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# Calculation of the differential equations and harmonicity of the involute curve according to unit Darboux vector with a new method

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**Abstract.** In this study we first write the characterizations of involute of a curve by means of the unit Darboux vector of the involute curve. Then we make use of the Frenet formulas obtained by O. Çakır and S. Şenyurt to explain the characterizations of involute of a curve by means of Frenet apparatus of the main curve. Finally we examined the helix as an example.

## 1. Introduction and Preliminaries

To state a correlation between the invariants of a curve and characterizations of the curve in Euclidean space and non-Euclidean spaces and then to interpret it from the language of geometry has been the focus of interest for many researchers. Some curves are well-known by their explorers such as involute and evolute curves,[2]. Afterwards, many studies have been conducted in Euclidean and non-Euclidean spaces closely related to involute curves, [3, 4]. Later it has been revealed that curves can be classified, [5, 6, 8]. In this paper, we first take a regular curve, that is, a main curve, then write the characterizations of the involute curve by means of Frenet apparatus of the main curve. This work is one of the applications of [1] by which looking from such a point of view that we make the complex calculations more elementary. Eventually we put the example which support our assumption.
Now we may look at the main concepts related to the curve theory. Frenet vector fields can be expressed by means of covariant derivative of these vectors and this relation is known as Frenet formulas, see [9]

$$T' = \vartheta\kappa N, \qquad N' = -\vartheta\kappa T + \vartheta\tau B\,, \qquad B' = -\vartheta\tau N\,. \tag{1}$$

Frenet vectors $T\,,N\,,B$ form a Frenet frame and every Frenet frame moves along an instantaneous rotation axis which is called a Darboux vector and given by, see [9]

$$W = \tau T + \kappa B. \tag{2}$$

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When we denote the angle between $W$ and $B$ by $\phi$, the Darboux vector can be expressed as a unit Darboux vector $C$ given by, see [10]

$$C = sin\phi T + cos\phi B\,,\ \ sin\phi = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\ ,\ cos\phi = \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\,. \tag{3}$$

**Definition 1.1.** *Let $\alpha$ and $\beta$ be two differentiable curves. If the tangent vector of $\alpha$ is perpendicular to the tangent vector of $\beta$, then we call $\beta$ as the involute of $\alpha$. According to this definition, following parametrization can be given*

$$\beta(s) = \alpha(s) + \lambda(s)T(s)\,,\quad \lambda(s) = c - s\,,\quad c \in \mathbb{R}. \tag{4}$$

*When $\beta$ is the involute of $\alpha$, we have $d(\alpha(s), \beta(s)) =\mid c - s \mid$, $\forall s \in I$ and $c = const$. The relationship between the Frenet apparatus of the curves $\alpha$ and $\beta$ is given by*

$$T_\beta = N\,,\qquad N_\beta = \frac{-\kappa T + \tau B}{\sqrt{\kappa^2+\tau^2}}\,,\qquad B_\beta = \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}}\,,\qquad \kappa_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}\,,\qquad \tau_\beta = \frac{\kappa\tau' - \kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}\,. \tag{5}$$

By this definition, Darboux vector of the curve $\beta$ is given by, see [9]

$$W_\beta = \tau_\beta T_\beta + \kappa_\beta B_\beta. \tag{6}$$

There is still another way to express Darboux vector named as unit Darboux vector in [10]

$$C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta\,,\quad sin\phi_\beta = \frac{\tau_\beta}{\sqrt{\kappa_\beta^2+\tau_\beta^2}}\ ,\ cos\phi_\beta = \frac{\kappa_\beta}{\sqrt{\kappa_\beta^2+\tau_\beta^2}}\,. \tag{7}$$

with the angle $\phi_\beta$ between the vectors $W_\beta$ and $B_\beta$. It is also worth noting the relation here is that, see [11]

$$\begin{aligned} sin\phi_\beta &= \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\ ,\qquad cos\phi_\beta = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\ , \\ \phi_\beta' &= \Big(\frac{\phi'}{\sqrt{\phi'^2+\kappa^2+\tau^2}}\Big)'\,\frac{\sqrt{\phi'^2+\kappa^2+\tau^2}}{\sqrt{\kappa^2+\tau^2}}\,. \end{aligned} \tag{8}$$

This leads us the following relation, see [11]

$$C_\beta = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}N + \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}C\,. \tag{9}$$

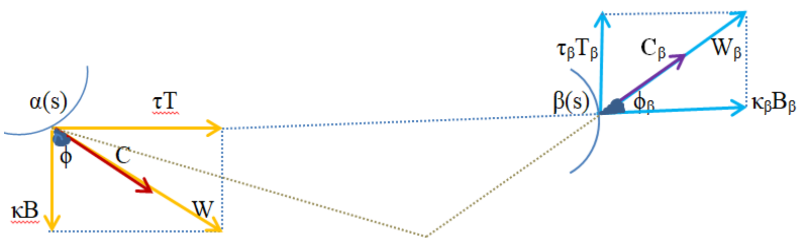


Figure 1: Unit Darboux vectors of the curves $\alpha$ and $\beta$.

**Definition 1.2.** *Let $\alpha$ be the unit speed curve, then the mean curvature vector field $H$ along the curve $\alpha$ is defined as,* see [7]

$$H = D_{\alpha'}\alpha' = \kappa N \tag{10}$$

where $D$ is the Levi-Civita connection. According to this definition the mapping

$$\Delta : \chi^{\perp}(\alpha(I)) \to \chi(\alpha(I))\,, \qquad \Delta H = -D_T^2 H \tag{11}$$

is called a Laplace operator. Let us denote the normal bundle of a curve $\alpha = \alpha(s)$ by $\chi^{\perp}(\alpha(s))$. Then the normal connection $D^{\perp}$ is given as

$$D_T^{\perp} : \chi^{\perp}(\alpha(I)) \to \chi^{\perp}(\alpha(I))\,, \quad D_T^{\perp}X = D_T X - \left\langle D_T X, T\right\rangle T \tag{12}$$

and the normal Laplace operator $\Delta^{\perp}$ is given by the following mapping

$$\Delta_T^{\perp}X = -D_T^{\perp}D_T^{\perp}X, \ \ \forall X \in \chi^{\perp}(\alpha(I)). \tag{13}$$

**Theorem 1.3.** *Let $\alpha$ be the unit speed curve and $H$, $W$ be the mean curvature and Darboux vector along the curve $\alpha$, respectively. Then we have the following propositions,* see [8]

*a*) $\Delta C = 0$ then $\alpha$ is a biharmonic curve.

*b*) $\Delta C = \mu C$, $\lambda, \mu \in \mathbb{R}$, then $\alpha$ is a 1-type harmonic curve.

*c*) $\Delta^{\perp}C^{\perp} = 0$ then $\alpha$ is a weak biharmonic curve.

*d*) $\Delta^{\perp}C^{\perp} = \mu C^{\perp}$, $\lambda, \mu \in \mathbb{R}$, then $\alpha$ is a 1-type harmonic curve.

**Theorem 1.4.** *Let $\alpha$ be a differentiable curve with unit Darboux vector $C$, then the differential equation characterizing $\alpha$ according to unit Darboux vector is given as,* see [8]

$$D_T^3 C + \lambda_1 D_T^2 C + \lambda_2 D_T C + \lambda_3 C = 0 \tag{14}$$

with the coefficients $\lambda_1$, $\lambda_2$, $\lambda_3$

$$\lambda_1 = -\left(\frac{\phi''}{\phi'} + \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel \phi'}\right), \qquad \lambda_2 = (\vartheta \parallel W \parallel)^2 + (\phi')^2 - \left(\frac{\phi''}{\phi'}\right)' + \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel (\phi')^2}\phi''\,,$$

$$\lambda_3 = ((\phi')^2)' - \frac{(\phi'\vartheta \parallel W \parallel)'}{\vartheta \parallel W \parallel}\phi'.$$

**Theorem 1.5.** *Let $\alpha$ be a differentiable curve with unit normal Darboux vector $C^{\perp}$, then the differential equation characterizing $\alpha$ according to unit normal Darboux vector is given as,* see [8]

$$\lambda_2 D_T^{\perp}D_T^{\perp}C^{\perp} + \lambda_1 D_T^{\perp}C^{\perp} + \lambda_0 C^{\perp} = 0 \tag{15}$$

with the coefficients $\lambda_0$, $\lambda_1$, $\lambda_2$

$\lambda_0 = \phi' sin\phi(\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)') + \vartheta\tau cos\phi(\vartheta^2\tau^2 cos\phi + (\phi' sin\phi)')\,,$

$\lambda_1 = cos\phi(\phi' sin\phi\vartheta\tau - (\vartheta\tau cos\phi)')\,,$

$\lambda_2 = \vartheta\tau cos^2\phi\ .$

**Theorem 1.6.** [1] Let $\beta$ be the involute of a unit speed curve $\alpha$. Then the Frenet formulas for the curve $\beta$ with respect to Levi-Civita connection $D$ and normal Levi-Civita connection $D^{\perp}$ are given, respectively, as

$$D_N T = \kappa N, \quad D_N N = -\kappa T + \tau B, \quad D_N B = -\tau N\,, \tag{16}$$

$$D^{\perp}_N T = 0, \quad D^{\perp}_N B = 0\,. \tag{17}$$

## 2. Calculation of the differential equations and harmonicity of the involute curve according to unit Darboux vector with a new method

When we say $\alpha$, unless we stated otherwise, we mean a unit speed curve in Euclidean 3-space with the Frenet apparatus of $T, N, B, \kappa, \tau$ and when we mention $\beta$, it stands for the involute of the curve $\alpha$ in the same space with the Frenet apparatus of $T_\beta, N_\beta, B_\beta, \kappa_\beta, \tau_\beta$ and $\vartheta = \| \frac{d}{ds}\beta(s) \|$. Throughout the paper we use $C$ to denote the unit Darboux vector of $\alpha$ and $C_\beta$ to express the unit Darboux vector of $\beta$ respectively.

**Theorem 2.1.** *Let $\beta$ be the involute of the curve $\alpha$. Then the differential equation with respect to connection characterizing the curve $\beta$ by means of the unit Darboux vector $C_\beta$ is given as*

$$D^3_{T_\beta} C_\beta + \mu_{\beta 1} D^2_{T_\beta} C_\beta + \mu_{\beta 2} D_{T_\beta} C_\beta + \mu_{\beta 3} C_\beta = 0 \tag{18}$$

*with the coefficients* $\mu_{\beta 1}$ , $\mu_{\beta 2}$ , $\mu_{\beta 3}$

$$\mu_{\beta 1} = -\Big(\frac{(\phi_\beta)''}{(\phi_\beta)'} + \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel (\phi_\beta)'}\Big), \quad \mu_{\beta 3} = (((\phi_\beta)')^2)' - \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel}(\phi_\beta)',$$

$$\mu_{\beta 2} = (\vartheta \parallel W_\beta \parallel)^2 + ((\phi_\beta)')^2 - \Big(\frac{(\phi_\beta)''}{(\phi_\beta)'}\Big)' + \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel ((\phi_\beta)')^2}(\phi_\beta)''\,.$$

*Proof.* *From equ.(3) we have* $C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta \cdot$ *Taking the derivative with respect to $T_\beta$ gives us*

$$D_{T_\beta} C_\beta = \phi'_\beta(cos\phi_\beta T_\beta - sin\phi_\beta B_\beta)\cdot \tag{19}$$

*From the equalities (3) and (19) we write the equivalents of $T_\beta$ and $B_\beta$ as,*

$$T_\beta = sin\phi_\beta C_\beta + \frac{cos\phi_\beta}{(\phi_\beta)'} D_{T_\beta} C_\beta\,,$$

$$B_\beta = cos\phi_\beta C_\beta - \frac{sin\phi_\beta}{(\phi_\beta)'} D_{T_\beta} C_\beta\,\cdot$$

*Second derivative of $C_\beta$ with respect to $T_\beta$ gives us*

$$D^2_{T_\beta} C_\beta = \frac{(\phi_\beta)''}{(\phi_\beta)'} D_{T_\beta} C_\beta - ((\phi_\beta)')^2 C_\beta + (\phi_\beta)'\vartheta \parallel W_\beta \parallel N_\beta\,\cdot$$

*From this equality we derive $N_\beta$ as,*

$$N_\beta = \frac{1}{\vartheta((\phi_\beta)')^2 \parallel W_\beta \parallel}((\phi_\beta)' D^2_{T_\beta} C_\beta - (\phi_\beta)'' D_{T_\beta} C_\beta + ((\phi_\beta)')^3 C_\beta)\,\cdot$$

*After third derivative of $C_\beta$ we find*

$$D^3_{T_\beta}C_\beta = (\frac{(\phi_\beta)''}{(\phi_\beta)'} + \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel (\phi_\beta)'})D^2_{T_\beta}C_\beta + ((\frac{(\phi_\beta)''}{(\phi_\beta)'})' - (\vartheta \parallel W_\beta \parallel)^2 - ((\phi_\beta)')^2 - \frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel ((\phi_\beta)')^2}(\phi_\beta)'')D_{T_\beta}C_\beta$$

$$+(\frac{((\phi_\beta)'\vartheta \parallel W_\beta \parallel)'}{\vartheta \parallel W_\beta \parallel}(\phi_\beta)' - (((\phi_\beta)')^2)')C_\beta \cdot$$

*It remains only to rearrange the above equality as a linear combinations of $D^3_{T_\beta}C_\beta$, $D^2_{T_\beta}C_\beta$, $D_{T_\beta}C_\beta$ and $C_\beta\cdot$ Then we obtain the required equation which completes the proof.* □

**Theorem 2.2.** *Let $\alpha$ be a differentiable curve with principal normal $N$, unit Darboux vector $C$ and $\beta$ be the involute of $\alpha$. Then the differential equation characterizing the curve $\beta$ with respect to connection is given as*

$$c_1D^3_NC + \left(3c_1' + \mu_1c_1\right)D^2_NC + \left(3c_1'' + 2\mu_1c_1' + \mu_2c_1\right)D_NC$$

$$+\left(c_1''' + \mu_1c_1'' + \mu_2c_1' + \mu_3c_1\right)C + c_2D^3_NN + \left(3c_2' + \mu_1c_2\right)D^2_NN$$

$$+\left(3c_2'' + 2\mu_1c_2' + \mu_2c_2\right)D_NN + \left(c_2''' + \mu_1c_2'' + \mu_2c_2' + \mu_3c_2\right)N = 0 \tag{20}$$

*with the coefficients* $c_1$, $c_2$, $\mu_1$, $\mu_2$, $\mu_3$

$$c_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\ , \qquad c_2 = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\ ,$$

$$\mu_1 = -\frac{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})''}{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'} - \frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\left(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'}\ ,$$

$$\mu_2 = (\phi')^2+\kappa^2+\tau^2 + ((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})')^2 - (\frac{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})''}{(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'})'$$

$$+\frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\right)^2}\cdot(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})''\ ,$$

$$\mu_3 = \left(((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})')^2\right)'$$

$$-\frac{\left((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\sqrt{(\phi')^2+\kappa^2+\tau^2}\right)'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}(arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'.$$

*Proof.* *We can compute the equivalents of coefficients $\mu_{\beta 1}$, $\mu_{\beta 2}$, $\mu_{\beta 3}$ and the angle $\phi_\beta$ in the equation (18) by taking equations (5), (8) and (9) into consideration as $\mu_1$, $\mu_2$, $\mu_3$ and the angle $\phi$. It follows from the equ.(9) we have*

$$c_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \quad c_2 = \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}.$$

*Making use of the equalities (5), (8) and (9) again, we can write the equivalents of coefficients $\mu_{\beta 1}$, $\mu_{\beta 2}$, $\mu_{\beta 3}$ and the Darboux vector $W_\beta$ as*

$$W_\beta = \frac{\sin\phi\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}T + \frac{\kappa\tau'-\kappa'\tau}{\lambda\kappa(\kappa^2+\tau^2)}N + \frac{\cos\phi\sqrt{\kappa^2+\tau^2}}{\lambda\kappa}B.$$

*By referring the equalities (8) and (14) we can write that*

$$C_\beta = \frac{1}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}(\sin\phi\sqrt{\kappa^2+\tau^2}\,T + \phi' N + \cos\phi\sqrt{\kappa^2+\tau^2}\,B).$$

*Applying the equ.(16) we may write the counterparts of $D_{T_\beta}C_\beta$, $D^2_{T_\beta}C_\beta$, $D^3_{T_\beta}C_\beta$ as in the following form*

$$\begin{aligned}
D_{T_\beta}C_\beta &= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_N C + \left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D_N N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' N, \\
D^2_{T_\beta}C_\beta &= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^2_N C + 2\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N C \\
&+\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^2_N N \\
&+2\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D_N N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' N, \qquad (21) \\
D^3_{T_\beta}C_\beta &= \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^3_N C + 3\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D^2_N C \\
&+3\left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N C + \left(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)''' C \\
&+\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^3_N N + 3\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)' D^2_N N \\
&+3\left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)'' D_N N + \left(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\right)''' N.
\end{aligned}$$

*Finally setting the equivalents of coefficients and derivatives with respect to N into the first equation we get desired result which completes the proof.* □

**Theorem 2.3.** *Let β be the involute of the curve α. Then the differential equation with respect to normal connection characterizing the curve β by means of the unit Darboux vector $C_\beta^\perp$ is given as*

$$\lambda_{\beta 2} D_{T_\beta}^\perp D_{T_\beta}^\perp C_\beta^\perp + \lambda_{\beta 1} D_{T_\beta}^\perp C_\beta^\perp + \lambda_{\beta 0} C_\beta^\perp = 0 \tag{22}$$

*with the coefficients* $\lambda_{\beta 0}$ , $\lambda_{\beta 1}$ , $\lambda_{\beta 2}$

$\lambda_{\beta 2} = \vartheta\tau_\beta cos^2\phi_\beta, \quad \lambda_{\beta 1} = cos\phi_\beta\Big(\phi_\beta' sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big),$

$\lambda_{\beta 0} = \phi_\beta' sin\phi_\beta\Big(\phi_\beta' sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big) + \vartheta\tau_\beta cos\phi_\beta\Big(\vartheta^2(\tau_\beta)^2 cos\phi_\beta + (\phi_\beta' sin\phi_\beta)'\Big).$

*Proof.* *From equ. (13) we write the normal component of* $C_\beta$ *as*

$$C_\beta^\perp = cos\phi_\beta B_\beta \cdot \tag{23}$$

*Taking the first and second derivatives of this equality with respect to normal connection gives us,*

$$D_{T_\beta}^\perp C_\beta^\perp = -\vartheta\tau_\beta cos\phi_\beta N_\beta - \phi_\beta' sin\phi_\beta B_\beta, \tag{24}$$

$$D_{T_\beta}^\perp D_{T_\beta}^\perp C_\beta^\perp = \Big(\phi_\beta' sin\phi_\beta \vartheta\tau_\beta - (\vartheta\tau_\beta cos\phi_\beta)'\Big)N_\beta - \Big(\vartheta^2(\tau_\beta)^2 cos\phi_\beta + (\phi_\beta' sin\phi_\beta)'\Big)B_\beta. \tag{25}$$

*If we extract the vectors* $N_\beta$ *and* $B_\beta$ *from equ.(23) , (24) we have*

$$B_\beta = \frac{1}{cos\phi_\beta} C_\beta^\perp ,$$

$$N_\beta = \frac{-1}{\vartheta\tau_\beta cos\phi_\beta} D_{T_\beta}^\perp C_\beta^\perp - \frac{\phi_\beta' sin\phi_\beta}{\vartheta\tau_\beta cos^2\phi_\beta} C_\beta^\perp \cdot$$

*Putting the equivalents of* $B_\beta$ *and* $N_\beta$ *into the equ.(25) we obtain the desired equation which completes the proof.* □

**Theorem 2.4.** *Let α be a differentiable curve with principal normal N, unit Darboux vector C and β be the involute of α. Then the differential equation characterizing the curve β with respect to normal connection is given as*

$$\Big(\rho\lambda_2\Big) D_N^\perp D_N^\perp C + \Big(2\rho'\lambda_2 + \rho\lambda_1\Big) D_N^\perp C + \Big(\rho''\lambda_2 + \rho'\lambda_1 + \rho\lambda_0\Big) C = 0 \tag{26}$$

*with the coefficients* $\rho,\ \lambda_0,\ \lambda_1,\ \lambda_2$

$$\rho = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}, \quad \lambda_2 = \frac{\kappa\tau'-\kappa'\tau}{(\phi')^2+\kappa^2+\tau^2},$$

$$\lambda_1 = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\cdot$$

$$\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}-(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}})'\Big),$$

$$\lambda_0 = (arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\cdot$$

$$\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\frac{\kappa\tau'-\kappa'\tau}{\kappa^2+\tau^2}$$

$$-(\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}})'\Big)$$

$$+\frac{\kappa\tau'-\kappa'\tau}{\sqrt{((\phi')^2+\kappa^2+\tau^2)(\kappa^2+\tau^2)}}\Big((\frac{\kappa\tau'-\kappa'\tau}{(\kappa^2+\tau^2)})^2\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}$$

$$+\Big((arcsin\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}})'\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'\Big).$$

*Proof. From equ.(3) we have* $cos\phi = \kappa/\sqrt{\kappa^2+\tau^2}$ *and* $sin\phi = \tau/\sqrt{\kappa^2+\tau^2}$ *it follows from the equalities (8) and (14)*

*we figure out that* $sin\phi_\beta = \phi'/\sqrt{(\phi')^2+\kappa^2+\tau^2}$ , $cos\phi_\beta = \sqrt{\kappa^2+\tau^2}/\sqrt{(\phi')^2+\kappa^2+\tau^2}$. *Then we get,*

$$C_\beta^\perp = \frac{\tau}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}T + \frac{\kappa}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}B.$$

*On the other hand we can evaluate the equivalents of coefficients of the equation (22) by using the equalities (5), (8) and (17) as* $\lambda_0,\ \lambda_1,\ \lambda_2$. *By the same way we can make use of the equalities (5), (8) and (17)again, in order to write* *the equivalents of derivatives of* $D^{\perp}_{T_\beta}C^{\perp}_{\beta}$ *and* $D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta}$ *with respect to N. It follows that*

$$D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'C,$$

$$D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}D^{\perp}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C. \quad (27)$$

*Setting the equivalents of coefficients of the equation with the aid of equ.(5) and then the derivatives with respect to N into the equation above we get desired result which completes the proof.* □

**Theorem 2.5.** *Let* $\beta$ *be the involute of a differentiable curve* $\alpha$ *with the unit Darboux vector* $C_\beta$*. According to connection, harmonicity (biharmonic or 1-type harmonic) of the curve* $\beta$ *may not be expressed by means of the Frenet apparatus of the main curve* $\alpha$*.*

*Proof.* *From equ.(21), it is obvious that we have the following*

$$D^{2}_{T_\beta}C_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{2}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D_{N}C$$

$$+\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C + \frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{2}_{N}N$$

$$+2\Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D_{N}N + \Big(\frac{\phi'}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''N\cdot$$

*Considering the case* $\Delta C_\beta = 0$ *or* $\Delta C_\beta = \lambda C_\beta$*, from Theorem 1.3 of a and b we get* $D_N N = 0$ *and* $D_N C = 0$*.*
*Hence we cannot decide whether the curve* $\beta$ *is biharmonic or 1-type harmonic.* □

**Theorem 2.6.** *Let* $\beta$ *be the involute of a differentiable curve* $\alpha$ *with the normal Darboux vector* $C^{\perp}_{\beta}$*. According to normal connection, harmonicity (weak biharmonic or 1-type harmonic) of the curve* $\beta$ *may not be expressed by means of the Frenet apparatus of the main curve* $\alpha$*.*

*Proof.* *From equ.(27), it is clear that we have the following*

$$D^{\perp}_{T_\beta}D^{\perp}_{T_\beta}C^{\perp}_{\beta} = \frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}D^{\perp}_{N}D^{\perp}_{N}C + 2\Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)'D^{\perp}_{N}C + \Big(\frac{\sqrt{\kappa^2+\tau^2}}{\sqrt{(\phi')^2+\kappa^2+\tau^2}}\Big)''C\cdot$$

*Considering the case* $\Delta C^{\perp}_{\beta} = 0$ *or* $\Delta C^{\perp}_{\beta} = \lambda C^{\perp}_{\beta}$*, from Theorem 1.3 of c and d we get* $D_N C = 0$*.*
*Hence we cannot decide whether the curve* $\beta$ *is weak biharmonic or 1-type harmonic.* □

**Example 2.7.** *Let a curve* $\alpha(s) = \frac{1}{\sqrt{2}}(coss, sins, s)$ *be given. Then we have an involute of* $\alpha$*, that is, curve* $\beta$*,*

$\beta(s) = \frac{1}{\sqrt{2}}(coss - (c-s)sins,\ sins + (c-s)coss,\ c)$, $c \in \mathbb{R}$. *It follows that* $C_\beta = sin\phi_\beta T_\beta + cos\phi_\beta B_\beta$ *with*

$sin\phi_\beta = 0$, $cos\phi_\beta = 1$. *By the equ.(9) also we get* $B_\beta = C$*. Hence we obtain,* $D_N C = 0$ *and* $D^{\perp}_{N}C = 0$.

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