(TN)$, where $\mathcal{V}$ and $\mathcal{H}$ are the vertical and horizontal projections [23]. Using (2.4) and (2.5), we have

$$\overset{N}{\nabla}_V W = T_V W + \hat{\nabla}_V W, \tag{2.6}$$

$$\overset{N}{\nabla}_V X = T_V X + \mathcal{H}(\overset{N}{\nabla}_V X), \tag{2.7}$$

$$\overset{N}{\nabla}_X V = \mathcal{V}(\overset{N}{\nabla}_X V) + A_X V, \tag{2.8}$$

$$\overset{N}{\nabla}_X Y = A_X Y + \mathcal{H}(\overset{N}{\nabla}_X Y) \tag{2.9}$$

for any $V, W \in \Gamma(ker\pi_*)$ and $X, Y \in \Gamma((ker\pi_*)^\perp)$. Furthermore, if X is a basis then

$$\mathcal{H}(\overset{N}{\nabla}_V X) = A_X V. \tag{2.10}$$

We note that for $V, W \in \Gamma(ker\pi_*)$, $T_V W$ coincides with the second fundamental form of the immersion of the fiber submanifolds and $\mathcal{T}$ is symmetric on the vertical distribution: $\mathcal{T}_V W = \mathcal{T}_W V$, for $V, W \in \Gamma(ker\pi_*)$. Furhermore, $\mathcal{A}_X Y = \frac{1}{2}\mathcal{V}[X,Y]$, which shows the complete integrability of the horizontal distribution $\mathcal{H}$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$. Moreover, A alternates on the horizontal distribution, $\mathcal{A}_X Y = -\mathcal{A}_Y X$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$.

Lemma 2.1 ([14],[23]). *If $\pi : N \to B$ is a Riemannian submersion and U, V are basic vector fields on N, $\pi-$related to U' and V' on B, then we get*

(1) *$\mathcal{H}[U,V]$ is a basic vector field and $\pi_*\mathcal{H}[U,V] = [U', V'] \circ \pi$,*
(2) *$\mathcal{H}(\overset{N}{\nabla}_U V)$ is a basic vector field $\pi-$related to $(\overset{B}{\nabla}_{U'} V')$, where ∇^N and ∇^B are the Levi-Civita connections on N and B, respectively,*
(3) *$[E,K] \in \Gamma(ker\pi_*)$, for any $K \in \Gamma(ker\pi_*)$ and for any basic vector field E.*

Let $\pi : N \to B$ be a Riemannian submersion. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(U,V) = \overset{N}{\nabla}_U \pi_* V - \pi_*(\overset{M}{\nabla}_U V) \tag{2.11}$$

for $U, V \in \Gamma(TN)$. Moreover, π is called a *totally geodesic* map if $(\nabla\pi_*)(U,V) = 0$ for $U, V \in \Gamma(TN)$ [9].

Let $\pi : N \to B$ be a Riemannian submersion. O'Neill ([23]), defined the Riemannian curvature R of N, such that

$$R(V,W,U,S) = \widehat{R}(V,W,U,S) + g(\mathcal{T}_V U, \mathcal{T}_W S) - g(\mathcal{T}_W U, \mathcal{T}_V S) \tag{2.12}$$

and

$$R(V,W,U,X) = g((\nabla_V \mathcal{T})_U W, X) - g((\nabla_U \mathcal{T})_V W, X) \tag{2.13}$$

where $\widehat{R}$ is Riemannian curvature tensor of any fiber $(\pi^{-1}(x), g_x)$. If $\{V, W\}$ is an orthonormal basis of the vertical 2-plane, then from equation (2.12), we have

$$K(V,W) = \widehat{K}(V,W) + \|T_V W\| - g(T_V V, T_W W) \tag{2.14}$$

where K and $\widehat{K}$ are sectional curvatures of M and $\pi^{-1}(x)$.

A plane section in the tangent space T_pN at $p \in N$ is called a φ-section if it is spanned by a vector X orthogonal to ξ and φX. The sectional curvature of a φ-section is called the φ-sectional curvature. A Sasakian manifold with constant φ-sectional curvature c is called a Sasakian space form and is denoted by $N(c)$. The Riemannian curvature tensor of a Sasakian space form $N(c)$ is given by

$$\begin{aligned} R(V,W,U,S) \quad = \quad & \frac{c+3}{4}\{g_N(W,U)g_N(V,S) - g_N(V,U)g_N(W,S)\} \\ & +\frac{c-1}{4}\{g_N(W,S)\eta(V)\eta(U) - g_N(V,S)\eta(W)\eta(U) \\ & +g_N(V,U)\eta(W)\eta(S) - g_N(W,U)\eta(V)\eta(S) \\ & +g_N(\varphi W,U)g_N(\varphi V,S) - g_N(\varphi V,U)g_N(\varphi W,S) \\ & -2g_N(\varphi V,W)g_N(\varphi U,S)\} \end{aligned} \tag{2.15}$$

for all $V,W,U,S \in \Gamma(TN)$[10].

3. Generic $\xi^{\perp}$-Riemannian submersions

We define generic $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold with examples. We begin with the following definition:

Let $\pi : (N,\varphi,\xi,\eta,g) \longrightarrow (B,g)$ be a Riemannian submersion such that N is a Sasakian manifold, B is a Riemannian manifold and ξ is normal to $ker\pi_*$. Then, the complex subspace of the vertical subspace $\mathcal{V}_x$ is defined by

$$D_x = (\ker \pi_{*x} \cap \varphi(\ker \pi_{*x}))$$

where $x \in N$.

Definition 3.1. Let $\pi : (N,\varphi,\xi,\eta,g) \longrightarrow (B,g)$ be a Riemannian submersion such that N is a Sasakian manifold, B is a Riemannian manifold and ξ is normal to $ker\pi_*$. For $D \subset \ker \pi_*$ such that

$$\ker \pi_* = D \oplus D_{\perp}, \ \ \varphi(D) = D$$

π is called a generic $\xi^{\perp}-$Riemannian submersion, where $D_{\perp}$ is the orthogonal complement of D in $\ker \pi_*$, and purely real distribution on the fibers of the submersion π.

We provide examples that guarantee the existence of generic $\xi^{\perp}$-Riemannian submersions in Sasakian manifolds and demonstrate the effectiveness of the method presented. Note that, $(\mathbb{R}^{2n+1},\varphi,\eta,\xi,g)$ denotes the manifold $\mathbb{R}^{2n+1}$, with its Sasakian structure given by

$$\eta = \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i), \ \ \xi = 2\frac{\partial}{\partial z},$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\varphi(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}) = \sum_{i=1}^{n}(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i})$$

where $(x_1,..,x_n,y_1,...,y_n,z)$ denotes the Cartesian coordinates on $\mathbb{R}^{2n+1}$, and will be used throughout this section.

Example 3.2. Every semi-invariant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $D_{\perp}$ is total real distribution.

Example 3.3. Every slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $D = \{0\}$ and $D_{\perp}$ is a slant distribution.

Example 3.4. Every semi-slant $\xi^{\perp}$-Riemannian submersion from a Sasakian manifold onto a Riemannian manifold is a generic $\xi^{\perp}$-Riemannian submersion such that $D_{\perp}$ is a slant distribution.

Example 3.5. Let $\mathbb{R}^9$ and $\mathbb{R}^5$ be a Sasakian and Riemannian manifold, respectively. We denote Riemannian metric and Cartesian coordinates on $\mathbb{R}^5$ such that $g_{\mathbb{R}^5} = \frac{1}{4}\sum_{i=1}^{2}(du^i \otimes du^i + dv^i \otimes dv^i) + dz \otimes dz$ and (u_1, u_2, v_1, v_2, z), respectively. We define a map π by

$$\begin{array}{cccc} \pi : & \mathbb{R}^9 & \longrightarrow & \mathbb{R}^5 \\ & (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z) & & (y_1, x_1, \frac{y_2+x_3+y_3}{\sqrt{3}}, \frac{x_3-y_3}{\sqrt{2}}, z) \end{array}.$$

Then it follows that

$$\ker \pi_* = Sp\{V_1 = \frac{\partial}{\partial y_4}, V_2 = \frac{\partial}{\partial x_4}, V_3 = \frac{\partial}{\partial x_2}, V_4 = -2\frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}\}$$

and

$$\ker \pi_*^{\perp} = Sp\{W_1 = \frac{\partial}{\partial x_1}, W_2 = \frac{\partial}{\partial y_1}, W_3 = \frac{\partial}{\partial y_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}, W_4 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial y_3}, W_5 = \frac{\partial}{\partial z}\}.$$

After some basic calculations, we get that π is submersion. Hence we have $\varphi V_1 = V_2, \varphi V_3 = \frac{1}{3}(W_3 - V_4)$ and $\varphi V_4 = 2V_3 + W_4$. Thus it follows that $D = sp\{V_1, V_2\}$ and $D_{\perp} = sp\{V_3, V_4\}$. Also direct computations, we obtain

$$g_{\mathbb{R}^9}(W_i, W_i) = g_{\mathbb{R}^5}(\pi_* W_i, \pi_* W_i)$$

where $g_{\mathbb{R}^9}$ and $g_{\mathbb{R}^5}$ are metrics of $\mathbb{R}^9$ and $\mathbb{R}^5$, for all $i = 1, ..., 5$. Thus, π is a generic $\xi^{\perp}$-Riemannian submersion.

Example 3.6. Let $\mathbb{R}^7$ and $\mathbb{R}^3$ be a Sasakian and Riemannian manifold, respectively. We denote Riemannian metric and Cartesian coordinates on $\mathbb{R}^3$ such that $g_{\mathbb{R}^3} = \frac{1}{4}(du \otimes du + dv \otimes dv) + dz \otimes dz$ and (u, v, z), respectively. We define a map by

$$\begin{array}{cccc} \pi : & \mathbb{R}^7 & \longrightarrow & \mathbb{R}^3 \\ & (x_1, x_2, x_3, y_1, y_2, y_3, z) & & (x_1 + x_2 + y_2, x_1 - y_2, z) \end{array}$$

Then

$$\ker \pi_* = Sp\{V_1 = \frac{\partial}{\partial y_1}, V_2 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2}, V_3 = \frac{\partial}{\partial x_3}, V_4 = \frac{\partial}{\partial y_3}\}$$

and

$$\ker \pi_*^{\perp} = Sp\{W_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, W_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_2}, W_3 = \frac{\partial}{\partial z}\}$$

After some computation, we get that π is Riemannian submersion. Also, we have $\varphi V_1 = \frac{1}{2}W_2 + \frac{1}{2}V_2$, $\varphi V_2 = -V_1 - W_1 + W_2$ and $\varphi V_3 = -V_4$. Then, π is generic $\xi^{\perp}$-Riemannian submersion such that $D = sp\{V_3, V_4\}$ and $D_{\perp} = sp\{V_1, V_2\}$.

Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. For $Z \in \Gamma(TN)$, we have

$$Z = \mathcal{V}Z + \mathcal{H}Z \tag{3.1}$$

where $\mathcal{V}Z \in \Gamma(ker\pi_*)$ and $\mathcal{H}Z \in \Gamma(ker\pi_*)^{\perp}$. For $K \in \Gamma(ker\pi_*)$, we write

$$\varphi K = \phi K + \omega K \tag{3.2}$$

where ϕK and ωK are vertical (resp. horizontal) components of φK, respectively.

Alternately, let μ be the complementary distribution to $wD_{\perp}$ in $(ker\pi_*)^{\perp}$. Then, we give

$$\phi D_{\perp} \subset D_{\perp}, \qquad (ker\pi_*)^{\perp} = wD_{\perp} \oplus \mu,$$

where $\varphi(\mu) \subset \mu$. Hence, μ contains ξ. Similarly, for $X \in \Gamma((kerF_*)^{\perp})$, we have

$$\varphi X = \mathcal{B}X + \mathcal{C}X, \tag{3.3}$$

where $\mathcal{B}X$ and $\mathcal{C}X$ are vertical (resp. horizontal) components of φX, respectively.

Then using (2.6), (2.7), (3.2) and (3.3), we get

$$(\nabla^{M}_{V}\phi)W = \mathcal{B}T_{V}W - T_{V}\omega W, \tag{3.4}$$

$$(\nabla^{M}_{V}\omega)W = \mathcal{C}T_{V}W - T_{V}\phi W \tag{3.5}$$

for $V, W \in \Gamma(ker\pi_{*})$, where

$$(\nabla^{M}_{V}\phi)W = \hat{\nabla}_{V}\phi W - \phi\hat{\nabla}_{V}W$$

and

$$(\nabla^{M}_{V}\omega)W = \mathcal{H}\nabla^{M}_{V}\omega W - \omega\hat{\nabla}_{V}W.$$

4. Geometry of foliations

This section, investigates the integrability and totally geodesicness of distributions involved in the definition of a generic $\xi^{\perp}$-Riemannian submersion. Furhermore, we obtain decomposition theorems of this submersion.

Theorem 4.1. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the distribution D is integrable if and only if*

$$T_{K}\varphi L = T_{L}\varphi K$$

for any $K, L \in \Gamma(D)$.

Proof. For $K, L \in \Gamma(D), Z \in \Gamma(D_{\perp}), X \in \Gamma((ker\pi_{*})^{\perp})$, since $[K, L] \in \Gamma(ker\pi_{*})$, we have that $g_N([K, L], X) = 0$. Thus, D is integrable if and only if $g_N([K, L], X) = 0$. Firstly, for any $U, V, W \in \Gamma(TN)$, from (2.2) and (2.3), we have

$$g_N(\nabla_{U}V, W) = g_N(\nabla_{U}\varphi V, \varphi W). \tag{4.1}$$

For $K, L \in \Gamma(D)$, $Z \in \Gamma(D_{\perp})$,using (2.2) and (4.1) we have

$$g_N([K, L], Z) = g_N(\nabla_{K}\varphi L, \varphi Z) - g_N(\nabla_{L}\varphi K, \varphi Z). \tag{4.2}$$

Then, by (2.3),(2.6) and(4.2) we conclude that,

$$\begin{aligned} g_N([K, L], Z) \;\; &= \;\; g_N(\hat{\nabla}_{K}\varphi L + T_{K}\varphi L - g_N(K, L)\xi - \eta(L)K - \hat{\nabla}_{K}\varphi L \\ &\quad -T_{K}\varphi L + g_N(K, L)\xi + \eta(L)K, Z). \end{aligned}$$

By elementary calculations, we get

$$g_N([K, L], Z) = g_N(T_{K}\varphi L - T_{L}\varphi K, Z)$$

which gives proof. □

Theorem 4.2. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the distribution $D_{\perp}$ is integrable if and only if*

$$\widehat{\nabla}_{W}\phi Z - \widehat{\nabla}_{Z}\phi W + \mathcal{T}_{W}wZ - \mathcal{T}_{Z}wW \in \Gamma(D_{\perp})$$

for any $Z, W \in \Gamma(D_{\perp}), K \in \Gamma(D)$.

Proof. For $Z, W \in \Gamma(D_{\perp}), K \in \Gamma(D)$, using (2.2),(2.6),(2.7) and (4.2), we have,

$$\begin{aligned} g_N([Z, W], K) \;\; &= \;\; -g_N(\varphi(\widehat{\nabla}_{Z}\phi W + \mathcal{T}_{Z}\phi W + \mathcal{H}(\nabla_{Z}wW) + \mathcal{T}_{Z}wW), K) \\ &\quad +g_N(\varphi(\widehat{\nabla}_{W}\phi Z + \mathcal{T}_{W}\phi Z + \mathcal{H}(\nabla_{W}wZ) + \mathcal{T}_{W}wZ), K). \end{aligned}$$

By virtue of (3.2),(3.3), we arrive

$$\begin{aligned} g_N([Z,W],K) &= g_N(\phi(-\widehat{\nabla}_Z\phi W - \mathfrak{T}_Z wW + \widehat{\nabla}_W\phi Z + \mathfrak{T}_W wZ) \\ &\quad +B(-\mathfrak{T}_Z\phi W - \mathcal{H}(\nabla_Z wW) + \mathfrak{T}_W\phi Z + \mathcal{H}(\nabla_W wZ)), K) \\ &\quad +g_N(w(-\widehat{\nabla}_Z\phi W - \mathfrak{T}_Z wW + \widehat{\nabla}_W\phi Z + \mathfrak{T}_W wZ) \\ &\quad +C(-\mathfrak{T}_Z\phi W - \mathcal{H}(\nabla_Z wW) + \mathfrak{T}_W\phi Z + \mathcal{H}(\nabla_W wZ)), K). \end{aligned}$$

After some calculations, we get

$$\begin{aligned} g_N([Z,W],K) &= g_N(\phi(-\widehat{\nabla}_Z\phi W - \mathfrak{T}_Z wW + \widehat{\nabla}_W\phi Z + \mathfrak{T}_W wZ) \\ &\quad +B(-\mathfrak{T}_Z\phi W - \mathcal{H}(\nabla_Z wW) + \mathfrak{T}_W\phi Z + \mathcal{H}(\nabla_W wZ)), K). \end{aligned}$$

Since $B(-\mathfrak{T}_Z\phi W - \mathcal{H}(\nabla_Z wW) + \mathfrak{T}_W\phi Z + \mathcal{H}(\nabla_W wZ)) \in \Gamma(D^\perp)$, we conclude that

$$g_N([Z,W],K) = g_N(\phi(-\widehat{\nabla}_Z\phi W - \mathfrak{T}_Z wW + \widehat{\nabla}_W\phi Z + \mathfrak{T}_W wZ), K)$$

which proves the assertion. □

Theorem 4.3. *Let $\pi : (N,\varphi,\xi,\eta,g_N) \longrightarrow (B,g_B)$ be a generic $\xi^\perp$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the distribution D defines a totally geodesic foliation on N if and only if*

$$\phi(\widehat{\nabla}_K\phi L - T_K wL) = -B(T_K wL + H\nabla_K wL)$$

and

$$g_B((\nabla\pi_*)(K,\varphi L), \pi_* CX) = g_N(\nabla_K\varphi L, BX)$$

for any $K, L \in \Gamma(D), X \in \Gamma((\ker\pi_*)^\perp)$.

Proof. The distribution D defines a totally geodesic foliation on N if and only if $g_N(\nabla_K L, Z) = 0$ and $g_N(\nabla_K L, X) = 0$ for any $K, L \in \Gamma(D), Z \in \Gamma(D_\perp), X \in \Gamma((\ker\pi_*)^\perp)$. For $K, L \in \Gamma(D), Z \in \Gamma(D_\perp)$ using (2.2),(2.3) and (3.2) we have

$$g_N(\nabla_K L, Z) = g_N(\varphi\nabla_K L, \varphi Z).$$

By virtue of (2.3) and (3.2) imply that

$$g_N(\nabla_K L, Z) = g_N(\varphi(\nabla_K\phi L + \nabla_K wL), Z).$$

Then, from (2.6),(2.7),(3.2) and (3.3), we have

$$g_N(\nabla_K L, Z) = -g_N(\phi\widehat{\nabla}_K\phi L + B\mathfrak{T}_K\phi L + \phi\mathfrak{T}_K wL + B\mathcal{H}\nabla_K wL, Z).$$

On the other hand, for $X \in \Gamma((\ker\pi_*)^\perp)$, using (2.7),(2.11) and (3.3), we arrive

$$\begin{aligned} g_N(\nabla_K L, X) &= g_N(\nabla_K\varphi L, \varphi X) \\ &= g_N(\nabla_K\varphi L, BX + CX). \end{aligned}$$

Since π is generic $\xi^\perp$-Riemannian submersion, we have

$$g_N(\nabla_K L, X) = g_N(\nabla_K\varphi L, BX) - g_B(\pi_*\nabla_K\varphi L, \pi_* CX).$$

Then, using (2.11), we get

$$g_N(\nabla_K L, X) = g_N(\nabla_K\varphi L, BX) - g_B(\nabla\pi_*)(K,\varphi L), \pi_* CX)$$

which proves the assertion. □

Theorem 4.4. *Let $\pi : (N,\varphi,\xi,\eta,g_N) \longrightarrow (B,g_B)$ be a generic $\xi^\perp$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the distribution $D_\perp$ defines a totally geodesic foliation on N if and only if*

$$(\nabla\pi_*)(Z,\varphi W) \in \Gamma(\mu)$$

and

$$g_B(\pi_*(Z,CX), \pi_* wW) = g_N(\widehat{\nabla}_Z BX, \phi W) + g_N(\mathfrak{T}_Z BX, wW) + g_N(\mathfrak{T}_Z CX, \phi W)$$

for any $Z, W \in \Gamma(D_{\perp}), X \in \Gamma((\ker \pi_*)^{\perp})$.

Proof. For $Z, W \in \Gamma(D_{\perp}), K \in \Gamma(D), X \in \Gamma((\ker \pi_*)^{\perp})$ using (2.2) and (3.2), we have

$$g_N(\nabla_Z W, K) = g_N(\nabla_Z \varphi W, \phi K + wK).$$

Then, from (3.1), we arrive

$$g_N(\nabla_Z W, K) = g_N(H\nabla_Z \varphi W, wK).$$

Taking into account that π is generic $\xi^{\perp}$-Riemannian submersion and from (2.11), we conclude that

$$g_N(\nabla_Z W, K) = g_B((\nabla \pi_*)(Z, \varphi W), \pi_* wK).$$

Similarly, by virtue of (2.2) and (4.1), we have

$$g_N(\nabla_Z W, X) = g_N(\nabla_Z \varphi X, \varphi W).$$

Then, using (2.6),(2.7) and (3.3), we arrive

$$\begin{aligned} g_N(\nabla_Z W, X) &= g_N(\widehat{\nabla}_Z BX, \phi W) + g_N(\mathcal{T}_Z BX, wW) + g_N(\mathcal{H}\nabla_Z CX, wW) \\ &\quad + g_N(\mathcal{T}_Z CX, \phi W). \end{aligned}$$

Furthermore, using (2.11), we get

$$\begin{aligned} g_N(\nabla_Z W, X) &= g_N(\widehat{\nabla}_Z BX, \phi W) + g_N(\mathcal{T}_Z BX, wW) + g_B(\pi_*(\nabla_Z CX), \pi_* wW) \\ &\quad + g_B(\mathcal{T}_Z CX, \phi W) \end{aligned}$$

which gives proof. $\square$

Corollary 4.5. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the fibers of π are the locally product Riemannian manifold of leaves of D and $D_{\perp}$ if and only if*

$$\phi(\widehat{\nabla}_K \phi L - T_K wL) = -B(T_K wL + H\nabla_K wL),$$
$$g_B((\nabla \pi_*)(K, \varphi L), \pi_* CX) = g_N(\nabla_K \varphi L, BX)$$

and

$$(\nabla \pi_*)(Z, \varphi W) \in \Gamma(\mu),$$
$$g_B(\pi_*(Z, CX), \pi_* wW) = g_N(\widehat{\nabla}_Z BX, \phi W) + g_N(\mathcal{T}_Z BX, wW) + g_N(\mathcal{T}_Z CX, \phi W)$$

for any $K, L \in \Gamma(D),\ Z, W \in \Gamma(D_{\perp}), X \in \Gamma((\ker \pi_*)^{\perp})$.

Theorem 4.6. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then $\ker \pi_*$ defines a totally geodesic foliation on N if and only if*

$$\widehat{\nabla}_U \phi V + \mathcal{T}_U wV \in \Gamma(D),$$

and

$$\mathcal{T}_U \phi V + \mathcal{H}\nabla_U wV \in \Gamma(D^{\perp})$$

for any $U, V \in \Gamma(\ker \pi_*)$.

Proof. For all $U, V \in \Gamma(\ker \pi_*)$, we have

$$(\nabla_U \varphi)V = \nabla_U \varphi V - \varphi \nabla_U V.$$

Then, using (2.1) and (2.3), we arrive

$$\nabla_U V = -\varphi \nabla_U \varphi V. \tag{4.3}$$

On the other hand, using (3.2),(2.6),(2.7),(3.3) in (4.3), we conclude that

$$\begin{aligned} \nabla_U V &= -(\phi \widehat{\nabla}_U \phi V + w\widehat{\nabla}_U \phi V + B\mathcal{T}_U \phi V + C\mathcal{T}_U \phi V \\ &\quad + B\mathcal{H}\nabla_U wV + C\mathcal{H}\nabla_U wV + \phi \mathcal{T}_U wV + w\mathcal{T}_U wV). \end{aligned}$$

Then, $\ker\pi_*$ defines a totally geodesic foliation if and only if

$$C(\mathcal{T}_U\phi V+\mathcal{H}\nabla_U wV)+w(\widehat{\nabla}_U\phi V+\mathcal{T}_U wV)=0.$$

Hence, $\nabla_U V\in\Gamma(\ker\pi_*)$ if and only if

$$C(\mathcal{T}_U\phi V+\mathcal{H}\nabla_U wV)=0$$

and

$$w(\widehat{\nabla}_U\phi V+\mathcal{T}_U wV)=0.$$

□

Theorem 4.7. *Let $\pi:(N,\varphi,\xi,\eta,g_N)\longrightarrow(B,g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, $(\ker\pi_*)^{\perp}$ defines a totally geodesic foliation on N if and only if*

$$\mathcal{V}\nabla_X BY+A_X CY\in\Gamma(D^{\perp}),$$

and

$$A_X BY+\mathcal{H}\nabla_X CY\in\Gamma(\mu)$$

for any $X,Y\in\Gamma((\ker\pi_*)^{\perp})$.

Proof. For $X,Y\in\Gamma((\ker\pi_*)^{\perp})$, by virtue of (4.3)

$$\nabla_X Y=-\varphi\nabla_X\varphi Y.$$

By using (2.8),(2.9), (3.1),(3.3), we arrive

$$\begin{aligned}\nabla_X Y &= -(BA_XBY+CA_XBY+\phi\mathcal{V}\nabla_XBY+w\nabla_XBY+B\mathcal{H}(\nabla_XY)\\ &\quad +C\mathcal{H}(\nabla_XY)+\phi A_XCY+wA_XCY).\end{aligned}$$

On the other hand, from $\nabla_X Y\in\Gamma((\ker\pi_*)^{\perp})$, we have

$$BA_XBY+\phi\mathcal{V}\nabla_XBY+B\mathcal{H}(\nabla_XY)+\phi A_XCY=0.$$

Then, $\nabla_X Y\in\Gamma((\ker\pi_*)^{\perp})$ if and only if

$$B(A_XBY+\mathcal{H}(\nabla_XY))=0$$

and

$$\phi(\mathcal{V}\nabla_XBY+A_XCY)=0.$$

□

Corollary 4.8. *Let $\pi:(N,\varphi,\xi,\eta,g_N)\longrightarrow(B,g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then, the total space N is a locally product manifold of the leaves of $\ker\pi_*$ and $(\ker\pi_*)^{\perp}$, i.e. $N=N_{\ker\pi_*}\times N_{(\ker\pi_*)^{\perp}}$ if and only if*

$$\widehat{\nabla}_U\phi V+\mathcal{T}_UwV\in\Gamma(D),\qquad \mathcal{T}_U\phi V+\mathcal{H}\nabla_UwV\in\Gamma(D^{\perp})$$

and

$$\mathcal{V}\nabla_XBY+A_XCY\in\Gamma(D^{\perp}),\qquad A_XBY+\mathcal{H}\nabla_XCY\in\Gamma(\mu)$$

for any $U,V\in\Gamma(\ker\pi_*),X,Y\in\Gamma((\ker\pi_*)^{\perp})$.

5. Totally umbilical and totally geoedesicness of π

This section, we investigate new conditions for generic $\xi^{\perp}$-Riemannian submersion to be totally geodesic and totally umbilical.

A Riemannian submersion between two Riemannian manifolds is called totally geodesic if and only if $\nabla\pi_* = 0$. On the other hand, let π be Riemannian submersion. Then π is called Riemannian submersion with totally umbilical fiber if

$$\mathcal{T}_U V = g(U,V)H \tag{5.1}$$

for all $U, V \in \Gamma(\ker \pi_*)$ and H is mean curvature vector fields of fiber.

Theorem 5.1. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian and Riemannian manifolds, respectively. Then π is totally geodesic if*

$$\begin{aligned}\nabla_X \pi_* Z &= \pi_*(w(A_X\mathcal{H}Z) - \mathcal{V}\nabla_X\mathcal{V}\mathcal{Z}) + C(\mathcal{H}\nabla_X Z - A_X\mathcal{V}\mathcal{Z}) \\ &\quad -\eta(Z)CX + \eta(A_X\mathcal{V}\mathcal{Z})\xi + \eta(\mathcal{H}\nabla_X Z)\xi)\end{aligned}$$

for any $X \in \Gamma((\ker\pi_*)^{\perp})$, $Z \in \Gamma(TN)$.

Proof. For $Z \in \Gamma(TN), X \in \Gamma((\ker\pi_*)^{\perp})$ using (2.3), (2.11), and (3.1), we have

$$(\nabla_X\pi_*)Z = \nabla_X\pi_*Z - \pi_*(-\eta(Z)\varphi X - \varphi\nabla_X\mathcal{V}\mathcal{Z} + \varphi\nabla_X\mathcal{H}\mathcal{Z} + \eta(\nabla_X Z)\xi).$$

Then, by virtue of (2.8),(2.9),(3.2),(3.3), we have

$$\begin{aligned}(\nabla_X\pi_*)Z &= \nabla_X\pi_*Z - \pi_*(-\eta(Z)CX - \phi\mathcal{V}\nabla_X\mathcal{V}\mathcal{Z} - w\mathcal{V}\nabla_X\mathcal{V}\mathcal{Z} - \mathcal{B}A_X\mathcal{V}\mathcal{Z} \\ &\quad -\mathcal{C}A_X\mathcal{V}\mathcal{Z} + \phi A_X\mathcal{H}Z + wA_X\mathcal{H}Z + \mathcal{B}\mathcal{H}\nabla_X Z + C\mathcal{H}\nabla_X Z \\ &\quad +g(\mathcal{V}\nabla_X\mathcal{V}\mathcal{Z} + A_X\mathcal{V}\mathcal{Z}, \xi)\xi + g(A_X\mathcal{V}\mathcal{Z} + \mathcal{H}\nabla_X Z, \xi)\xi.\end{aligned}$$

Thus, taking into account that the vertical parts, we get

$$\begin{aligned}(\nabla_X\pi_*)Z &= \nabla_X\pi_*Z - \pi_*(-\eta(Z)\mathcal{C}X - w\mathcal{V}\nabla_X\mathcal{V}\mathcal{Z} - \mathcal{C}A_X\mathcal{V}\mathcal{Z} + wA_X\mathcal{H}Z \\ &\quad +\mathcal{C}\mathcal{H}\nabla_X Z + g(\mathcal{V}\nabla_X\mathcal{V}\mathcal{Z} + A_X\mathcal{V}\mathcal{Z}, \xi)\xi \\ &\quad +g(A_X\mathcal{V}\mathcal{Z} + \mathcal{H}\nabla_X Z, \xi)\xi\end{aligned}$$

which proves the assertion. □

Theorem 5.2. *Let $\pi : (N, \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion with totally umbilcal fibres such that N and B are Sasakian and Riemannian manifolds, respectively, then $H \in \Gamma(wD_{\perp})$.*

Proof. For any $U, V \in \Gamma(D)$, using (2.3), (2.6), (3.2) and (3.3), we have

$$g_N(U,V)\xi - \eta(V)U = T_U\phi V + \hat{\nabla}_U\phi V - BT_U\phi V - CT_U\phi V - \phi\hat{\nabla}_U V - w\hat{\nabla}_U V.$$

Taking inner product in above equation with $Z \in \Gamma(\mu)$, we arrive

$$g_N(U,V)\eta(Z) = g_N(T_U\phi V, Z) - g_N(CT_U\phi V, Z).$$

Since π is totally umbilical, using (3.2), we conclude that

$$g_N(U,V)\eta(Z) = g_N(U, \phi V)g_N(H, Z) - g_N(U,V)g_N(H, \varphi Z).$$

Interchanging U and V in last equation and subtracting this two equation, we get

$$g_N(H, Z) = 0$$

which completed that proof. □

Theorem 5.3. *Let $\pi : (N(c), g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion with totally umbilical fibres such that $N(c)$ and B are Sasakian space form and Riemannian manifold, respectively. Then $c = 1$.*

Proof. Taking into account that (2.13), (2.15) and (5.1) we obtain our assertion. □

6. Generic $\xi^\perp$-Riemannian submersions with Sasakian space form

This section, we study curvature properties and Einstein conditions of generic $\xi^\perp$-Riemannian submersion.

Let $\pi : (N(c), g_N) \longrightarrow (B, g_B)$ be a generic $\xi^\perp$-Riemannian submersion such that $N(c)$ and B are Sasakian space form and Riemannian manifold, respectively. Then, using (2.12) and (2.15)

$$\begin{aligned}\widehat{R}(V,W,U,S) &= \frac{c+3}{4}\{g_N(W,U)g_N(V,S) - g_N(V,U)g_N(W,S)\} \\ &\quad + \frac{c-1}{4}\{g_N(W,S)\eta(V)\eta(U) - g_N(V,S)\eta(W)\eta(U) \\ &\quad + g_N(V,U)\eta(W)\eta(S) - g_N(W,U)\eta(V)\eta(S) \\ &\quad + g_N(\varphi W,U)g_N(\varphi V,S) - g_N(\varphi V,U)g_N(\varphi W,S) \\ &\quad - 2g_N(\varphi V,W)g_N(\varphi U,S)\} \\ &\quad + g(\mathfrak{T}_V U, \mathfrak{T}_W S) - g(\mathfrak{T}_W U, \mathfrak{T}_V S) \qquad (6.1)\end{aligned}$$

for all $V, W, U, S \in \Gamma(TN)$.

Now, we choose an orthonormal frame on N by $\{e_1, ..., e_{2p}, e_{2p+1}, ..., e_{2p+2q}, e_{2p+2q+1}\}$. Then, we get,

$$D = sp\{e_1, ..., e_{2p}\},\ D_\perp = sp\{e_{2p+1}, ..., e_{2p+2q}\} \text{ and } \xi = sp\{e_{2p+2q+1}\}$$

where $dimD = 2p$ and $dimD_\perp = 2q$.

Theorem 6.1. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^\perp$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. Then, we have*

$$\begin{aligned}\widehat{R}(V,W,U,S) &= \frac{c+3}{4}\{g_N(W,S)g_N(V,U) - g_N(V,S)g_N(W,U)\} \\ &\quad + \frac{c-1}{4}\{g_N(\phi W,U)g_N(\phi V,S) - g_N(\phi V,U)g_N(\phi W,S) \\ &\quad - 2g_N(\phi V,W)g_N(\phi U,S)\} \\ &\quad + g_N(T_W U, T_V S) - g_N(T_V U, T_W S) \qquad (6.2)\end{aligned}$$

and

$$\begin{aligned}\widehat{K}(V,W) &= \frac{c+3}{4}\{g_N(V,W)^2 - 1\} + \frac{c-1}{4}3g(V,\phi W) \\ &\quad + g_N(T_W V, T_V W) - g_N(T_V V, T_W W) \qquad (6.3)\end{aligned}$$

for any $V, W, U, S \in \Gamma(D_\perp)$.

Proof. For $V, W, U, S \in \Gamma(D_\perp)$, using (3.2), (6.1) and $\eta(V) = 0$, then we have

$$\begin{aligned}R(V,W,U,S) &= \frac{c+3}{4}\{g_N(W,S)g_N(V,U) - g_N(V,S)g_N(W,U)\} \\ &\quad + \frac{c-1}{4}\{g_N(\phi W,U)g_N(\phi V,S) - g_N(\phi V,U)g_N(\phi W,S) \\ &\quad - 2g_N(\phi V,W)g_N(\phi U,S)\}. \qquad (6.4)\end{aligned}$$

Therefore, from (2.12) and (6.4), we obtain (6.2).

Similarly, by elementary calculations in (6.2), we obtain (6.3). □

Theorem 6.2. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^\perp$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. If $D_\perp$ is totally geodesic, the distribution $D_\perp$ is Einstein.*

Proof. For any $V, W \in \Gamma(D_{\perp})$, we recall

$$\widehat{S}_{\perp}(V,W) = \sum_{i=1}^{2q} \widehat{R}(E_i, V, W, E_i)$$

where $\widehat{S}$ is Ricci tensor. Let $D_{\perp}$ is totally geodesic. Then, using (6.2), we have

$$\begin{aligned}\widehat{S}_{\perp}(V,W) \quad = \quad & \sum_{i=1}^{2q}\{\frac{c+3}{4}\{g_N(V,E_i)g_N(E_i,W) - g_N(E_i,E_i)g_N(V,W)\}\\ & +\frac{c-1}{4}\{g_N(\phi V,W)g_N(\phi E_i,E_i) - g_N(\phi E_i,W)g_N(\phi V,E_i)\\ & -2g_N(\phi E_i,V)g_N(\phi W,E_i)\}\}.\end{aligned}$$

Then, by elementary calculations, we get

$$\widehat{S}_{\perp}(V,W) = \frac{(c+3)(1-2q)+3(c-1)}{4} g_N(V,W) \tag{6.5}$$

which proves the assertion. □

Proposition 6.3. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. If the distribution $D_{\perp}$ is totally geodesic, then*

$$\widehat{\tau}_{\perp} = q\frac{(c+3)(1-2q)+3(c-1)}{2}$$

where $\widehat{\tau}_{\perp}$ is the scalar curvature of $D_{\perp}$.

Theorem 6.4. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. Then, we get*

$$\begin{aligned}\widehat{R}(K,L,M,N) \quad = \quad & \frac{c+3}{4}\{g_N(L,M)g_N(K,N) - g_N(K,M)g_N(L,N)\}\\ & +\frac{c-1}{4}\{g_N(\varphi L,M)g_N(\varphi K,N) - g_N(\varphi K,M)g_N(\varphi L,N)\\ & -2g_N(\varphi K,L)g_N(\varphi M,N)\}\\ & +g_N(T_L M, T_K N) - g_N(T_K M, T_L N)\end{aligned} \tag{6.6}$$

and

$$\begin{aligned}\widehat{K}(K,L) \quad = \quad & \frac{c+3}{4}\{g_N(K,L)^2 - 1\} - 3\frac{c-1}{4}g_N(\varphi K,L)^2\\ & +g_N(T_L K, T_K L) - g_N(T_K K, T_L L)\end{aligned} \tag{6.7}$$

for any $K, L, M, N \in \Gamma(D)$.

Proof. For $K, L, M, N \in \Gamma(D)$, using (2.12),(2.14),(6.1), we obtain equations (6.6) and (6.7). □

Theorem 6.5. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. If D is totally geodesic, the distribution D is Einstein.*

Proof. For any $K, L \in \Gamma(D)$, using (6.6), we get

$$\widehat{S}(K,L) = \frac{c+3}{4}(2p-1)g_N(K,L) + 3\frac{c-1}{4}g_N(K,L) \tag{6.8}$$

which gives proof. □

Proposition 6.6. *Let $\pi : (N(c), \varphi, \xi, \eta, g_N) \longrightarrow (B, g_B)$ be a generic $\xi^{\perp}$-Riemannian submersion such that N and B are Sasakian space form and Riemannian manifold, respectively. If the distirbution D is totally geodesic, then we get*

$$\widehat{\tau} = p\frac{(c+3)(2p-1)+3(c-1)}{2},$$

where $\widehat{\tau}$ is scalar curvature of D.

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