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4-Dimensional Euler-Totient Matrix Operator and Some Double Sequence Spaces

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Abstract
Our main purpose in this study is to investigate the matrix domains of the 4-dimensional Euler-totient matrix operator on the classical double sequence spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$. Besides these, we examine their topological and algebraic properties and give inclusion relations about the new spaces. Also, the $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals of these spaces are determined and finally, some matrix classes are characterized.

Keywords: *Euler function, Möbius function, 4-dimensional Euler-totient matrix operator, matrix domain, double sequence space, α-, $\beta(\vartheta)$- and γ-duals, matrix transformations.*

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1. Preliminaries, Background and Notations

The function f described by $f : \mathbb{N} \times \mathbb{N} \to \wp$, $(t, u) \mapsto f(t, u) = x_{tu}$ is entitled as *double sequence*, where $\wp$ denotes any nonempty set and $\mathbb{N} = \{1, 2, ...\}$. Ω stands for the set of all complex valued double sequences. It is well known that this set is a vector space with coordinatewise addition and scalar multiplication. Any linear subspace of Ω is called as *double sequence space*. The set of all bounded complex valued double sequences is symbolized with $\mathcal{M}_u$, that is,

$$\mathcal{M}_u = \left\{ x = (x_{tu}) \in \Omega : \|x\|_\infty = \sup_{t,u\in\mathbb{N}} |x_{tu}| < \infty \right\}.$$

It should be noted that $\mathcal{M}_u$ is a Banach space with the norm $\|x\|_\infty$. We say that the double sequence $x = (x_{tu})$ is *convergent* in the *Pringsheim's sense* provided that for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $|x_{tu} - L| < \varepsilon$ whenever $t, u > n_\varepsilon$. In that case, $L \in \mathbb{C}$ is called the *Pringsheim limit* of x and stated by $p - \lim_{t,u\to\infty} x_{tu} = L$; where $\mathbb{C}$ denotes the complex field. $\mathcal{C}_p$ represents the space of all such x which are called shortly as *p-convergent*. Of particular interest is unlike single sequences, *p-convergent* double sequences need not be bounded. For example, if we consider the sequence $x = (x_{tu})$ identified by

$$x_{tu} = \begin{pmatrix} 1 & 2 & 3 & \cdots & u & \cdots \\ 2 & 0 & 0 & \cdots & 0 & \cdots \\ 3 & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \\ t & 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \ldots & \vdots & \ldots \end{pmatrix},$$

it can easily seen that $p - \lim x_{tu} = 0$ but $\|x\|_\infty = \infty$. As a conclusion $x \in \mathcal{C}_p - \mathcal{M}_u$. The bounded sequences which are also p-convergent are indicated by $\mathcal{C}_{bp}$, that is, $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A double sequence $x = (x_{tu}) \in \mathcal{C}_p$ is called as *regularly convergent* if the limits $x_t := \lim_u x_{tu}$, $(t \in \mathbb{N})$ and $x_u := \lim_t x_{tu}$, $(u \in \mathbb{N})$ exist, and the limits $\lim_t \lim_u x_{tu}$

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and $\lim_u \lim_t x_{tu}$ exist and are equivalent to the $p-\lim$ of x. The space of all regularly convergent double sequences is denoted by $\mathcal{C}_r$. Obviously, the regular convergence of a double sequence x implies the convergence in Pringsheim's sense as well as the boundedness of the terms of x, but the converse implication fails. A sequence $x = (x_{tu})$ is called *double null sequence* if it converges to zero. Additionally, all double null sequences in the spaces $\mathcal{C}_{bp}$ and $\mathcal{C}_r$ are denoted by $\mathcal{C}_{bp0}$ and $\mathcal{C}_{r0}$, respectively. Móricz [25] showed that the spaces $\mathcal{C}_{bp}$, $\mathcal{C}_{bp0}$, $\mathcal{C}_r$ and $\mathcal{C}_{r0}$ are Banach spaces endowed with the norm $\|.\|_\infty$.

Let us take any $x \in \Omega$ and describe the sequence $K = (k_{rs})$ defined by

$$k_{rs} := \sum_{t=1}^{r} \sum_{u=1}^{s} x_{tu}, \quad (r, s \in \mathbb{N}).$$

In that case, the pair $((x_{rs}), (k_{rs}))$ is entitled as *double series*. Here, the sequence $K = (k_{rs})$ is the *sequence of partial sums* of the double series.

Consider the double sequence space Ψ converging with respect to some linear convergence rule $\vartheta - \lim : \Psi \to \mathbb{C}$. The sum of a double series $\sum_{t,u} x_{tu}$ relating to this rule is defined by $\vartheta - \sum_{t,u} x_{tu} = \vartheta - \lim_{r,s\to\infty} s_{rs}$. Here and thereafter, when needed we will use the summation $\sum_{t,u}$ instead of $\sum_{t=1}^{\infty}\sum_{u=1}^{\infty}$, assume that $\vartheta \in \{p, bp, r\}$ and p' denotes the conjugate of p, that is, $p' = p/(p-1)$ for $1 < p < \infty$. With the notation of Zeltser [42], we describe the double sequences $e^{rs} = (e^{rs}_{tu})$ and e by $e^{rs}_{tu} = 1$ if $(r, s) = (t, u)$ and $e^{r,s}_{t,u} = 0$ otherwise, and $e = \sum_{r,s} e^{r,s}$ (coordinatewise convergence) for every $r, s, t, u \in \mathbb{N}$.

The $\alpha-$*dual* Ψ^α, $\beta(\vartheta)-$*dual* $\Psi^{\beta(\vartheta)}$ with respect to the $\vartheta-$convergence and the $\gamma-$*dual* Ψ^γ of a double sequence space Ψ are described as

$$\begin{aligned}
\Psi^\alpha &:= \left\{ c = (c_{tu}) \in \Omega : \sum_{t,u} |c_{tu}x_{tu}| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi \right\}, \\
\Psi^{\beta(\vartheta)} &:= \left\{ c = (c_{tu}) \in \Omega : \vartheta - \sum_{t,u} c_{tu}x_{tu} \quad \text{exists for all} \quad (x_{tu}) \in \Psi \right\}, \\
\Psi^\gamma &:= \left\{ c = (c_{tu}) \in \Omega : \sup_{r,s\in\mathbb{N}} \left| \sum_{t,u=1}^{r,s} c_{tu}x_{tu} \right| < \infty \quad \text{for all} \quad (x_{tu}) \in \Psi \right\},
\end{aligned}$$

respectively. It is well known that $\Psi^\alpha \subset \Psi^\gamma$ and if $\Psi \subset \Lambda$, then $\Lambda^\alpha \subset \Psi^\alpha$ for the double sequence spaces Ψ and Λ.

Let us remember the definition of triangle matrix. If $b_{rstu} = 0$ for $t > r$ or $u > s$ or both for every $r, s, t, u \in \mathbb{N}$, it is said that $B = (b_{rstu})$ is a *triangular matrix* and also if $b_{rsrs} \neq 0$ for every $r, s \in \mathbb{N}$, then the 4-dimensional matrix B is called *triangle*. It should be noted by [11] that, every triangle has a unique inverse which is also a triangle.

Now, we shall deal with matrix mapping. Let us consider double sequence spaces Ψ and Λ and the 4-dimensional complex infinite matrix $B = (b_{rstu})$. In that case, we say that B defines a *matrix mapping* from Ψ into Λ and it is written as $B : \Psi \to \Lambda$, if for every sequence $x = (x_{tu}) \in \Psi$, the B-transform $Bx = \{(Bx)_{rs}\}_{r,s\in\mathbb{N}}$ of x exists and is in Λ; where

$$(Bx)_{rs} = \vartheta - \sum_{t,u} b_{rstu}x_{tu}, \tag{1.1}$$

for each $r, s \in \mathbb{N}$. $(\Psi : \Lambda)$ stands for the class of all 4-dimensional complex infinite matrices from a double sequence space Ψ into a double sequence space Λ. In that case, $B \in (\Psi : \Lambda)$ if and only if $B_{rs} \in \Psi^{\beta(\vartheta)}$, where $B_{rs} = (b_{rstu})_{t,u\in\mathbb{N}}$ for all $r, s \in \mathbb{N}$.

The ϑ-summability domain $\Psi_B^{(\vartheta)}$ of a 4-dimensional complex infinite matrix B in a double sequence space Ψ consists of whose B-transforms are in Ψ; that is,

$$\Psi_B^{(\vartheta)} := \left\{ x = (x_{tu}) \in \Omega : Bx := \left(\vartheta - \sum_{tu} b_{rstu}x_{tu} \right)_{r,s\in\mathbb{N}} \text{exists and is in } \Psi \right\}.$$

In the past, many authors were interested in double sequence spaces. Now, let us give some information about these studies. Zeltser [41] has fundamentally examined both the topological structure and the theory of summability of double sequences in her doctoral dissertation. Recently, Altay and Başar [3] defined the double sequence spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double series whose sequences of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those spaces. Later, in [5], Başar and Sever have defined the set $\mathcal{L}_p$ of all absolutely p-summable double sequences which is a Banach space with the norm $\|.\|_{\mathcal{L}_p}$ defined in the following way:

$$\|.\|_{\mathcal{L}_p} = \left(\sum_{t,u} |x_{tu}|^p \right)^{\frac{1}{p}}.$$

It is also significant that the double sequence space $\mathcal{L}_u$ which was defined by Zeltser [42] is the special case of the space $\mathcal{L}_p$ for $p = 1$. For more details about the double sequences and related topics, the reader may refer to [1, 3–5, 13, 25–28, 31, 35–40, 43] and references therein.

In the rest of the study, φ and μ represent Euler function and the Möbius function, respectively. For every $r \in \mathbb{N}$ with $r > 1$, $\varphi(r)$ is the number of positive integers less than r which are coprime with r and $\varphi(1) = 1$. If ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3}...{a_m}^{b_m}$ is the prime factorization of a natural number $r > 1$, then

$$\varphi(r) = r(1 - \frac{1}{a_1})(1 - \frac{1}{a_2})(1 - \frac{1}{a_3})...(1 - \frac{1}{a_m}).$$

Also, the equality

$$r = \sum_{t|r} \varphi(t)$$

holds for every $r \in \mathbb{N}$ and $\varphi(r_1 r_2) = \varphi(r_1)\varphi(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime. Given any $r \in \mathbb{N}$ with $r > 1$, μ is defined as

$$\mu(r) := \begin{cases} (-1)^m & , \quad \text{if } r = a_1 a_2 ... a_m, \text{ where } a_1 a_2 ... a_m \text{ are} \\ & \quad\ \ \text{non-equivalent prime numbers} \\ 0 & , \quad \text{if } a^2 \mid r \text{ for some prime number} \quad a, \end{cases}$$

and $\mu(1) = 1$. If ${a_1}^{b_1}{a_2}^{b_2}{a_3}^{b_3}...{a_m}^{b_m}$ is the prime factorization of a naturel number $r > 1$, in this fact,

$$\sum_{t|r} t\mu(t) = (1 - a_1)(1 - a_2)(1 - a_3)...(1 - a_m).$$

If $r \neq 1$, then the equality

$$\sum_{t|r} \mu(t) = 0$$

holds and $\mu(r_1 r_2) = \mu(r_1)\mu(r_2)$, where $r_1, r_2 \in \mathbb{N}$ are coprime.

By using the regular 2-dimensional Euler-totient matrix Φ, the Euler-totient sequence spaces $\ell_p(\Phi)$ and $\ell_\infty(\Phi)$ which consist of all sequences whose Φ-transforms are in the spaces ℓ_p of absolutely p-summable and ℓ_∞ of bounded single sequences are introduced and examined by İlkhan and Kara [18].

The target of the existing study is to acquaint the matrix domains of the 4-dimensional Euler-totient matrix on some classical double sequence spaces.

2. Domain of Euler-Totient Matrix in Some Spaces of Double Sequences

In this section, we introduce the double sequence spaces $\Phi^\star(\mathcal{M}_u)$, $\Phi^\star(\mathcal{C}_p)$, $\Phi^\star(\mathcal{C}_{bp})$ and $\Phi^\star(\mathcal{C}_r)$ by using the 4-dimensional Euler-totient matrix $\Phi^\star$ and give some properties and results on theese spaces.

In [14], we have defined the 4-dimensional matrix $\Phi^\star = (\phi^\star_{rstu})$ which is called Euler-totient matrix operator as follows:

$$\phi^\star_{rstu} := \begin{cases} \dfrac{\varphi(t)\varphi(u)}{rs} & , \quad t \mid r\ ,\ u \mid s, \\ \\ 0 & , \quad \text{otherwise}, \end{cases} \tag{2.1}$$

for every $r, s, t, u \in \mathbb{N}$. Thus, it is clear that $\Phi^\star$ is a triangle and the $\Phi^\star$-transform of a double sequence $x = (x_{rs})$ is given by

$$y_{rs} := (\Phi^\star x)_{rs} = \frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u)x_{tu}, \tag{2.2}$$

for every $r, s \in \mathbb{N}$. Throughout the article, we suppose that the terms of the double sequences $x = (x_{rs})$ and $y = (y_{rs})$ are connected with the relation (2.2).

The inverse $(\Phi^\star)^{-1} = (\phi^{\star -1}_{rstu})$ of the triangle matrix $\Phi^\star$ is calculated as

$$\phi^{\star -1}_{rstu} := \begin{cases} \dfrac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tu & , \quad t \mid r \ , \ u \mid s, \\ \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. We introduce the sequence spaces $\Phi^\star(\mathcal{M}_u)$, $\Phi^\star(\mathcal{C}_p)$, $\Phi^\star(\mathcal{C}_{bp})$ and $\Phi^\star(\mathcal{C}_r)$ as the sets of all double sequences such that $\Phi^\star$-transforms of them are in the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, that is,

$$\begin{aligned}
\Phi^\star(\mathcal{M}_u) &= \left\{ x = (x_{rs}) \in \Omega : \sup_{r,s \in \mathbb{N}} \left| \frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u) x_{tu} \right| < \infty \right\}, \\
\Phi^\star(\mathcal{C}_p) &= \left\{ x = (x_{rs}) \in \Omega : \exists L \in \mathbb{C} \ni p - \lim_{r,s \to \infty} \left| \frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u) x_{tu} - L \right| = 0 \right\}, \\
\Phi^\star(\mathcal{C}_{bp}) &= \left\{ x = (x_{rs}) \in \Omega : \left(\frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u) x_{tu} \right) \in \mathcal{C}_{bp} \right\}, \\
\Phi^\star(\mathcal{C}_r) &= \left\{ x = (x_{rs}) \in \Omega : \left(\frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u) x_{tu} \right) \in \mathcal{C}_r \right\}.
\end{aligned}$$

It is immediately seen that $\Phi^\star(\mathcal{M}_u)$, $\Phi^\star(\mathcal{C}_p)$, $\Phi^\star(\mathcal{C}_{bp})$ and $\Phi^\star(\mathcal{C}_r)$ are the domains of the 4-dimensional Euler-totient matrix $\Phi^\star$ in the spaces $\mathcal{M}_u$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively.

If Ψ is any normed double sequence space, then we call the matrix domain $\Phi^\star(\Psi)$ as the double Euler-totient sequence space.

Definition 2.1 (See [16],[31])**.** A 4-dimensional matrix B is said to be RH-regular if it maps every bounded p-convergent sequence into a p-convergent sequence with the same p-limit.

Lemma 2.1 (See [16],[31])**.** *A 4-dimensional triangle matrix $B = (b_{rstu})$ is RH-regular iff*

$$\begin{aligned}
RH_1 &: \ p - \lim_{r,s \to \infty} b_{rstu} = 0 \quad \textit{for each} \quad t, u \in \mathbb{N}, \\
RH_2 &: \ p - \lim_{r,s \to \infty} \sum_{t,u} b_{rstu} = 1, \\
RH_3 &: \ p - \lim_{r,s \to \infty} \sum_{t} |b_{rstu}| = 0 \quad \textit{for each} \quad u \in \mathbb{N}, \\
RH_4 &: \ p - \lim_{r,s \to \infty} \sum_{u} |b_{rstu}| = 0 \quad \textit{for each} \quad t \in \mathbb{N}, \\
RH_5 &: \ \textit{There exists finite positive integers} \quad M_1 \quad \textit{and} \quad M_2 \quad \textit{such that}
\end{aligned}$$

$$\sum_{t,u > M_1} |b_{rstu}| < M_2.$$

It should be noted that the 4-dimensional Euler-totient matrix $\Phi^\star$ described by (2.1) is RH-regular [14].

Now, we may continue with the following two theorems which are the essential in the study.

Theorem 2.1. *The sets $\Phi^\star(\mathcal{M}_u)$, $\Phi^\star(\mathcal{C}_{bp})$ and $\Phi^\star(\mathcal{C}_r)$ are the linear spaces which are linearly norm isomorphic to the spaces $\mathcal{M}_u$, $\mathcal{C}_{bp}$ and $\mathcal{C}_r$, respectively, and are the Banach spaces with the norm*

$$\|x\|_{\Phi^\star(\mathcal{M}_u)} = \|\Phi^\star x\|_\infty = \sup_{r,s \in \mathbb{N}} \left| \frac{1}{rs} \sum_{t|r, u|s} \varphi(t)\varphi(u) x_{tu} \right|. \tag{2.3}$$

Proof. To avoid the repetition of the similar statements, we give the proof only for the space $\Phi^\star(\mathcal{M}_u)$. Since the initial assertion is routine verification and is easy to prove, we ignore its proof in here. To confirm the fact that $\Phi^\star(\mathcal{M}_u)$ is linearly norm isomorphic to the space $\mathcal{M}_u$, we need to be sure the existence of a linear and norm preserving bijection between the spaces $\Phi^\star(\mathcal{M}_u)$ and $\mathcal{M}_u$. For this purpose, let us take the transformation B defined from $\Phi^\star(\mathcal{M}_u)$ into $\mathcal{M}_u$ by $x \mapsto y = Bx$, where $y = (y_{rs})$ is the $\Phi^\star$-transform of the sequence $x = (x_{tu})$. The linearity of B is clear. Consider the equality $Bx = \theta$ which yields us that $x_{tu} = 0$ for every $t, u \in \mathbb{N}$. So, $x = \theta$. Therefore, B is injective. Let us consider $y \in \mathcal{M}_u$ and describe the double sequence $x = (x_{rs})$ by

$$x_{rs} = \sum_{t|r,u|s} \frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tuy_{tu} \tag{2.4}$$

for every $r, s \in \mathbb{N}$. By taking supremum over $r, s \in \mathbb{N}$ on the following equality

$$|(\Phi^\star x)_{rs}| = \left| \frac{1}{rs} \sum_{t|r,s|u} \varphi(t)\varphi(u)x_{tu} \right| = |y_{rs}|,$$

it can be derived that B is surjective and norm preserving.

Now, we may prove that $\Phi^\star(\mathcal{M}_u)$ is a Banach space with the norm $\|.\|_{\Phi^\star(\mathcal{M}_u)}$ described by (2.3). Since $\mathcal{M}_u$ is a Banach space from [25], we obtain the desired result from Section (b) of Corollary 6.3.41 in [6]. □

Theorem 2.2. *The set $\Phi^\star(\mathcal{C}_p)$ is linearly isomorphic to the space $\mathcal{C}_p$ and is a complete semi-normed space with the semi-norm*

$$\|x\|_{\Phi^\star(\mathcal{C}_p)} = \lim_{i\to\infty} \left(\sup_{r,s\geq i} |(\Phi^\star x)_{rs}| \right).$$

Proof. Since the proof of the theorem is similar to the proof of Theorem 2.1, we ignore it.

□

Now, let us give our results about inclusion relations.

Theorem 2.3. *The inclusion $\mathcal{M}_u \subset \Phi^\star(\mathcal{M}_u)$ holds.*

Proof. Let us take a sequence $x = (x_{tu}) \in \mathcal{M}_u$. In that case, there exists a positive real number M_3 such that $\sup_{t,u\in\mathbb{N}} |x_{tu}| \leq M_3$. Therefore, one can immediately see that

$$\begin{aligned} \|x\|_{\Phi^\star(\mathcal{M}_u)} &= \sup_{r,s\in\mathbb{N}} \left| \frac{1}{rs} \sum_{t|r,u|s} \varphi(t)\varphi(u)x_{tu} \right| \\ &\leq \sup_{r,s\in\mathbb{N}} \left| \frac{1}{rs} \sum_{t|r,u|s} \varphi(t)\varphi(u) \right| |x_{tu}| \\ &\leq M_3 \sup_{r,s\in\mathbb{N}} \left| \frac{1}{rs} \sum_{t|r,u|s} \varphi(t)\varphi(u) \right| = M_3. \end{aligned}$$

Thus, the inclusion is valid. □

Theorem 2.4. *The inclusion $\mathcal{C}_{bp} \subset \Phi^\star(\mathcal{C}_p)$ holds.*

Proof. Let us take the sequence $x = (x_{tu}) \in \mathcal{C}_{bp}$ with $p\lim_{t,u\to\infty} x_{tu} = L$. Since 4-dimensional Euler-totient matrix is RH-regular, $p\lim_{t,u\to\infty} y_{tu} = L$, where $(y_{tu}) = (\Phi^\star x)_{tu}$. Hence, we see that $\mathcal{C}_{bp} \subset \Phi^\star(\mathcal{C}_p)$. □

3. Dual Spaces

In the current section, we tend to compute the $\alpha-$, $\beta(\vartheta)-$ and $\gamma-$duals of the new double Euler-totient sequence spaces.

Theorem 3.1. *The α-dual of the space $\Phi^\star(\mathcal{M}_u)$ is $\mathcal{L}_u$.*

Proof. Suppose that $c = (c_{rs}) \in \{\Phi^\star(\mathcal{M}_u)\}^\alpha$ but $c \notin \mathcal{L}_u$. Then, $\sum_{r,s} |c_{rs}x_{rs}| < \infty$ for all $x = (x_{rs}) \in \Phi^\star(\mathcal{M}_u)$. If we consider $e \in \Phi^\star(\mathcal{M}_u)$, in that case $ce = c \notin \mathcal{L}_u$, that is $c \notin \{\Phi^\star(\mathcal{M}_u)\}^\alpha$ and it is seen that this is a contradiction. Thus, c must be in $\mathcal{L}_u$.

Conversely, let us take sequences $c = (c_{rs}) \in \mathcal{L}_u$ and $x = (x_{rs}) \in \Phi^\star(\mathcal{M}_u)$. In that case, there exists a double sequence $y = (y_{rs}) \in \mathcal{M}_u$ such that $y = \Phi^\star x$ and $\sup_{r,s} |y_{rs}| < M_4$, where $M_4 \in \mathbb{R}^+$. Then, we have from the following inequality

$$\begin{aligned}
\sum_{r,s} |c_{rs}x_{rs}| &= \sum_{r,s} |c_{rs}| \left| \sum_{t|r,u|s} \frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tuy_{tu} \right| \\
&\leq M_4 \sum_{r,s} |c_{rs}| \left| \sum_{t|r,u|s} \frac{\mu(\frac{r}{t})\mu(\frac{s}{u})}{\varphi(r)\varphi(s)} tu \right| \\
&= M_4 \sum_{r,s} |c_{rs}| < \infty,
\end{aligned}$$

that $c \in (\Phi^\star(\mathcal{M}_u))^\alpha$ and this completes the proof. □

Now, we give some lemmas which characterize the classes of 4-dimensional matrix mappings(see [16], [42] and [43]). With the help of these lemmas, we will calculate the $\beta(\vartheta)$, $\beta(bp)$, $\beta(p)$ and γ-duals of our new double sequence spaces.

Lemma 3.1. *Suppose that $B = (b_{rstu})$ is a 4-dimensional infinite matrix. Then, $B \in (\mathcal{C}_{bp} : \mathcal{C}_\vartheta)$ iff following conditions hold:*

$$\sup_{r,s\in\mathbb{N}} \sum_{t,u} |b_{rstu}| < \infty, \tag{3.1}$$

$$\exists b_{tu} \in \mathbb{C} \ni \vartheta - \lim_{r,s\to\infty} b_{rstu} = b_{tu} \textit{ for all } \quad t,u \in \mathbb{N}, \tag{3.2}$$

$$\exists L \in \mathbb{C} \ni \vartheta - \lim_{r,s\to\infty} \sum_{t,u} b_{rstu} = L \quad \textit{exists}, \tag{3.3}$$

$$\exists t_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s\to\infty} \sum_{u} |b_{rst_0u} - b_{t_0u}| = 0, \tag{3.4}$$

$$\exists u_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s\to\infty} \sum_{t} |b_{rstu_0} - b_{tu_0}| = 0. \tag{3.5}$$

In the case of (3.5), $b = (b_{tu}) \in \mathcal{L}_u$ and

$$\vartheta - \lim_{r,s\to\infty} [Bx]_{rs} = \sum_{t,u} b_{tu}x_{tu} + \left(L - \sum_{t,u} b_{tu} \right) bp - \lim_{r,s\to\infty} x_{rs}$$

satisfies for $x \in \mathcal{C}_{bp}$.

Lemma 3.2. *Suppose that $B = (b_{rstu})$ is a 4-dimensional infinite matrix. Then, $B \in (\mathcal{C}_p : \mathcal{C}_\vartheta)$ iff (3.1)-(3.3) hold and the following conditions hold, too:*

$$\forall t \in \mathbb{N}, \quad \exists u_0 \in \mathbb{N} \ni b_{rstu} = 0 \quad \textit{for every} \quad u > u_0 \quad \textit{and} \quad r,s \in \mathbb{N}, \tag{3.6}$$

$$\forall u \in \mathbb{N}, \quad \exists t_0 \in \mathbb{N} \ni b_{rstu} = 0 \quad \textit{for every} \quad t > t_0 \quad \textit{and} \quad r,s \in \mathbb{N}. \tag{3.7}$$

In the case of (3.7), $\exists t_0, u_0 \in \mathbb{N}$ such that $b = (b_{tu}) \in \mathcal{L}_u$ and $(b_{tu_0})_{t\in\mathbb{N}}$, $(b_{t_0u})_{u\in\mathbb{N}} \in \zeta$, where ζ represents the space of every finitely sequences which are non-equivalent zero and

$$\vartheta - \lim_{r,s\to\infty} [Bx]_{rs} = \sum_{t,u} b_{tu}x_{tu} + \sum_{t} \left(L - \sum_{t,u} b_{tu} \right) p - \lim_{r,s\to\infty} x_{rs}$$

satisfies for $x \in \mathcal{C}_p$.

Lemma 3.3. *Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. Then,* $B \in (\mathcal{C}_r : \mathcal{C}_\vartheta)$ *iff (3.1)-(3.3) hold and the following conditions hold, too:*

$$\exists u_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s\to\infty} \sum_t b_{rstu_0} = \rho_{u_0}, \tag{3.8}$$

$$\exists t_0 \in \mathbb{N} \ni \vartheta - \lim_{r,s\to\infty} \sum_u b_{rst_0u} = \varrho_{t_0}. \tag{3.9}$$

In the case of (3.9), $b = (b_{tu}) \in \mathcal{L}_u$ *and* $\rho_u, \varrho_t \in \ell_1$ *and*

$$\begin{aligned} \vartheta - \lim_{r,s\to\infty} [Bx]_{rs} &= \sum_{t,u} b_{tu}x_{tu} + \sum_t \left(\varrho_t - \sum_u b_{tu}\right) x_t + \sum_u \left(\rho_u - \sum_t b_{tu}\right) x_u \\ &+ \left(L + \sum_{t,u} b_{tu} - \sum_t \varrho_t - \sum_u \rho_u\right) r - \lim_{r,s\to\infty} x_{rs} \end{aligned}$$

satisfies for $x \in \mathcal{C}_r$.

Lemma 3.4. *[36] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. Then,* $B \in (\mathcal{C}_{bp} : \mathcal{M}_u)$ *iff the condition (3.1) hold.*

Lemma 3.5. *[12] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. Then,* $B \in (\mathcal{M}_u : \mathcal{C}_{bp})$ *iff the conditions (3.1), (3.2) hold and the following conditions hold, too:*

$$\exists b_{tu} \in \mathbb{C} \ni bp - \lim_{r,s\to\infty} \sum_{tu} |b_{rstu} - b_{tu}| = 0, \tag{3.10}$$

$$bp - \lim_{r,s\to\infty} \sum_{u=0}^{s} b_{rstu} \quad \textit{exists for each} \quad t \in \mathbb{N}, \tag{3.11}$$

$$bp - \lim_{r,s\to\infty} \sum_{t=0}^{r} b_{rstu} \quad \textit{exists for each} \quad u \in \mathbb{N}, \tag{3.12}$$

$$\sum_{t,u} |b_{rstu}| \quad \textit{converges.} \tag{3.13}$$

Lemma 3.6. *[38] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. Then,* $B \in (\mathcal{M}_u : \mathcal{M}_u)$ *iff the condition (3.1) holds.*

Lemma 3.7. *[39] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. Then,* $B \in (\mathcal{M}_u : \mathcal{C}_p)$ *iff the conditions (3.2), (3.6) and (3.7) hold.*

Lemma 3.8. *[40] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. In that case:*

(i) If $0 < p \leq 1$*, then* $B \in (\mathcal{L}_p : \mathcal{M}_u)$ *iff*

$$\sup_{r,s,t,u\in\mathbb{N}} |b_{rstu}| < \infty, \tag{3.14}$$

(ii) If $1 < p < \infty$*, then* $B \in (\mathcal{L}_p : \mathcal{M}_u)$ *iff*

$$\sup_{r,s\in\mathbb{N}} \sum_{t,u} |b_{rstu}|^{p'} < \infty. \tag{3.15}$$

Lemma 3.9. *[40] Suppose that* $B = (b_{rstu})$ *is a 4-dimensional infinite matrix. In that case:*

(i) If $0 < p \leq 1$*, then* $B \in (\mathcal{L}_p : \mathcal{C}_{bp})$ *iff the conditions (3.2) and (3.14) hold with* $\vartheta = bp$*,*

(ii) If $1 < p < \infty$*, then* $B \in (\mathcal{L}_p : \mathcal{C}_{bp})$ *iff the conditions (3.2) and (3.15) hold.*

Theorem 3.2. *Consider the set w_1 defined by*

$$w_1 = \left\{ c = (c_{rs}) \in \Omega : \sup_{r,s} \sum_{t,u} |\sigma(r,s,t,u,m,n)| < \infty \right\},$$

where

$$\sigma(r,s,t,u,m,n) = \sum_{m=t,t|m}^{r} \sum_{n=u,u|n}^{s} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tuc_{mn}.$$

Then, $(\Phi^{\star}(\mathcal{C}_{bp}))^{\gamma} = w_1 = (\Phi^{\star}(\mathcal{M}_u))^{\gamma}$.

Proof. Suppose that $c = (c_{rs}) \in \Omega$ and $x = (x_{rs}) \in \Phi^{\star}(\mathcal{C}_{bp})$. Then, we can conclude from (2.2) that $y = (y_{rs}) \in \mathcal{C}_{bp}$. Now, let us define the 4-dimensional matrix $O = (o_{rstu})$ by

$$o_{rstu} := \begin{cases} \sigma(r,s,t,u,m,n) & , \quad t \mid m \quad , \quad u \mid n, \\ 0 & , \quad \text{otherwise}, \end{cases}$$

for every $r, s, t, u \in \mathbb{N}$. Therefore, we obtain by using the relation (2.4) that

$$\begin{aligned} z_{rs} &= \sum_{t,u=1}^{r,s} c_{tu} x_{tu} \\ &= \sum_{t,u=1}^{r,s} c_{tu} \left[\sum_{m|t,n|u} \frac{\mu(\frac{t}{m})\mu(\frac{u}{n})}{\varphi(t)\varphi(u)} mny_{mn} \right] \\ &= \sum_{t,u=1}^{r,s} \left[\sum_{m=t,t|m}^{r} \sum_{n=u,u|n}^{s} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tuc_{mn} \right] y_{tu} \\ &= (Oy)_{rs} \end{aligned} \tag{3.16}$$

for every $r, s \in \mathbb{N}$. Then, by considering the equality (3.16), we deduce that $cx = (c_{rs}x_{rs}) \in \mathcal{BS}$ whenever $x \in \Phi^{\star}(\mathcal{C}_{bp})$ iff $z = (z_{rs}) \in \mathcal{M}_u$ whenever $y \in \mathcal{C}_{bp}$. This leads us to the fact that $c = (c_{rs}) \in (\Phi^{\star}(\mathcal{C}_{bp}))^{\gamma}$ iff $O \in (\mathcal{C}_{bp} : \mathcal{M}_u)$. Hence, we achieve that $(\Phi^{\star}(\mathcal{C}_{bp}))^{\gamma} = w_1$. The other part of the theorem can be proven by using similar technique. So, we omit it. $\square$

Theorem 3.3. *Consider the sets $w_2 - w_{13}$ defined by*

$$\begin{aligned}
w_2 &= \left\{c=(c_{rs})\in\Omega : \exists b_{tu}\in\mathbb{C}\ni \vartheta-\lim_{r,s\to\infty}\sigma(r,s,t,u,m,n)=b_{tu}\right\},\\
w_3 &= \left\{c=(c_{rs})\in\Omega : \exists L\in\mathbb{C}\ni \vartheta-\lim_{r,s\to\infty}\sum_{t,u}\sigma(r,s,t,u,m,n)=L \quad \textit{exists}\right\},\\
w_4 &= \left\{c=(c_{rs})\in\Omega : \exists u_0\in\mathbb{N}\ni \vartheta-\lim_{r,s\to\infty}\sum_{t}|\sigma(r,s,t,u_0,m,n)-b_{tu_0}|=0\right\},\\
w_5 &= \left\{c=(c_{rs})\in\Omega : \exists t_0\in\mathbb{N}\ni \vartheta-\lim_{r,s\to\infty}\sum_{u}|\sigma(r,s,t_0,u,m,n)-b_{t_0u}|=0\right\},\\
w_6 &= \left\{c=(c_{rs})\in\Omega : \forall t\in\mathbb{N},\exists u_0\in\mathbb{N}\ni \sigma(r,s,t,u,m,n)=0,\forall u>u_0,\forall r,s\in\mathbb{N}\right\},\\
w_7 &= \left\{c=(c_{rs})\in\Omega : \forall u\in\mathbb{N},\exists t_0\in\mathbb{N}\ni \sigma(r,s,t,u,m,n)=0,\forall t>t_0,\forall r,s\in\mathbb{N}\right\},\\
w_8 &= \left\{c=(c_{rs})\in\Omega : \exists u_0\in\mathbb{N}\ni \vartheta-\lim_{r,s\to\infty}\sum_{t}\sigma(r,s,t,u_0,m,n)=b_{u_0}\right\},\\
w_9 &= \left\{c=(c_{rs})\in\Omega : \exists t_0\in\mathbb{N}\ni \vartheta-\lim_{r,s\to\infty}\sum_{u}\sigma(r,s,t_0,u,m,n)=b_{t_0}\right\},\\
w_{10} &= \left\{c=(c_{rs})\in\Omega : \exists b_{tu}\in\mathbb{C}\ni bp-\lim_{r,s\to\infty}\sum_{t,u}|\sigma(r,s,t,u,m,n)-b_{tu}|=0\right\},\\
w_{11} &= \left\{c=(c_{rs})\in\Omega : \forall t\in\mathbb{N}\ni bp-\lim_{r,s\to\infty}\sum_{u=1}^{s}\sigma(r,s,t,u,m,n) \quad \textit{exists}\right\},\\
w_{12} &= \left\{c=(c_{rs})\in\Omega : \forall u\in\mathbb{N}\ni bp-\lim_{r,s\to\infty}\sum_{t=1}^{r}\sigma(r,s,t,u,m,n) \quad \textit{exists}\right\},\\
w_{13} &= \left\{c=(c_{rs})\in\Omega : \sum_{t,u}|\sigma(r,s,t,u,m,n)| \quad \textit{converges}\right\}.
\end{aligned}$$

In that case, following statements are satisfied:

(i) $(\Phi^\star(\mathcal{C}_{bp}))^{\beta(\vartheta)} = \bigcap_{k=1}^{5} w_k$,

(ii) $(\Phi^\star(\mathcal{C}_{p}))^{\beta(\vartheta)} = \bigcap_{k=1}^{3} w_k \cap w_6 \cap w_7$,

(iii) $(\Phi^\star(\mathcal{C}_{r}))^{\beta(\vartheta)} = \bigcap_{k=1}^{3} w_k \cap w_8 \cap w_9$,

(iV) $(\Phi^\star(\mathcal{M}_{u}))^{\beta(bp)} = w_1 \cap w_2 \bigcap_{k=10}^{13} w_k$,

(V) $(\Phi^\star(\mathcal{M}_{u}))^{\beta(p)} = w_2 \cap w_6 \cap w_7$.

Proof.

(i) Suppose that $c=(c_{rs})\in\Omega$ and $x=(x_{rs})\in\Phi^\star(\mathcal{C}_{bp})$. In that case, there exists a double sequence $y=(y_{rs})\in\mathcal{C}_{bp}$ with $\Phi^\star x = y$. Since (3.16) holds, we deduce that $cx\in\mathcal{CS}_\vartheta$ whenever $x\in\Phi^\star(\mathcal{C}_{bp})$ iff $z\in\mathcal{C}_\vartheta$ whenever $y\in\mathcal{C}_{bp}$. This leads us to the fact that $c=(c_{rs})\in(\Phi^\star(\mathcal{C}_{bp}))^{\beta(\vartheta)}$ iff $O\in(\mathcal{C}_{bp}:\mathcal{C}_\vartheta)$. Therefore, the conditions of Lemma 3.1 are satisfied with $O=(o_{rstu})$ defined as in Theorem 3.2. Hence, we achieve that the $\beta(\vartheta)$ -dual of the space $\Phi^\star(\mathcal{C}_{bp})$ is $\bigcap_{k=1}^{5} w_k$.

The other parts of the Theorem can be done analogously by using the Lemmas 3.2, 3.3, 3.5 and 3.7, respectively. So, we pass the details. ∎

4. Charactarization of Some Classes of 4-Dimensional Matrices

In the current section, we deal with some 4-dimensional matrix mapping classes related to the double sequence spaces $\Phi^{\star}(\mathcal{M}_u)$, $\Phi^{\star}(\mathcal{C}_p)$, $\Phi^{\star}(\mathcal{C}_{bp})$ and $\Phi^{\star}(\mathcal{C}_r)$ by using dual summability methods for double sequences which have been presented and examined by Başar [4] and Yeşilkayagil and Başar [37] and which have been applied by Tuğ [36].

Theorem 4.1. *Assume that the elements of 4-dimensional infinite matrices $B = (b_{rstu})$ and $H = (h_{rstu})$ are connected with the relation*

$$h_{rstu} = \sum_{m=t,t|m}^{\infty} \sum_{n=u,u|n}^{\infty} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tub_{rsmn}. \tag{4.1}$$

Then, $B \in (\Phi^{\star}(\Psi) : \Lambda)$ iff $B_{rs} \in [\Phi^{\star}(\Psi)]^{\beta(\vartheta)}$ for every $r, s \in \mathbb{N}$ and $H \in (\Psi : \Lambda)$, where Ψ and $\Lambda \in \{\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r\}$.

Proof. Assume that $B \in (\Phi^{\star}(\Psi) : \Lambda)$. In that case, Bx exists and is in Λ for every $x \in \Phi^{\star}(\Psi)$ and it also implies that $B_{rs} \in [\Phi^{\star}(\Psi)]^{\beta(\vartheta)}$ for every $r, s \in \mathbb{N}$. Thus, we have the following equality derived from partial sums of the series $\sum_{t,u} b_{rstu}x_{tu}$ with relation (2.4)

$$\begin{aligned} \sum_{t,u=1}^{i,j} b_{rstu}x_{tu} &= \sum_{t,u=1}^{i,j} b_{rstu} \left[\sum_{m|t,n|u} \frac{\mu(\frac{t}{m})\mu(\frac{u}{n})}{\varphi(t)\varphi(u)} mny_{mn} \right] \\ &= \sum_{t,u=1}^{i,j} \left[\sum_{m=t,t|m}^{i} \sum_{n=u,u|n}^{j} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tub_{rsmn} \right] y_{tu} \end{aligned}$$

for every $i, j \in \mathbb{N}$. In that case, if we take ϑ-limit on equality above as $i, j \to \infty$, we have $Bx = Hy$. Therefore, we obtain that $Hy \in \Lambda$ whenever $y \in \Psi$, that is $H \in (\Psi : \Lambda)$.

Conversely, suppose that $B_{rs} \in [\Phi^{\star}(\Psi)]^{\beta(\vartheta)}$ for every $r, s \in \mathbb{N}$, $H \in (\Psi : \Lambda)$ and $x \in \Phi^{\star}(\Psi)$ such that $y = \Phi^{\star}x$. In that case, Bx exists and therefore, the (k,l)th rectangular partial sums of the series $\sum_{t,u} b_{rstu}x_{tu}$ obtained as

$$\begin{aligned} (Bx)_{rs}^{[k,l]} &= \sum_{t,u=1}^{k,l} b_{rstu}x_{tu} \\ &= \sum_{t,u=1}^{k,l} b_{rstu} \left[\sum_{m|t,n|u} \frac{\mu(\frac{t}{m})\mu(\frac{u}{n})}{\varphi(t)\varphi(u)} mny_{mn} \right] \\ &= \sum_{t,u=1}^{k,l} \left[\sum_{m=t,t|m}^{k} \sum_{n=u,u|n}^{l} \frac{\mu(\frac{m}{t})\mu(\frac{n}{u})}{\varphi(m)\varphi(n)} tub_{rsmn} \right] y_{tu} \end{aligned} \tag{4.2}$$

for every $r, s, k, l \in \mathbb{N}$. By taking ϑ-limit on (4.2) while $k, l \to \infty$, it can be easily obtain from the following equality

$$\sum_{t,u} b_{rstu}x_{tu} = \sum_{t,u} h_{rstu}y_{tu}$$

for every $r, s \in \mathbb{N}$ that $Bx = Hy$ which leads us to the fact that $B \in (\Phi^{\star}(\Psi) : \Lambda)$. □

Corollary 4.1. *Suppose that $B = (b_{rstu})$ is a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $B \in (\Phi^{\star}(\mathcal{C}_p) : \mathcal{C}_{\vartheta})$ iff the conditions (3.1)-(3.3), (3.6) and (3.7) are satisfied with h_{rstu} in place of b_{rstu},

(ii) $B \in (\Phi^{\star}(\mathcal{C}_{bp}) : \mathcal{C}_{\vartheta})$ iff the conditions (3.1)-(3.5) are satisfied with h_{rstu} in place of b_{rstu},

(iii) $B \in (\Phi^{\star}(\mathcal{C}_{bp}) : \mathcal{M}_u)$ iff the condition (3.1) is satisfied with h_{rstu} in place of b_{rstu},

(iv) $B \in (\Phi^{\star}(\mathcal{C}_r) : \mathcal{C}_{\vartheta})$ iff the conditions (3.1)-(3.3), (3.8) and (3.9) are satisfied with h_{rstu} in place of b_{rstu},

(v) $B \in (\Phi^{\star}(\mathcal{M}_u) : \mathcal{C}_{bp})$ iff the conditions (3.1), (3.2), (3.10)-(3.13) are satisfied with h_{rstu} in place of b_{rstu},

(vi) $B \in (\Phi^\star(\mathcal{M}_u) : \mathcal{C}_p)$ *iff the conditions (3.2), (3.6) and (3.7) are satisfied with* h_{rstu} *in place of* b_{rstu}.

Lemma 4.1. *[40] Let* Ψ *and* Λ *be two double sequence spaces,* $B = (b_{rstu})$ *be any 4-dimensional matrix and* $F = (f_{rstu})$ *also be a 4-dimensional triangle matrix such that* $f_{rstu} = 0$ *if* $t > r$ *and* $u > s$ *for every* $r, s, t, u \in \mathbb{N}$. *In that case,* $B \in (\Psi : \Lambda_F)$ *iff* $FB \in (\Psi : \Lambda)$.

Now, let us define the 4-dimensional matrix $G = (g_{rstu})$ by

$$g_{rstu} = \sum_{m|r,n|s} \phi^\star_{rsmn} b_{mntu}$$

for every $r, s, t, u \in \mathbb{N}$ and give following corollary.

Corollary 4.2. *Suppose that* $B = (b_{rstu})$ *is a 4-dimensional matrix. In that case the following statements are satisfied:*

(i) $B \in (\mathcal{C}_p : \Phi^\star(\mathcal{C}_\vartheta))$ *iff the conditions (3.1)-(3.3), (3.6) and (3.7) are satisfied with* g_{rstu} *in place of* b_{rstu},

(ii) $B \in (\mathcal{C}_{bp} : \Phi^\star(\mathcal{C}_\vartheta))$ *iff the conditions (3.1)-(3.5) are satisfied with* g_{rstu} *in place of* b_{rstu},

(iii) $B \in (\mathcal{C}_r : \Phi^\star(\mathcal{C}_\vartheta))$ *iff the conditions (3.1)-(3.3), (3.8) and (3.9) are satisfied with* g_{rstu} *in place of* b_{rstu},

(iv) $B \in (\mathcal{L}_p : \Phi^\star(\mathcal{C}_{bp}))$ *iff the conditions (3.2) and (3.14) are satisfied for* $0 < p \leq 1$ *and* $\vartheta = bp$ *with* g_{rstu} *in place of* b_{rstu},

(v) $B \in (\mathcal{L}_p : \Phi^\star(\mathcal{C}_{bp}))$ *iff the conditions (3.2) and (3.15) are satisfied for* $1 < p < \infty$ *and* $\vartheta = bp$ *with* g_{rstu} *in place of* b_{rstu},

(vi) $B \in (\mathcal{L}_p : \Phi^\star(\mathcal{M}_u))$ *iff the condition (3.14) is satisfied for* $0 < p \leq 1$ *with* g_{rstu} *in place of* b_{rstu},

(vii) $B \in (\mathcal{L}_p : \Phi^\star(\mathcal{M}_u))$ *iff the condition (3.15) is satisfied for* $1 < p < \infty$ *with* g_{rstu} *in place of* b_{rstu},

(viii) $B \in (\mathcal{M}_u : \Phi^\star(\mathcal{C}_{bp}))$ *iff the conditions (3.1),(3.2), (3.10)-(3.13) are satisfied with* g_{rstu} *in place of* b_{rstu},

(ix) $B \in (\mathcal{M}_u : \Phi^\star(\mathcal{C}_p))$ *iff the conditions (3.2), (3.6) and (3.7) are satisfied with* g_{rstu} *in place of* b_{rstu},

(x) $B \in (\mathcal{C}_{bp} : \Phi^\star(\mathcal{M}_u))$ *iff the condition (3.1) is satisfied with* g_{rstu} *in place of* b_{rstu}.

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