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Application of Legendre Polynomial Basis Function on the Solution of Volterra Integro-Differential Equations Using Collocation Method

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Abstract: In this paper, we presented an efficient numerical method of solving Volterra integro-differential equations by applying Legendre as basis function for the solution of initial value problem of Integro-differential equations. We assumed appropriate solutions in terms of Legendre polynomial as basis function which was substituted into the class of integro-differential equations considered. This transformed the integro-differential equations and the given initial conditions into matrix equations. By collocating at point $x = x_k$ corresponding to N- systems of equations, the results obtained for some numerical examples justified the efficiency and reliability of the proposed method.

Keywords: Collocation method, Legendre polynomial basis function, Volterra Integro-differential equations.

1. Introduction

Volterra integro differential equations have been the focus of many studies due to their frequent appearance in various applications, such as in fluid mechanics and viscoelasticity. Volterra integral equations in the order hand, arise in engineering, physics, chemistry and biological problems such as parabolic boundary value problems, population dynamics and semi-conductor device. Many initial and boundary value problems associated with the ordinary and partial differential equations can be model into the Volterra integral equation types.

This equation was first used by [1], who presented the theory of functional of integral and integro-differential equations. The Volterra integral equation of second kind is of the form:

$$u(x) = f(x) + \lambda \int_0^x k(x,t)u(t)dt \tag{1}$$

where $k(x, t)$ is the kernel of the integral equation and λ is a parameter.

A variety of analytic and numerical methods have been used to solve Volterra integral equations. For Example, the Taylor series expansion method is used for the second kind Volterra integral equation as presented in [2]. In [3-4], application of collocation method on Volterra integral equations were investigated. Chebyshev polynomials was used to find numerical solutions of nonlinear Volterra integral equations of the second kind in [5]. Numerical solution of the second kind Volterra integral equation using an expansion method is found in [6]. A new approach to solve Volterra integral equation by using Bernstein's approximation is employed in [7]. The application of Adomian decomposition method to solve integral equations was also presented in [8-9].

The Lagrange interpolation method is applied to solve the integro differential equation in [10]. In [11], the application of Adomian's decomposition method on the Integro-differential equation is investigated. In [12], Taylor polynomials was used to solve high-order Volterra integro-differential equation. In [13], the rationalized Haar functions method was applied to the system of linear integro-differential equations. In [14], the authors investigated the integro-differential equation by using the differential transform method. The solution of the fourth-order integro-differential equation using the variational iteration method can be found in [15].

In [16], numerical solutions of a class of integro-differential equations ware presented. Variation Iteration method was used to solve Emden-Fowler type equation in [17]. Also, [18] presented the numerical treatment of differential equations using collocation methods for ordinary differential equations and illustrated its application with some numerical examples. This particular method has brought into literature terminologies like; Spilt-range collocation methods, exponentially fitted collocation methods Segmented Domain collocation method [19]. Integral collocation method was presented in [20].

Legendre polynomial is an important orthogonal polynomial with interval of orthogonality between -1 and 1, and also is considered as the eigen-functions of singular sturm-Liouville. Mathematically, Legendre polynomials are solutions to Legendre's differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}Pn(x)\right]\lambda Pn(x)=0 \tag{2}$$

Where the eigenvalue $\lambda = n(n+1)$ .The recurrence relation of Legendre polynomial is

$$(n+1)Pn+1(x)=(2x+1)xPn(x)nPn-1(x); \quad n\geq 1 \tag{3}$$

Collocation is a method for evaluating a given differential equation after assuming an approximate solution which is then substituted back into the given problem and collocated at some equally spaced interior points in order to nullify the unknown constants in the assumed solution.

2. Review of Legendre and shifted Legendre polynomials

The legendre polynomials Lm(x); m=0,1,2,…, are eigen functions of the singular sturm-Liouville problem

$$((1-x^2)L'm(t))'m(m+1)Lm(t)=0,\ x\in[-1,1] \tag{4}$$

The Legendre polynomials satisfy the recursion relation:

$$L_{m+1}(t)=\frac{2x+1}{m+1}xL_m(t)-\frac{m}{m+1}L_{m-1}(t),\ \text{m=1,2,…} \tag{5}$$

where L_0(t)=1 and L_1(t)= x which are thus generated by the Legendre relation

$$L_m(t)=\frac{1}{2^m m!}\ \frac{d^m}{dx^m}(x^2-1)^m;\ \text{m=0,1,2,…,} \tag{6}$$

In order to use these polynomials on the interval $[0,1]$, we define the so called shifted Legendre polynomials by introducing the change of variable $t=2x-1$. Let the shifted Legendre polynomials $L_m(2x-1)$ be denoted by $L^*{}_m(x)$. Then $L^*{}_m(x)$ can be obtained as follows

$$L^*{}_{m+1}(x)=\frac{(2m+1)(2x-1)}{m+1}L^*{}_m(x)-\frac{m}{m+1}L^*{}_{m-1}(x), \tag{7}$$

where

$$L^*{}_0(x)=1$$

$$L^*{}_1(x)=2x-1$$

$$L^*_2(x) = 6x^2 - 6x + 1$$

$$L^*_3(x) = 20x^3 - 30x^2 + 12x - 1$$

$$L^*_4(x) = 70x^4 - 140x^3 + 90x^2 - 20x + 1$$

$$L^*_5(x) = 252x^5 - 630x^4 + 560x^3 - 210x^2 + 30x - 1$$

The analytical form of the legandre polynomial $L^*_m(x)$ of degree m is given by:

$$L^*_m(x) = \sum_{i=0}^{m} (-1)^{m+i} \frac{(m+i)!}{(m-i)!(i!)^2} x^i \text{ , m = 2,3,...} \tag{8}$$

Note that $L^*_m(0) = (-1)^m$ and $L^*_m(0) = (-1)^m$. The orthogonality condition is

$$\int_0^1 L^*_i(x) L^*_j(x)dx = \begin{cases} \frac{1}{2i+1}, & for\ i = j, \\ 0, & for\ i \neq j, \end{cases} \tag{9}$$

To consider the differential equation of n^{th} order, integration collocation method using truncated Legendre series of degree k to represent the n^{th} derivative of the unknown functions $u(x)$ in the following manner.

$$\frac{d^n u(x)}{dx^n} \equiv \sum_{m=0}^{k} a_m L^*_m(x) = \sum_{m=0}^{k} a_m I^{(n)}_m(x) \tag{10}$$

where $L_m(2x-1) = L^*_m(x) = I^n_m(x)$ is the shifted Legendre polynomial using the integration, we can obtain the lower order derivatives and the function itself as follows:

$$\frac{d^{n-1} u(x)}{dx^{n-1}} \equiv \sum_{m=0}^{k} a_m I_m^{(n-1)}(x) + c_1 \tag{11}$$

$$\frac{d^{n-2} u(x)}{dx^{n-2}} \equiv \sum_{m=0}^{k} a_m I_m^{(n-2)}(x) + c_1 x + c_2 \ldots \tag{12}$$

$$\frac{du(x)}{dx} = \sum_{m=0}^{k} a_m I_m^{(1)}(x) + c_1 \frac{x^{n-2}}{(n-2)!} + c_2 \frac{x^{n-3}}{(n-3)!} + \ldots + c_{n-2} x + c_{n-1,} \tag{13}$$

$$u(x) = \sum_{m=0}^{k} a_m I_m{}^{(0)}(x) + c_1 \frac{x^{n-1}}{(n-1)!} + c_2 \frac{x^{n-2}}{(n-2)!} + \ldots c_{n-1}x + c_n, \tag{14}$$

From equations (8) and (10) we have:

$$I_m{}^{(n)}(x) = \sum_{i=0}^{m} \frac{(-1)^{m+i}(m+i)!}{(k-i)!(i!)^2} x^i \tag{15}$$

$$I_m{}^{(n-1)}(x) = \int I_m{}^{(n)}(x)dx = \sum_{i=0}^{m} \frac{(-1)^{m+i}(m+i)!}{(m-i)!(i!)^2(i+1)} x^{i+1} \tag{16}$$

$$I_m{}^{(n-2)}(x) = \int I_m{}^{(n-1)}(x)dx = \sum_{i=0}^{m} \frac{(-1)^{m+i}(m+i)!}{(m-i)!(i!)^2(i+1)(i+2)} x^{i+2} \tag{17}$$

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$$I_m{}^{(0)}(x) = \int I_m{}^{(1)}(x)dx = \sum_{i=0}^{m} \frac{(-1)^{m+i}(m+i)!}{(m-i)!(i!)^2(i+1)\ldots(i+n-1)(i+n)} x^{i+n} \tag{18}$$

Hence, we collocate equations (10) and (14) at (k+1) points x^p, m=0,1,…,m as

$$\frac{d^n u(x_p)}{dx^n} = \Gamma^{(n)}S, \ \frac{d^{n-1}u(x^p)}{dx^{n-1}} = \Gamma^{(n-1)}S \tag{19}$$

$$\frac{du(x^p)}{dx} = \Gamma^{(1)}S, \ u(x^p) = \Gamma^{(0)}S \tag{20}$$

Where $S = [a_0, a, \ldots a_m, c_1, c_2, \ldots, c_n]^T$ and $\Gamma^n, \Gamma^{n-1}, \ldots, \Gamma^0$ are integrated matrices.

3. Construction of the Method

This section, we discussed the Numerical application of Legendre polynomial basis function on the solution of Volterra Integro-Differential equations using Collocation Method.

Consider the general Volterra Integro-differential equation of the form:

$$y^n(x) = \int_0^x (x-t)y(x)dt = f(x) \tag{21}$$

With the initial condition:

$$y_k(0) = \phi_k \tag{22}$$

We assumed an approximate solution of the form:

$$y(x) = y_N(x) = \sum_{i=0}^{N} a_i L_i(x) \tag{23}$$

Where $a_i, i = 0(1)N$ are unknown constants to be determined and $L_i(x)$ is the Shifted Legendre Polynomial Basis function. Differentiating equation (17) n-times to obtain:

$$y'_N(x) = \sum_{i=0}^{N} a_i L_i(x)$$

$$y''_N(x) = \sum_{i=0}^{N} a_i L_i(x) \tag{24}$$

$$y'''_N(x) = \sum_{i=0}^{N} a_i L_i(x)$$

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$$y^n{}_N(x) = \sum_{i=0}^{N} a_i L_i(x)$$

Substitute the assumed approximate solution equation (24) into equation (21) to obtain:

$$\sum_{i=0}^{N} a_i L_i(x) - \int_0^x (x-t) \sum_{i=0}^{N} a_i L_i(x)dt = f(x) \tag{25}$$

Equation (24) is further simplified to give rise to $N - n$ linear algebraic systems of equations in $N - n + 1$ unknown constants. An extra equations are obtained from the initial conditions. Altogether we have $N + 1$ system of equations in $N + 1$ unknowns.

The values of the unknown constants are now substituted into the assumed approximate solution given in equation (22) to give rise to the required approximate solution.

4. Numerical Applications

Example 1: Consider the third order Volterra Integro-differential equation.

$$u'''(x) = 1 + x + \frac{x^3}{6} + \int_0^x (x-t)u(t)dt \tag{26}$$

With the initial condition,

$$u(0) = 1,\ u'(0) = 0, u''(0) = 1 \tag{27}$$

and the exact solution given by $u(x) = e^x - x$

TABLE 1. Table presenting the exact and approximate solutions of the given Example 1

X	Exact	Approx. N=2	Error N=2	Approx. N=3	Error N=3	Approx. N=4	Error N=4	Approx. N=5	Error N=5
0.0	1.00000	1.00000	0	0.99999	5.0000e-10	0.99999	2.0000e-10	0.99999	1.0000e-10
0.1	1.00517	1.00500	1.7092e-04	1.00553	3.5938e-04	1.00506	1.0532e-04	1.00520	3.2687e-05
0.2	1.02140	1.02000	1.4028e-03	1.02424	2.8397e-03	1.02061	7.8887e-04	1.02163	2.3002e-04
0.3	1.04985	1.04500	4.8588e-03	1.05930	9.4594e-03	1.04737	2.4862e-03	1.05054	6.8239e-04
0.4	1.09182	1.08000	1.1825e-02	1.11393	2.2115e-02	1.08633	5.4881e-03	1.09324	1.4235e-03
0.5	1.14872	1.12500	2.3721e-02	1.19128	4.2567e-02	1.13877	9.9530e-03	1.15117	2.4559e-03
0.6	1.22211	1.18000	4.2119e-02	1.29454	7.2427e-02	1.20619	1.5922e-02	1.22589	3.7752e-03
0.7	1.31375	1.24500	6.8753e-02	1.42689	1.1314e-01	1.29041	2.3333e-02	1.31914	5.3929e-03
0.8	1.42554	1.32000	1.0554e-01	1.59151	1.6597e-01	1.39350	3.2040e-02	1.43289	7.3753e-03
0.9	1.55960	1.40500	1.5460e-01	1.79159	2.3199e-01	1.51777	4.1832e-02	1.56937	9.7731e-03
1.0	1.71828	1.50000	2.1828e-01	2.03030	3.1202e-01	1.66582	5.2452e-02	1.73109	1.2818e-02

Example 2: Consider the fourth order Volterra Integro-differential difference equation of the form:

$$u^{iv}(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \int_0^x (x-t)u(t)dt \tag{28}$$

With the initial condition,

$$u(0) = 2,\ u'(0) = 2,\ \ u''(0) = 1,\ u'''(0) = 1. \tag{29}$$

And the exact solution given by $u(x) = e^x + x + 1$

TABLE 2. Table presents the exact and the approximate solutions of the given Example 2.

X	**Exact**	**Approx. N=3**	**Error N=3**	**Approx. N=4**	**Error N=4**	**Approx. N=5**	**Error N=5**
0.0	2.0000000	2.0000000	0	2.0000000	0	1.9999999	1.0000e-09
0.1	2.2051709	2.2051666	4.2513e-06	2.2051822	1.1352e-05	2.2051660	4.6971e-06
0.2	2.4214027	2.4213333	6.9425e-05	2.4215829	1.8022e-04	2.4213310	7.1392e-05
0.3	2.6498580	2.6495000	3.5881e-04	2.6507638	9.0502e-04	2.6495161	3.4269e-04
0.4	2.8918246	2.8906666	1.1580e-03	2.8946609	2.8363e-03	2.8908000	1.0247e-03
0.5	3.1487212	3.1458333	2.8879e-03	3.1555851	6.8638e-03	3.1463601	2.3611e-03
0.6	3.4221188	3.4160000	6.1188e-03	3.4362212	1.4102e-02	3.4175095	4.6092e-03
0.7	3.7137527	3.7021660	1.1586e-02	3.7396290	2.5876e-02	3.7057362	8.0165e-03
0.8	4.0255400	4.0053333	2.0208e-02	4.0692425	4.3702e-02	4.0127413	1.2800e-02
0.9	4.3596031	4.3265000	3.3103e-02	4.4288702	6.9267e-02	4.3404780	1.9125e-02
1.0	4.7182818	4.6666666	5.1615e-02	4.8226950	1.0441e-01	4.6911908	2.7091e-02

Example 3: Consider the second order Volterra Integro-differential difference equation [9].

$$u''(x) = -x - \frac{x^3}{6} + \int_0^x (x-t)u(t)dt, \tag{30}$$

With the initial condition,

$$u(0) = 0, u'(0) = 2 \tag{31}$$

And the exact solution given by $u(x) = x + \sin(x)$

TABLE 3. Table presents the exact and the approximate solutions of the given Example 3.

x	Exact	Approx N=4	Error N=4	Approx N=5	Error N=5	Approx N=7	Error N=7	Approx N=10	Error N=10
0.0	0.00000	0.00000	4.2000e-11	0	0	0	0	0	0
0.1	0.199833	0.18852	1.1305e-02	0.194060	5.7641e-03	0.19980	1.5841e-07	0.19983	3.3315e-07
0.2	0.39866	0.35400	4.4645e-02	0.376239	2.2430e-02	0.39867	1.0211e-05	0.39867	1.0644e-05
0.3	0.59500	0.49600	9.9220e-02	0.546300	4.9149e-02	0.59560	7.9922e-05	0.59560	8.0615e-05
0.4	0.78940	0.61510	1.7431e-01	0.704200	8.5189e-02	0.78970	3.3753e-04	0.78970	3.3848e-04
0.5	0.97940	0.71010	2.6931e-01	0.849000	1.2993e-01	0.98040	1.0270e-03	0.98040	1.0283e-03
0.6	1.16460	0.78090	3.8369e-01	0.981700	1.8286e-01	1.16718	2.5434e-03	1.16710	2.5449e-03
0.7	1.34420	0.82710	5.1708e-01	1.100600	2.4356e-01	1.34960	5.4654e-03	1.34960	5.4667e-03
0.8	1.51735	0.84810	6.6918e-01	1.205600	3.1171e-01	1.52790	1.0586e-02	1.52790	1.0586e-02
0.9	1.68330	0.84340	8.3987e-01	1.296200	3.8708e-01	1.70227	1.8943e-02	1.70226	1.8937e-02
1.0	1.84147	0.81230	1.0291e+00	1.371900	4.6949e-01	1.87330	3.1841e-02	1.87320	3.1826e-02

5. Conclusion

The Legendre Polynomial have been employed successfully in solving Volterra integro-differential equations. The solution obtained by the means of the basis function used yielded the desired accuracy when compared with the exact solution. The simplicity is an added advantage to the method and hence it is reliable and powerful numerical tools for the class of the problem considered.

Authorship contribution statement

M. O. Olayiwola: Supervision, Conceptualization, Methodology, Reviewing and Editing, **A. F. Adebisi:** Investigation, Visualization, **Y. S. Arowolo:** Investigation, Typing, Software.

Declaration of Competing Interest

The authors declare that there is no competing financial interests or personal relationships that influence the work in this paper.

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