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An Inverse Problem for the Forced Transverse Vibration of a Rectangular Membrane with Time Dependent Potential

Zamana Bağlı Potansiyeli Olan Dikdörtgen Bir Zarın Zorlanmış Çapraz Titreşimi için Bir Ters Problem

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Abstract

In this paper, an initial-boundary value problem for a two-dimensional wave equation which arises in the equation of motion for the forced transverse vibration of a rectangular membrane is considered. Giving an additional condition, a time-dependent coefficient is determined and existence and uniqueness theorem for small times is proved. Moreover, characterization of the conditional stability is given and numerical solution of the inverse problem investigated by using finite difference method.

Keywords: *Inverse problem, Fourier method, Two dimensional wave equation, Finite difference method*

Öz

Bu çalışmada, dikdörtgen bir zarın zorlanmış enine titreşimi için hareket denkleminde ortaya çıkan iki boyutlu bir dalga denklemi için başlangıç-sınır değer problemi ele alınmıştır. Verilmiş bir ek koşul ile zamana bağlı katsayı belirlenmiştir ve yeteri kadar küçük zaman değerleri için varlık ve teklik teoremi ispatlanmıştır. Ayrıca, koşullu kararlılığın karakterizasyonu verilmiş ve ters problemin sayısal çözümü sonlu farklar yöntemi kullanılarak incelenmiştir.

Anahtar Kelimeler: *Ters problem, Fourier yöntemi, İki boyutlu dalga denklemi, Sonlu farklar yöntemi*

1. Introduction

The wave equation, which arises in fields like acoustics, electro magnetics, elasticity, and fluid dynamics, is an important second order hyperbolic partial differential equation. Vibrations of structures (as buildings and beams) are modelled by hyperbolic partial differential equations. For instance; the vibrating string is a basic one-dimensional vibrational problem and its two-dimensional analogue, namely, the motion of a membrane which is a perfectly flexible thin plate or lamina that is subject to tension as drumhead and diagrams of condenser microphones, [1].

In this paper, we consider the two dimensional wave equation

$$u_{tt} = u_{xx} + u_{yy} + a(t)u(t,x,y) + f\,(t,x,y), (t,x,y) \in D_T\,, \quad (1)$$

with the initial conditions

$$\begin{cases} u(0,x,y) = \varphi(x,y), \\ u_t(0,x,y) = \psi(x,y), \end{cases} \quad (x,y) \in \Omega \qquad (2)$$

boundary conditions

$$\begin{cases} u(t,0,y) = u(t,1,y) = 0, \\ u(t,x,0) = u(t,x,1) = 0, \end{cases} \qquad (3)$$
$$t \in [0,T], (x,y) \in \bar{\Omega},$$

where $D_T = [0,T] \times \Omega, \Omega = \{(x,y): 0 < x,y < 1\}$ for some fixed $T > 0$. $u = u(t,x,y)$ represents the displacement at the instant t of the point located at (x,y), $a(t)$ is the time dependent potential, the functions $\varphi(x,y)$ and $\psi(x,y)$ are the displacement and velocity at t = 0, respectively.

This model can be used for the equation of motion for the forced transverse vibration of a rectangular membrane with time dependent potential which is clamped or fixed on all the edges.

For a given function $a(t), 0 \le t \le T$ the problem (1)-(3) for the unknown function $u(t,x,y)$ is called direct (forward) problem. The well-posedness of the direct problem for the two-dimensional linear wave equation has been established in [1, 2]. Moreover, the papers [3] and [4] are dedicated to the study of existence of classical solution of an initial-boundary value problem for one class of semi-linear and non-linear multidimensional wave equations, respectively. For numerical aspects of direct problem [5] applies the compact finite difference approximation which is combined collocation technique to two dimensional homogeneous wave equation.

If $a(t), 0 \le t \le T$ is unknown, finding the pair of solution $\{a(t), u(t,x,y)\}$ of the problem (1)-(3) with the additional condition

$$u(t,x_0,y_0) = h(t), (x_0,y_0) \in \Omega, \quad t \in [0,T] \qquad (4)$$

is called inverse problem.

The inverse problems for the one-dimensional wave equation with different boundary conditions and space dependent coefficients are considered in [6-9]. The inverse problem for the one-dimensional wave equation with time dependent coefficient with integral condition is investigated in [10] and with non-classical boundary condition is studied in [11].

For a multidimensional hyperbolic equation local solvability of inverse Cauchy problem is studied in [12] and the problem of regularization of a solution to the Cauchy problem for a two-dimensional hyperbolic equation on the half-plane is studied in [13]. Under a weak regularity assumption, the uniqueness and stability of the solution of inverse problem of finding space-dependent potential in a multidimensional wave equation is established in [14]. More recently, the global uniqueness and stability in determining the solely space-dependent coefficient $p(x), (x \in \Omega, \Omega \subset \mathbb{R}^n, n = 1,2,3)$ from the extra data is studied in [15].

For the some numerical aspects of inverse initial-boundary value problems for the two and multidimensional wave equations are considered in [16-19], and [20].

It is important to note that the paper [21] which considers a multidimensional inverse boundary-value problem of recovering three solely time-dependent functions for a linear wave equation in a bounded domain and proves the existence and uniqueness theorem for the inverse problem in a suitable Banach space.

In this paper, we consider an inverse initial-boundary value problem for a two-dimensional wave equation. We transform the inverse problem (1)-(4) to a fixed-point system and prove the existence and uniqueness of a solution on a sufficiently small time interval by means of the contraction principle. The fixed-point system is presented via Fourier series. Such a form of the system brings along computations that are technically more simple than in the case of the usual Green's function approach. Moreover, we give the theorem of continuous dependence upon the data and numerical solution of the inverse problem by using finite difference method.

The article is organized as following: In Section 2, we present auxiliary spectral problem of this problem and its properties. In Section 3, the series expansion method in terms of eigenfunctions converts the inverse problem to a fixed point problem in a suitable Banach space. Under some consistency, regularity conditions on initial and boundary data the existence and uniqueness of the solution of the inverse problem is shown by the way that the fixed point problem has a unique solution for small T. Also, we characterize the estimation of conditional stability of the solution of inverse problem. In section 4, the inverse problem of finding time-dependent coefficient is studied by using the finite difference method and we present three numerical examples intended to illustrate the behaviour of the proposed methods and the tests are performed by using MATLAB. While the second example's data does not provide the conditions of the existence and uniqueness theorem, the first example's data provides the conditions. The third example's data satisfies the conditions of the theorem but the coefficient solution is not smooth.

2. Auxiliary Spectral Problem

Since the function a is space independent and the boundary conditions are linear and homogeneous, the method of separation of variables is suitable. The auxiliary spectral problem of the problem (1)-(3) is

$$\begin{cases} Z_{xx}(x,y) + Z_{yy}(x,y) + \mu Z(x,y) = 0, \\ Z(0,y) = Z(1,y) = 0, 0 \le y \le 1, \\ Z(x,0) = Z(x,1) = 0, 0 \le x \le 1. \end{cases} \quad (5)$$

Let us present the solution of (5) in the form

$$Z(x,y) = X(x)Y(y). \quad (6)$$

Substituting the expression (6) into (5), we obtain following two problems:

$$\begin{cases} Y''(y) + \lambda Y(y) = 0, \quad 0 < y < 1, \\ Y(0) = Y(1) = 0, \end{cases} \quad (7)$$

$$\begin{cases} X''(x) + \gamma X(x) = 0, \quad 0 < x < 1, \\ X(0) = X(1) = 0, \end{cases} \quad (8)$$

where $\gamma = \mu - \lambda$. It is easy to see that the solutions of the problems (7) and (8) have the form $\lambda_k = (\pi k)^2$, $Y(y) = \sqrt{2}\sin(\pi k y)$, $k = 1,2,\ldots$ and $\gamma_m = (\pi m)^2$, $X(x) = \sqrt{2}\sin(\pi m x)$, $m = 1,2,\ldots$, respectively.

Thus, the eigenvalues (or natural frequencies of the membrane) and corresponding eigenfunctions (or mode shape) of the problem (5) have the form

$$\mu_{mk} = \gamma_m + \lambda_k = (\pi m)^2 + (\pi k)^2,$$

$$Z_{mk}(x,y) = 2\sin(\pi m x)\sin(\pi k y), \quad k,m = 1,2,\ldots.$$

Note that the system $Z_{mk}(x,y), k,m = 1,2,\ldots$ is bi-orthonormal on Ω, i.e. for any admissible indices m,l,k and p

$$(Z_{mk}, Z_{lp}) = \iint_{\Omega} Z_{mk}(x,y) Z_{lp}(x,y)\,dxdy = \begin{cases} 1, & m = l, k = p \\ 0, & otherwise \end{cases}$$

Let us introduce the functional space

$$B_{2,T}^{\frac{3}{2}} = \left\{ u(t,x,y) = \sum_{m,k=1}^{\infty} u_{mk}(t) Z_{mk}(x,y) : u_{mk}(t) \in C[0,T], \; J_T(u) = \left[\sum_{m,k=1}^{\infty} \left(\mu_{mk}^{\frac{3}{2}} \max_{0\le t\le T} |u_{mk}(t)| \right)^2 \right]^{\frac{1}{2}} < +\infty \right\}$$

with the norm $\|u(t,x,y)\|_{B_{2,T}^{\frac{3}{2}}} \equiv J_T(u)$ which relates the Fourier coefficients of the function $u(t,x,y)$ by the eigenfunctions $Z_{mk}(x,y), m,k = 1,2,\ldots$. It is shown in [22] that $B_{2,T}^{\frac{3}{2}}$ is Banach space. Obviously for the couple $z = \{a(t), u(t,x,y)\}$, $E_T^{\frac{3}{2}} = B_{2,T}^{\frac{3}{2}} \times C[0,T]$ with the norm $\|z\|_{E_T^{\frac{3}{2}}} = \|u(t,x,y)\|_{B_{2,T}^{\frac{3}{2}}} + \|a(t)\|_{C[0,T]}$ is also Banach space.

3. Solution of the Inverse Problem

In this section, we will examine the existence and uniqueness of the solution of the inverse initial-boundary value problem for the equation (1) with time-dependent coefficient.

Definition 1. *The pair* $\{a(t), u(t,x,y)\}$ *from the class* $C[0,T] \times \left(C^2(\bar{D}_T)\right)$ *for which the conditions (1)-(4) are satisfied is called the classical solution of the inverse problem (1)-(4).*

From this definition, the consistency conditions

$$A_0 = \begin{cases} \varphi(0,y) = \varphi(1,y) = 0, \\ \psi(0,y) = \psi(1,y) = 0, \\ h(0) = \varphi(x_0,y_0), h'(0) = \psi(x_0,y_0) \end{cases}$$

holds for the data $\varphi(x,y), \psi(x,y) \in C^1(\bar{\Omega})$, and $h(t) \in C^1[0,T]$, with $h(t) \neq 0, \forall t \in [0,T]$. Moreover, we will use the following assumptions on the data of problem (1)-(4) to guarantee the convergence of the majorizing series which arise in the solution:

$$A_1 = \begin{cases} \varphi(0,y) = \varphi(1,y) = 0, \\ \varphi_{xx}(0,y) = \varphi_{xx}(1,y) = 0, \\ \varphi_{xxx}(x,0) = \varphi_{xxx}(x,1) = 0, \\ \varphi_{xxxyy}(x,0) = \varphi_{xxxyy}(x,1) = 0, \end{cases}$$

$$A_2 = \begin{cases} \psi(0,y) = \psi(1,y) = 0, \\ \psi_{xx}(x,0) = \psi_{xx}(x,1) = 0, \end{cases}$$

$$A_3 = \begin{cases} h(t) \in C^2[0,T],\, h(0) = \phi(x_0,y_0), \\ h'(0) = \psi(x_0,y_0),\, h(t) \neq 0, \forall t \in [0,T], \end{cases}$$

$$A_4 = \begin{cases} f(t,x,y) \in C(D_T), \\ f(t,x,y) \in C^2(\bar{\Omega}), \forall t \in [0,T] \\ f(t,0,y) = f(t,1,y) = 0, \\ f_{xx}(t,x,0) = f_{xx}(t,x,1) = 0. \end{cases},$$

Theorem 1 (Existence and uniqueness). *Let the assumptions $(A_0) - (A_4)$ be satisfied. Then, the inverse problem (1)-(4) has a unique solution for small T.*

Proof. Let $a(t), t \in [0,T]$ is an unknown function. Since the function $a(t)$ is solely time dependent, seeking the solution of the problem (1)-(3) in the following form is suitable:

$$u(t,x,y) = \sum_{m,k=1}^{\infty} u_{mk}(t) Z_{mk}(x,y) \qquad (9)$$

where $u_{mk}(t) = \iint_{\bar{\Omega}} u(t,x,y) Z_{mk}(x,y)dxdy$, $m,k = 1,2,\ldots$.

From the equation (1) and initial conditions (2), we obtain

$$\begin{cases} u_{mk}(t) + \mu_{mk} u_{mk}(t) = F_{mk}(t; a, u_{mk}), \\ u_{mk}(0) = \varphi_{mk}, u'_{mk}(0) = \psi_{mk}, \end{cases} \qquad (10)$$

Where $F_{mk}(t; a, u_{mk}) = a(t) u_{mk}(t) + f_{mk}(t)$, $f_{mk}(t) = \iint_{\bar{\Omega}} f(t,x,y) Z_{mk}(x,y)dxdy$, $\varphi_{mk} = \iint_{\bar{\Omega}} \varphi(x,y) Z_{mk}(x,y)dxdy$, $\psi_{mk} = \iint_{\bar{\Omega}} \psi(x,y) Z_{mk}(x,y)dxdy$ and $m,k = 1,2,\ldots$.

Solving the Cauchy problems (10), we get

$$u_{mk}(t) = \varphi_{mk} \cos(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \psi_{mk} \sin(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \int_0^t F_{mk}(\tau; a, u_{mk}) \sin(\sqrt{\mu_{mk}}\,(t - \tau)) d\tau. \qquad (11)$$

Substituting (11) into (9), the second component of the pair $\{a(t), u(t,x,y)\}$ is

$$u(t,x,y) = \sum_{m,k=1}^{\infty} \left[\varphi_{mk} \cos(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \psi_{mk} \sin(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \int_0^t F_{mk}(\tau; a, u_{mk}) \sin(\sqrt{\mu_{mk}}\,(t - \tau)) d\tau \right] Z_{mk}(x,y). \qquad (12)$$

Considering the over-determination condition (4), into the equation (1) we get

$$a(t) = \frac{1}{h(t)} [h''(t) - f(t,x_0,y_0) - u_{xx}(t,x_0,y_0) - u_{yy}(t,x_0,y_0)].$$

By using this equality and equation (12), we obtain the first component of the pair as

$$a(t) = \frac{1}{h(t)} \left[h''(t) - f(t,x_0,y_0) + \sum_{m,k=1}^{\infty} \mu_{mk} \left[\varphi_{mk} \cos(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \psi_{mk} \sin(\sqrt{\mu_{mk}}\, t) + \frac{1}{\sqrt{\mu_{mk}}} \int_0^t F_{mk}(\tau; a, u_{mk}) \sin(\sqrt{\mu_{mk}}\,(t - \tau)) d\tau \right] Z_{mk}(x_0,y_0) \right]. \qquad (13)$$

We get the equalities of the pair $\{a(t), u(t,x,y)\}$. Thus the solution of problem (1)-(4) is reduced to the solution of system of equations (12) and (13) with respect to the unknown functions $\{a(t), u(t,x,y)\}$. It follows that to prove the uniqueness of the solution of the problem (1)-(4) is equivalent to prove the uniqueness of the solution of system of equations (12) and (13).

Let us denote $z = [a(t), u(t,x,y)]^T$ and consider the operator equation

$$z = \Phi(z). \tag{14}$$

The operator Φ is determined in the set of functions z and has the form $[\phi_1, \phi_2]^T$, where

$$\phi_1(z) = \frac{1}{h(t)}\Bigg[h''(t) - f(t,x_0,y_0) + \sum_{m,k=1}^{\infty} \mu_{mk}\Bigg[\varphi_{mk}\, cos\left(\sqrt{\mu_{mk}}\, t\right) + \frac{1}{\sqrt{\mu_{mk}}}\psi_{mk}\, sin\left(\sqrt{\mu_{mk}}\, t\right) + \frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau; a, u_{mk})\, sin\left(\sqrt{\mu_{mk}}\,(t-\tau)\right)d\tau\Bigg] Z_{mk}(x_0,y_0).\Bigg] \tag{15}$$

$$\phi_2(z) = \sum_{m,k=1}^{\infty}\Bigg[\varphi_{mk}\, cos\left(\sqrt{\mu_{mk}}\, t\right) + \frac{1}{\sqrt{\mu_{mk}}}\psi_{mk}\, sin\left(\sqrt{\mu_{mk}}\, t\right) + \frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau; a, u_{mk})\, sin\left(\sqrt{\mu_{mk}}\,(t-\tau)\right)d\tau\Bigg] Z_{mk}(x,y). \tag{16}$$

Let us show that Φ maps $E_T^{\frac{3}{2}}$ onto itself continuously. In other words, we need to show $\phi_1(z) \in C[0,T]$ and $\phi_2(z) \in B_{2,T}^{\frac{3}{2}}$ for arbitrary $z = [a(t), u(t,x,y)]^T$ with $a(t) \in C[0,T]$, $u(t,x,y) \in B_{2,T}^{\frac{3}{2}}$.

Using integration by parts under the assumptions $(A_0) - (A_4)$, we can derive that

$$\varphi_{mk} = \frac{1}{(\pi m)^3(\pi k)^3}\iint_{\bar{\Omega}} \varphi_{xxxyyy}(x,y) cos(\pi m x) cos(\pi k y)\, dxdy,$$

$$\psi_{mk} = \frac{1}{(\pi m)^2(\pi k)^2}\iint_{\bar{\Omega}} \psi_{xxyy}(x,y)\, sin(\pi m x) sin(\pi k y)\, dxdy,$$

$$f_{mk}(t) = \frac{1}{(\pi m)^2(\pi k)^2}\iint_{\bar{\Omega}} f_{xxyy}(t,x,y)\, sin(\pi m x) sin(\pi k y)\, dxdy.$$

First, let us show that $\phi_1(z) \in C[0,T]$. Under the assumptions $(A_0) - (A_4)$, we obtain from (15)

$$|\phi_1(z)| \le \frac{1}{\min\limits_{0\le t\le T}|h(t)|}\Bigg[|h''(t)| + |f(t,x_0,y_0)| + \sum_{m,k=1}^{\infty}\Bigg(\frac{\mu_{mk}}{(\pi m)^3(\pi k)^3}|\alpha_{mk}| + \frac{\mu_{mk}}{(\pi m)^2(\pi k)^2}|\beta_{mk}| + \frac{T\sqrt{\mu_{mk}}}{(\pi m)^2(\pi k)^2}|\eta_{mk}(t)| + \frac{T|a(t)|}{\mu_{mk}}\mu_{mk}^{\frac{3}{2}}|u_{mk}(t)|\Bigg)\Bigg], \tag{17}$$

where

$$\alpha_{mk} = \iint_{\bar{\Omega}} \varphi_{xxxyyy}(x,y) cos(\pi m x) cos(\pi k y)\, dxdy,$$

$$\beta_{mk} = \iint_{\bar{\Omega}} \psi_{xxyy}(x,y)\, sin(\pi m x) sin(\pi k y)\, dxdy,$$

$$\eta_{mk}(t) = \iint_{\bar{\Omega}} f_{xxyy}(t,x,y)\, sin(\pi m x) sin(\pi k y)\, dxdy.$$

Since $u(t,x,y) \in B_{2,T}^{\frac{3}{2}}$, the majorizing series of (17) is convergent by using Cauchy-Schwartz inequality and Bessel inequality. This implies that by the Weierstrass-M test, the series (15) is uniformly convergent in $[0,T]$. Thus $\phi_1(z)$ is continuous in $[0,T]$.

Now, let us show that $\phi_2(z) \in B^{\frac{3}{2}}_{2,T}$, i.e. we need to show

$$J_T(\phi_2) = \left[\sum_{m,k=1}^{\infty}\left(\mu_{mk}^{\frac{3}{2}} \max_{0\le t\le T}\left|\phi_{2_{mk}}(t)\right|\right)^2\right]^{\frac{1}{2}} < +\infty$$

where

$$\phi_{2_{mk}}(t) = \varphi_{mk}\, cos\left(\sqrt{\mu_{mk}}\; t\right) + \frac{1}{\sqrt{\mu_{mk}}}\psi_{mk}\, sin\left(\sqrt{\mu_{mk}}\; t\right) + \frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau; a, u_{mk})\, sin\left(\sqrt{\mu_{mk}}\;(t-\tau)\right)d\tau$$

After some manipulations under the assumptions $(A_0)-(A_4)$, we get

$$\sum_{m,k=1}^{\infty}\left(\mu_{mk}^{\frac{3}{2}} \max_{0\le t\le T}\left|\phi_{2_{mk}}(t)\right|\right)^2 \le \frac{36}{\pi^6}\sum_{m,k=1}^{\infty}|\alpha_{mk}|^2 + \frac{16}{\pi^4}\sum_{m,k=1}^{\infty}|\beta_{mk}|^2 + \frac{16T^2}{\pi^4}\sum_{m,k=1}^{\infty}\left(\max_{0\le t\le T}|\eta_{mk}(t)|\right)^2 + \frac{\left(\max_{0\le t\le T}|a(t)|\right)^2 T^2}{2\pi^4}\sum_{m,k=1}^{\infty}\left(\mu_{mk}^{\frac{3}{2}} \max_{0\le t\le T}|u_{mk}(t)|\right)^2 \tag{18}$$

From the Bessel inequality and $\sum_{m,k=1}^{\infty}\left(\mu_{mk}^{\frac{3}{2}} \max_{0\le t\le T}|u_{mk}(t)|\right)^2 < +\infty$, series on the right side of (18) are convergent. Thus $J_T(\phi_2) < +\infty$ and ϕ_2 is belongs to the space $B^{\frac{3}{2}}_{2,T}$.

Now, let z_1 and z_2 be any two elements of $E^{\frac{3}{2}}_T$. We know that $\|\Phi(z_1) - \Phi(z_2)\|_{E^{\frac{3}{2}}_T} = \|\Phi_1(z_1) - \Phi_1(z_2)\|_{C[0,T\,]} + \|\Phi_2(z_1) - \Phi_2(z_2)\|_{B^{\frac{3}{2}}_{2,T}}$.

Here $z_i = \left[a^i(t), u^i(t,x,y)\right]^T, i = 1,2$.

Under the assumptions $(A_0)-(A_4)$ and considering (17)-(18), we obtain

$$\|\Phi(z_1) - \Phi(z_2)\|_{E^{\frac{3}{2}}_T} \le A(T)C(a^1,u^2)\|z_1 - z_2\|_{E^{\frac{3}{2}}_T}$$

where $A(T) = T\left(\frac{1}{\min_{0\le t\le T}|h(t)|} + \frac{1}{\pi\sqrt{2}}\right)$ and $C(a^1,u^2)$is the constant includes the norms of $\|a^1(t)\|_{C[0,T\,]}$ and $\|u^2(t,x,y)\|_{B^{\frac{3}{2}}_{2,T}}$.

For sufficiently small T such that $0 < A(T) < 1$, the operator Φ is contraction mapping which maps $E^{\frac{3}{2}}_T$ onto itself continuously. Then according to Banach fixed point theorem there exists a unique solution of (14). Thus, the inverse problem (1)-(4) has a unique classical solution $a(t) \in C[0,T\,]$, $u(t,x,y) \in B^{\frac{3}{2}}_{2,T}$.

3.1. Continuous dependence of the data

Now, let us investigate the stability of the solution of the inverse problem. Because of the presence of the term $a(t)u(t,x,y)$ in the equation (1), finding the pair of solution $\{a(t), u(t,x,y)\}$ of the inverse problem (1)-(4) is non-linear. Therefore we can not apply the standard stability criteria but we can characterize the estimation of conditional stability. Thus we can obtain a stability estimate under a priori assumption on the smallness of $a(t)$. Now, let us characterize the estimation of conditional stability of the solution of inverse problem. Such an estimate can be obtained by setting a certain class of data $\Im(\alpha, N_0, N_1, N_2, N_3)$ for the functions $\varphi(x,y), \psi(x,y), h(t), f(t,x,y)$ and a class $\aleph(M_0)$ for the functions $a(t)$ if they satisfy

$$\|f\|_{C(\bar{D}_T)} \le N_0, \|\varphi\|_{C^3(\bar{\Omega})} \le N_1, \|\psi\|_{C^2(\bar{\Omega})} \le N_2,$$

$$\|h\|_{C^2[0,T]} \le N_3, 0 < \alpha < |h(t)|,$$

and

$$\|a(t)\|_{C[0,T]} \le M_0,$$

respectively.

Since $\varphi, \psi, h, f \in \Im(\alpha, N_0, N_1, N_2, N_3)$ and $a(t) \in \aleph(M_0)$, we get the estimate

$$\|u(t,x,y)\|_{B^{\frac{3}{2}}_{2,T}} \le M_1$$

where $M_1 = \frac{2\sqrt{2}}{\sqrt{2}\pi^2 - 4TM_0}\left(2TN_0 + \frac{3N_1}{\pi} + 2N_2\right)$.

Let $\{a(t), u(t,x,y)\}$ and $\{\bar{a}(t), \bar{u}(t,x,y)\}$ be the solutions of (1)-(4) corresponding to data $\varphi(x,y)$, $\psi(x,y)$, $h(t)$, $f(t,x,y)$ and $\bar{\varphi}(x,y)$, $\bar{\psi}(x,y)$, $\bar{h}(t), \bar{f}(t,x,y)$ respectively. Then, we obtain from (12) and (13)

$u(t,x,y)-\bar{u}(t,x,y,)=$
$\sum_{m,k=1}^{\infty}\left(\left[\varphi_{mk}\cos\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\psi_{mk}\sin\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau;a,u_{mk})\sin\left(\sqrt{\mu_{mk}}\,(t-\tau)d\tau\right]-\left[\bar{\varphi}_{mk}\cos\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\bar{\psi}_{mk}\sin\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau;\bar{a},\bar{u}_{mk})\sin\left(\sqrt{\mu_{mk}}\,(t-\tau)d\tau\right]\right)Z_{mk}(x,y),$

and

$a(t)-\bar{a}(t)=\frac{1}{h(t)\bar{h}(t)}\left(\bar{h}(t)\left[h''(t)-f(t,x_0,y_0)+\sum_{m,k=1}^{\infty}\mu_{mk}\left[\varphi_{mk}\cos\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\psi_{mk}\sin\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau;a,u_{mk})\sin\left(\sqrt{\mu_{mk}}\,(t-\tau)d\tau\right]Z_{mk}(x_0,y_0)\right]-h(t)\left[\bar{h}''(t)-\bar{f}(t,x_0,y_0)+\sum_{m,k=1}^{\infty}\mu_{mk}\left[\bar{\varphi}_{mk}\cos\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\bar{\psi}_{mk}\sin\left(\sqrt{\mu_{mk}}\,t\right)+\frac{1}{\sqrt{\mu_{mk}}}\int_0^t F_{mk}(\tau;\bar{a},\bar{u}_{mk})\sin\left(\sqrt{\mu_{mk}}\,(t-\tau)d\tau\right]Z_{mk}(x_0,y_0)\right]\right).$

Denote the difference between two functions with the tilde $(\sim)$, i.e. $\tilde{a}=a-\bar{a},\tilde{u}=u-\bar{u}$, etc. Then, under the conditions $(A_0)-(A_4)$ by using the estimates given above we obtain

$$\|\tilde{a}(t)\|_{C[0,T]}\le\frac{D_1}{\Delta(T)}\left\{\|\tilde{h}\|_{C^2[0,T]}+\|\tilde{f}\|_{C(\bar{D}_T)}+\|\tilde{\varphi}\|_{C^3(\bar{\Omega})}+\|\tilde{\psi}\|_{C^2(\bar{\Omega})}\right\},\tag{19}$$

$$\|\tilde{u}(t,x,y)\|_{B^{\frac{3}{2}}_{2,T}}\le\frac{D_2}{\Delta(T)}\left\{\|\tilde{h}\|_{C^2[0,T]}+\|\tilde{f}\|_{C(\bar{D}_T)}+\|\tilde{\varphi}\|_{C^3(\bar{\Omega})}+\|\tilde{\psi}\|_{C^2(\bar{\Omega})}\right\},\tag{20}$$

where $\Delta(T)=d_1d_4-d_2d_3\neq0$, $d_1=1-\frac{T}{\pi\alpha^2\sqrt{2}}N_3M_1$, $d_2=\frac{T}{\pi\alpha^2\sqrt{2}}N_3M_0$, $d_3=\frac{T}{\pi\sqrt{2}}M_1$, $d_4=1-\frac{T}{\pi\sqrt{2}}M_0$ and D_1 , D_2 are constants depend only the parameters $\alpha,N_0,N_1,N_2,N_3,M_0$ and M_1.

Thus we proved the follwing theorem:

Theorem 2 (continuous dependence upon the data). *Let $\{a(t),u(t,x,y)\}$ and $\{\bar{a}(t),\bar{u}(t,x,y)\}$ be two solutions of the inverse problem (1)-(4) with the data $\varphi(x,y)$, $\psi(x,y)$, $h(t)$, $f(t,x,y)$ and $\bar{\varphi}(x,y)$, $\bar{\psi}(x,y)$, $\bar{h}(t)$, $\bar{f}(t,x,y)$, respectively, which are satisfied the conditions of the Theorem 1. Then the estimates (19)-(20) are true for small T . The constants D_1 and D_2 depend only on the choice of the classes $\Im(\alpha,N_0,N_1,N_2,N_3)$ and $\aleph(M_0,M_1)$.*

4. Numerical Method and Examples

In this section,we describe the numerical method applied to the inverse initial-boundary value problem (1)-(4). As in the one dimensional case, the standard and most natural difference method for the two dimensional wave equation is an explicit central finite difference approximation which is very fast and effective.

The discrete form of our problem is as follows: We divide the domain $[0,T]\times[0,1]\times[0,1]$ into nt,nx and ny subintervals of equal length $\Delta t,\Delta x$ and Δy, where $\Delta t=\frac{T}{nt}$, $\Delta x=\frac{1}{nx}$ and $\Delta y=\frac{1}{ny}$, respectively. We denote by $U_{i,j}^k:=U(t_k,x_i,y_j)$, $a:=a(t_k)$ and $f_{i,j}^k:=f(t_k,x_i,y_j)$, where $t_k=k\Delta t$, $x_i=i\Delta x$, $y_j=j\Delta y$ for $k\ 0,\dots,nt$, $i=0,\dots,nx$, and $j=0,\dots,ny$. Then, an explicit central finite difference approximation to the equation (1) at the mesh points (t_k,x_i,y_j) is

$$\frac{U_{i,j}^{k+1}-2U_{i,j}^k+U_{i,j}^{k-1}}{(\Delta t)^2}=\frac{U_{i+1,j}^k-2U_{i,j}^k+U_{i-1,j}^k}{(\Delta x)^2}+\frac{U_{i,j+1}^k-2U_{i,j}^k+U_{i,j-1}^k}{(\Delta y)^2}+a^kU_{i,j}^k+f_{i,j}^k.$$

If we let $r_x=\left(\frac{\Delta t}{\Delta x}\right)^2$ and $r_y=\left(\frac{\Delta t}{\Delta y}\right)^2$, we can rewrite the above discrete form as

$$U_{i,j}^{k+1}=2\left(1-(r_x+r_y)\right)U_{i,j}^k+r_x\left(U_{i+1,j}^k+U_{i-1,j}^k\right)+r_y\left(U_{i,j+1}^k+U_{i,j-1}^k\right)-U_{i,j}^{k-1}+(\Delta t)^2\left(a^kU_{i,j}^k+f_{i,j}^k\right),$$
$$k=1,\dots,nt-1,\ i=1,\dots,nx-1,\ j=1,\dots,ny-1.\tag{21}$$

Discretizing the initial and boundary conditions (2) and (3) we obtain

$$U_{i,j}^0=\varphi_{i,j},\ \frac{U_{i,j}^1-U_{i,j}^{-1}}{2\Delta t}=\psi_{i,j},\quad i=0,\dots,nx,\ j=0,\dots,ny\tag{22}$$

$$U_{0,j}^k = U_{nx,j}^k = U_{i,0}^k = U_{i,ny}^k = 0, \quad k = 0, \ldots, nt. \tag{23}$$

Considering (22) into (21), we derive the finite difference approximation for $k = 0$ as

$$U_{i,j}^1 = \frac{1}{2}\left[2\left(1 - (r_x + r_y)\right)\varphi_{i,j} + r_x(\varphi_{i+1,j} + \varphi_{i-1,j}) + r_y(\varphi_{i,j+1} + \varphi_{i,j-1}) + 2\Delta t\psi_{i,j} + (\Delta t)^2(a^0\varphi_{i,j} + f_{i,j}^0)\right], \ i = 1, \ldots, nx-1, \ j = 1, \ldots, ny-1. \tag{24}$$

If $a(t), 0 \le t \le T$ is known, the system (21)-(24) can be easily solved explicitly and has a second order accuracy in both $(\Delta t)^2$, $(\Delta x)^2$ and $(\Delta y)^2$. Moreover, the stability of this explicit finite difference method for multidimensional wave equation is dictated by the Courant–Friedrichs–Lewy (CFL) condition $r \le \frac{1}{\sqrt{s}}$ where s is the space dimensions of the wave equation, and $r = r_x = r_y$. In the two dimensional case, the explicit scheme is stable for $r \le \frac{1}{\sqrt{2}}$. Now, let us construct the mechanism for the coefficients $a(t)$. Consider (14) into the equation (1), we obtain

$$a(t) = \frac{1}{h(t)}[h''(t) - f(t, x_0, y_0) - u_{xx}(t, x_0, y_0) - u_{yy}(t, x_0, y_0)].$$

The finite difference approximation of this equation is

$$a^k = \frac{1}{h^k}\left[\frac{h^{k+1} - 2h^k + h^{k-1}}{(\Delta t)^2} - f_{x_0 i, y_0 j}^k - \frac{U_{x_0 i+1, y_0 j}^k - 2U_{x_0 i, y_0 j}^k + U_{x_0 i-1, y_0 j}^k}{(\Delta x)^2} - \frac{U_{x_0 i, y_0 j+1}^k - 2U_{x_0 i, y_0 j}^k + U_{x_0 i, y_0 j-1}^k}{(\Delta y)^2}\right], \ k = 1, \ldots, nt-1, \tag{25}$$

where $x_0 i$ and $y_0 j$ are the mesh points according to known x_0 and y_0, respectively. For $k = 0$ and $k = nt$,

$$a^0 = \frac{1}{h^0}\left[\frac{h^2 - 2h^1 + h^0}{(\Delta t)^2} - f_{x_0 i, y_0 j}^0 - \frac{\varphi_{x_0 i+1, y_0 j} - 2\varphi_{x_0 i, y_0 j} + \varphi_{x_0 i-1, y_0 j}}{(\Delta x)^2} - \frac{\varphi_{x_0 i, y_0 j+1} - 2\varphi_{x_0 i, y_0 j} + \varphi_{x_0 i, y_0 j-1}}{(\Delta y)^2}\right], \tag{26}$$

$$a^{nt} = \frac{1}{h^{nt}}\left[\frac{h^{nt} - 2h^{nt-1} + h^{nt-2}}{(\Delta t)^2} - f_{x_0 i, y_0 j}^{nt} - \frac{U_{x_0 i+1, y_0 j}^{nt} - 2U_{x_0 i, y_0 j}^{nt} + U_{x_0 i-1, y_0 j}^{nt}}{(\Delta x)^2} - \frac{U_{x_0 i, y_0 j+1}^{nt} - 2U_{x_0 i, y_0 j}^{nt} + U_{x_0 i, y_0 j-1}^{nt}}{(\Delta y)^2}\right]. \tag{27}$$

Now let us consider (25) with the conditions (26)-(27) in the system (21)-(24), we obtain the system with respect to $U_{i,j}^k$, $k = 0, \ldots, nt$, $i = 0, \ldots, nx$, $j = 0, \ldots, ny$ which can be solved explicitly. Then using the calculated values of $U_{x_0 i, y_0 j}^k$ in (25), we obtain the values of a^k, $k = 0, \ldots, nt$.

Numerical examples for the inverse problem are presented below. We also calculate the absolute error (ae) to analyse the error between the exact and numerically obtained solution $u(t, x, y)$, and it is defined as $ae(u(t, x, y)) = |u_{num} - u_{exact}|$.

In the following examples we take $(x_0, y_0) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $T = 1$.

Example 1. Consider the inverse initial-boundary value problem (1)-(4) with the input data $\varphi(x, y) = sin(\pi x)sin(\pi y)$, $\psi(x, y) = sin(\pi x)sin(\pi y)$, $h(t) = e^t$, $f(t, x, y) = (1 + 2\pi^2)sin(\pi x)sin(\pi y)e^t - sin(\pi x)sin(\pi y)$, $(t, x, y) \in [0,1] \times [0,1] \times [0,1]$.

The data $\varphi(x, y) \in C^3(\bar{\Omega})$, $\psi(x, y) \in C^2(\bar{\Omega})$, $h(t) \in C^2[0, T]$, and $f(t, x, y) \in C(D_T)$, $f(t, x, y) \in C^2(\bar{\Omega}), \forall t \in [0, T]$ satisfy the conditions $(A_0) - (A_4)$. Hence, according to the Theorem 1 the solution of the inverse problem exists and unique. In fact, using the direct substitution the exact solution of the inverse problem (1)-(4) is given by

$$\{a(t), u(t, x, y)\} = \{e^{-t}, sin(\pi x)sin(\pi y)e^t\}$$

with $a(t) \in C[0,T]$, and $u(t,x,y) \in B^{\frac{3}{2}}_{2,T}$. Figure 1 and Figure 2 show the exact and numerical solutions of $\{a(t), u(t,x,y)\}$ for $nx = ny = 50$ and $nt = 50\sqrt{2}$ to satisfy CFL condition, respectively. It is seen from these figures that the exact and inverse numerical solutions are in a good agreement.

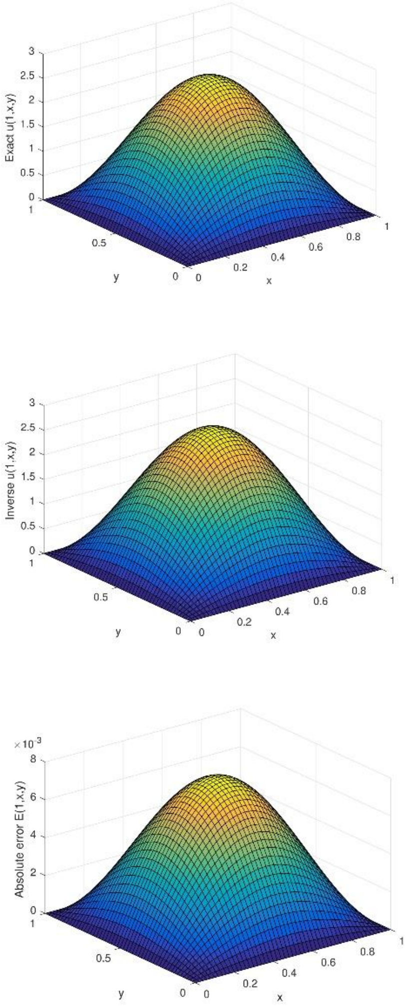

Figure 1. Exact and inverse numerical solutions of $u(t,x,y)$ at $t = 1$ and the absolute error for the direct and inverse numerical solutions for Example 1.

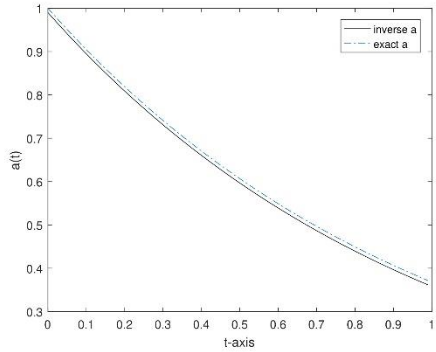

Figure 2. Exact and inverse numerical solutions of $a(t)$ for Example 1.

Example 2. In previous example, we consider the inverse problem where that $\varphi(x,y) \in C^3(\overline{\Omega})$, $\psi(x,y) \in C^2(\overline{\Omega})$, $h(t) \in C^2[0,T]$, and $f(t,x,y) \in C(D_T), f(t,x,y) \in C^2(\overline{\Omega}), \forall t \in [0,T]$ satisfy the conditions $(A_0)-(A_4)$. Now, consider the inverse initial-boundary value problem (1)-(4) with the input data $\varphi(x,y) = (x^2 - x)(y^2 - y)$, $\psi(x,y) = (x^2-x)(y^2-y)$, $h(t) = \frac{e^{\frac{t}{2}}}{16}$, $f(t,x,y) = \left(\frac{1}{4}(x^2-x)(y^2-y) - 2(x^2+y^2-x-y)\right)e^{t/2} - (x^2-x)(y^2-y)$, $(t,x,y) \in [0,1] \times [0,1] \times [0,1]$.

From the second derivative of the initial data we obtain $\varphi_{xx}(x,y) = \psi_{xx}(x,y) = 2(y^2-y)$ and $\varphi_{xx}(0,y), \varphi_{xx}(1,y) \neq 0$. Thus, the condition (A_1) is not satisfied. As the condition of Theorem 1 is not satisfied we can not conclude the unique solvability of the inverse problem. However, the solution at least exists and is given by $\{a(t), u(t,x,y)\} = \{e^{-t/2}, (x^2-x)(y^2-y)e^{t/2}\}$ which can be checked by direct substitution. Figure 3 and Figure 4 show the exact and numerical solutions of $\{a(t), u(t,x,y)\}$ for $nt = nx = 100$, respectively. Although the existence and uniqueness theorem is not satisfied, we can conclude from these figures that convergent exact and inverse numerical solutions are obtained.

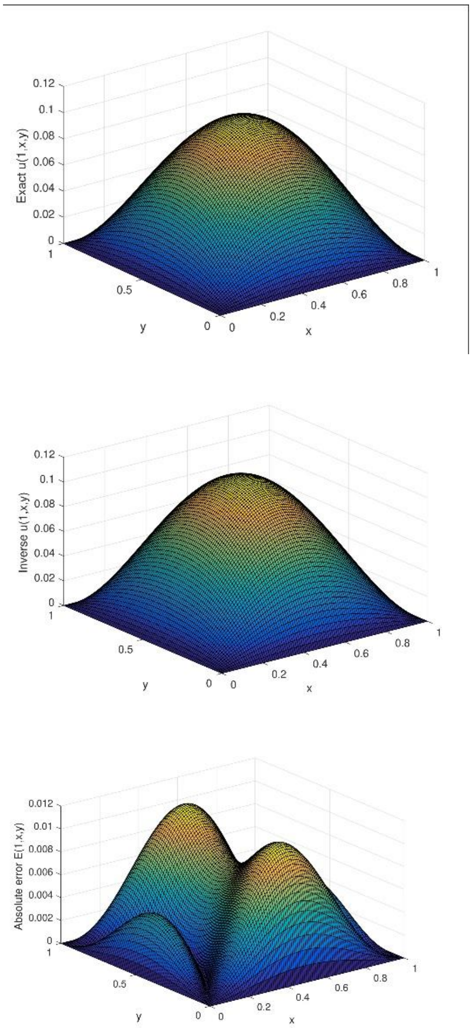

Figure 3. Exact and inverse numerical solutions of $u(t, x, y)$ at $t = 1$ and the absolute error for the direct and inverse numerical solutions for Example 2.

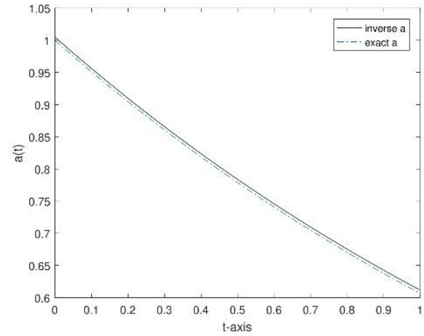

Figure 4. Exact and inverse numerical solutions of $a(t)$ for Example 2.

Example 3. In Example 1 and Example 2, we consider the inverse problem where the solution $a(t)$ is smooth. Now, consider the example with the data which derive a non-smooth coefficient. Consider the inverse initial-boundary value problem (1)-(4) with the input data $\varphi(x, y) = sin(\pi x)sin(\pi y)$, $\psi(x, y) = sin(\pi x)sin(\pi y)$, $h(t) = e^t$, $f(t, x, y) = \left(1 + 2\pi^2 - \frac{1}{\left|t-\frac{1}{2}\right|+1}\right) sin(\pi x)sin(\pi y)e^t$, $(t, x, y) \in [0,1] \times [0,1] \times [0, 1]$. Obviously, $\varphi(x, y) \in C^3(\bar{\Omega})$, $\psi(x, y) \in C^2(\bar{\Omega})$, $h(t) \in C^2[0, T\,]$, and $f(t, x, y) \in C(D_T)$, $f(t, x, y) \in C^2(\bar{\Omega}), \forall t \in [0, T\,]$ satisfy the conditions $(A_0) - (A_4)$. Then the exact solution of the problem (1)-(4) is

$$\{a(t), u(t, x, y)\} = \left\{\left|t - \frac{1}{2}\right| + 1,\; sin(\pi x)sin(\pi y)e^t\right\}$$

The corresponding exact and numerical non-smooth coefficient solution $a(t)$ of the problem (1)-(4) is presented in Figure 5. From this figure it can be seen that the recovered coefficient is in very good agreement with their corresponding exact solution. Since the numerical performance of the smooth coefficients is shown in previous example, we present the numerical performance for non-smooth coefficient $a(t)$.

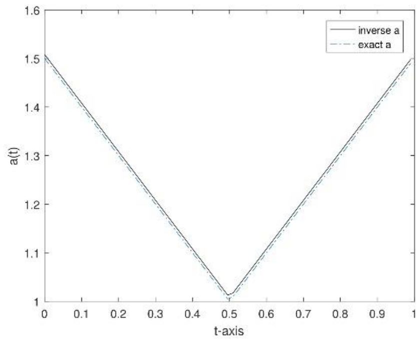

Figure 5. Exact and inverse numerical solutions of non-smooth $a(t)$ for Example 3.

5. Discussion and Conclusion

The paper considers the inverse problem of recovering a time-dependent potential in an initial-boundary value problem for a two dimensional wave equation. The series expansion method in terms of eigenfunction of a Sturm-Liouville problem converts the considered inverse problem to a fixed point problem in a suitable Banach space. Under some consistency and regularity conditions on initial and boundary data, the existence and uniqueness of the solution of inverse problem is shown by using the Banach fixed point theorem and conditional stability of the solution of the inverse problem is shown in a certain class of data. For the numerical solution of inverse problem finite difference approximations of the second order derivatives appearing in the numerical schemes are used. It is important to note that by using finite differences method we can solve the inverse problem which does not satisfy the theoretical conditions. The presented numerical examples for the inverse problem are solved accurately.

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