Matematik / *Mathematics* **DOI: 10.21597/jist.686462**

Araştırma Makalesi / *Research Article*

Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 10(2): 1247-1253, 2020
Journal of the Institute of Science and Technology, 10(2): 1247-1253, 2020
ISSN: 2146-0574, eISSN: 2536-4618

Fekete-Szegö Inequality for (p,q)-Starlike and (p,q)-Convex Functions of Complex Order

Feyza YATKIN[1], Ekrem KADIOĞLU[2*]

ABSTRACT: We have investigated Fekete-Szegö inequality in the classes of (p,q)-starlike and (p,q)-convex functions of complex order defined in the disc $U=\{z\in\mathbb{C}:|z|<1\}$. Our main theorems are also a generalization of the result obtained.

Keywords: Fekete-Szegö Inequality, (p,q)-Starlike Functions, (p,q)-Convex Functions, Complex Order

[1] Feyza YATKIN (**Orcid ID:** 0000-0002-9707-9424), Ataturk University, Institute of Science and Technology, Department of Mathematics, Erzurum, Turkey
[2] Ekrem KADIOĞLU (**Orcid ID:** 0000-0002-0039-4939), Atatürk University, Faculty of Science, Department of Mathematics, Erzurum, Turkey

*Sorumlu Yazar/Corresponding Author: Ekrem KADIOĞLU, e-mail: ekrem@atauni.edu.tr

* This study was produced from the ongoing PhD Thesis studies of Feyza YATKIN

Geliş tarihi / *Received:* 07-02-2020
Kabul tarihi / *Accepted:* 08-03-2020

INTRODUCTION

An application of q-calculus was firstly studied by Jakson in 1908 (Jackson, 1908). (p,q)-calculus was defined as a generelization of q-calculus. The (p,q)-integer was worked by Chakrabarti and Jagannathan in 1991 (Chakrabarti and Jaganathan, 1991). (p,q)-calculus is also recently studied in Geometric Function Theory (Seoudy and Aouf, 2016; Srivastava at al., 2019; Uçar, 2016). We have studied Fekete-Szegö inequality for (p,q)-starlike and (p,q)-convex functions of complex order.

MATERIALS AND METHODS

Let $\mathcal{A}$ be the class of functions which are the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

and analytic in disc $U = \{z \in \mathbb{C}: |z| < 1\}$. The function f is said to be subordinate to g, and denoted $f \prec g$ or $f(z) \prec g(z)$, if there exists a function w analytic in U and w provides the conditions $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = g(w(z))$ where f and g be analytic in U (Miller and Mocanu, 2000). This function w is called the Schwarz function.

Let $0 < q < p \leq 1$. Let $D_{p,q}f$ be the (p,q)-derivative of a function f and we define by

$$(D_{p,q}f)(z) = \frac{f(pz)-f(qz)}{pz-qz} \qquad (z \neq 0)$$

and

$$(D_{p,q}f)(0) = f'(0) \ \ (p = 1, q \to 1^-)$$

if f is differentiable at 0 where f' is the ordinary derivative (Jagannathan and Rao, 2006; Acar at al., 2016). It is also defined as $D_{p,q}^2 f(z) = D_{p,q}\left(D_{p,q}f(z)\right)$.

The following relation exists between the ordinary derivative f' and the (p,q)-derivative

$$f'(z) = \lim_{q\to 1^-}(D_{1,q}f)(z).$$

by an easy calculation we have

$$D_{p,q}(z^n) = [n]_{p,q} z^{n-1}$$

where

$$[n]_{p,q} = \frac{p^n - q^n}{p-q}.$$

Hence we can write

$$D_{p,q}f(z) = 1 + \sum_{n=2}^{\infty}[n]_{p,q} a_n z^{n-1}$$

such that $f(z) \in \mathcal{A}$.

The classes of q-starlike and q-convex functions, respectively, are defined by using the subordination priciple as

$$\mathcal{S}_q^*(\phi)=\left\{f\epsilon\mathcal{A}:\ \frac{zD_qf(z)}{f(z)}\prec\phi(z),z\epsilon U\right\},\qquad (0<q<1)$$

$$\mathcal{C}_q(\phi)=\left\{f\epsilon\mathcal{A}:\frac{D_q\big(zD_qf(z)\big)}{D_qf(z)}\prec\phi(z),z\epsilon U\right\}\qquad (0<q<1)$$

where the function $\phi(z)$ is analytic in U with Re $\phi(z)>0,\phi(0)=1$ and $\phi'(0)>0$ (Cetinkaya at al., 2018). On the other hand the classes of q-starlike and q-convex functions of complex order, respectively, are defined by using the subordination priciple as

$$\mathcal{S}_{q,b}(\phi)=\left\{f\epsilon\mathcal{A}:1+\frac{1}{b}\left[\frac{zD_qf(z)}{f(z)}-1\right]\prec\phi(z),z\epsilon U\right\}\qquad (0<q<1;\ b\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})$$

$$\mathcal{C}_{q,b}(\phi)=\left\{f\epsilon\mathcal{A}:1+\frac{1}{b}\left[\frac{D_q\left(zD_qf(z)\right)}{D_qf(z)}-1\right]\prec\phi(z),z\epsilon U\right\}\quad (0<q<1;\ b\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})$$

where the function $\phi(z)$ is analytic and univalent in U and $\phi(U)$ is convex with Re $\phi(z)>0,\phi(0)=1$, $\phi'(0)>0$ (Seoudy and Aouf, 2016).

RESULTS AND DISCUSSION

We define the classes of (p,q)-starlike and (p,q)-convex functions of complex order by using the (p,q)-derivative and subordination principle.

Definition 1. The class of (p,q)-starlike functions of complex order which denoted by $\mathcal{S}_{p,q}^b(\phi)$ is defined by

$$\mathcal{S}_{p,q}^b(\phi)=\left\{f\epsilon\mathcal{A}:1+\frac{1}{b}\left[\frac{zD_{p,q}f(z)}{f(z)}-1\right]\prec\phi(z),z\epsilon U\right\}\qquad (0<q<p\leq 1;\ b\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})$$

where the function $\phi(z)$ is analytic and univalent in U and $\phi(U)$ is convex with Re $\phi(z)>0,\phi(0)=1$, $\phi'(0)>0$.

Definition 2. The class of (p,q)-convex functions of complex order which denoted by $\mathcal{C}_{p,q}^b(\phi)$ is defined by

$$\mathcal{C}_{p,q}^b(\phi)=\left\{f\epsilon\mathcal{A}:1+\frac{1}{b}\left[\frac{D_{p,q}\big(zD_{p,q}f(z)\big)}{D_{p,q}f(z)}-1\right]\prec\phi(z),z\epsilon U\right\}\ (0<q<p\leq 1;\ b\in\mathbb{C}^*=\mathbb{C}\backslash\{0\})$$

where the function $\phi(z)$ is analytic and univalent in U and $\phi(U)$ is convex with Re $\phi(z)>0,\phi(0)=1$, $\phi'(0)>0$.

Now let's give two lemma which we use to prove our theorems:

Lemma 3. If $p(z)=1+c_1z+c_2z^2+\cdots$ is a function such that Re $p(z)>0$ in U and $\mu\in\mathbb{C}$ then

$$|c_2-\mu c_1^2|\leq 2\max\{1;|2\mu-1|\}.$$

The result sharp for

$$p(z)=\frac{1+z^2}{1-z^2}\ \ ve\ \ p(z)=\frac{1+z}{1-z}$$

(Ma and Minda, 1992).

Theorem 4. Let $\phi(z)=1+B_1z+B_2z^2+\cdots$ such that $B_1\neq 0$. If $f\in\mathcal{S}_{p,q}^b(\phi)$, then

$$|a_3-\mu a_2^2|\le\frac{|B_1b|}{[3]_{p,q}-1}\max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1b}{[2]_{p,q}-1}\left(1-\frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right|\right\} \tag{2.1}$$

where μ is a complex number.

Proof: If $f\in\mathcal{S}_{p,q}^b(\phi)$, then there is a Schwarz function w such that

$$1+\frac{1}{b}\left[\frac{zD_{p,q}f(z)}{f(z)}-1\right]=\phi\big(w(z)\big). \tag{2.2}$$

Let define the function $p(z)$ as

$$p(z)=\frac{1+w(z)}{1-w(z)}=1+c_1z+c_2z^2+\cdots. \tag{2.3}$$

Since $w(z)$ is a Schwarz function, we have that Re $p(z)>0$ and $p(0)=1$. Therefore, we have

$$\phi\big(w(z)\big)=\phi\left(\frac{p(z)-1}{p(z)+1}\right)$$

$$=\ \phi\left(\frac{1}{2}\left[c_1z+\left(c_2-\frac{c_1^2}{2}\right)z^2+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)z^3\right]+\cdots\right)$$

$$=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\cdots. \tag{2.4}$$

Now using (2.4) in (2.2), we have

$$1+\frac{1}{b}\left[\frac{zD_{p,q}f(z)}{f(z)}-1\right]=1+\frac{B_1c_1}{2}z+\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]z^2+\cdots.$$

From this equation, we can write

$$\frac{[2]_{p,q}-1}{b}a_2=\frac{B_1c_1}{2}$$

$$\frac{[3]_{p,q}-1}{b}a_3-\frac{[2]_{p,q}-1}{b}a_2^2=\frac{B_1c_2}{2}-\frac{B_1c_1^2}{4}+\frac{B_2c_1^2}{4}$$

or

$$a_2=\frac{B_1c_1b}{2\big([2]_{p,q}-1\big)}$$

$$a_3 = \frac{B_1 b}{2\left([3]_{p,q}-1\right)}\left\{c_2 - \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q}-1}\right]c_1^2\right\}.$$

Considering the complex number μ we have

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2\left([3]_{p,q}-1\right)}\{c_2 - v c_1^2\}, \tag{2.5}$$

where

$$v = \frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right]. \tag{2.6}$$

By application of Lemma 3, we get

$$|a_3 - \mu a_2^2| = \frac{|B_1 b|}{2\left([3]_{p,q}-1\right)}|c_2 - v c_1^2|$$

$$\leq \frac{|B_1 b|}{2\left([3]_{p,q}-1\right)}.2\max\{1; |2v-1|\}$$

$$= \frac{|B_1 b|}{\left([3]_{p,q}-1\right)}\max\left\{1; \left|2.\left(\frac{1}{2}\left[1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right]\right) - 1\right|\right\}$$

$$= \frac{|B_1 b|}{\left([3]_{p,q}-1\right)}\max\left\{1; \left|\frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}-1}{[2]_{p,q}-1}\mu\right)\right|\right\}.$$

Theorem 5. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ such that $B_1 \neq 0$. If $f \in \mathcal{C}_{p,q}^b(\phi)$, then

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|B_1 b|}{[3]_{p,q}\left([3]_{p,q}-1\right)}\max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}\left([3]_{p,q}-1\right)}{\left([2]_{p,q}\right)^2\left([2]_{p,q}-1\right)}\mu\right)\right|\right\} \tag{2.7}$$

where μ is a complex number.

Proof: If $f \in \mathcal{C}_{p,q}^b(\phi)$ then there is a Schwarz function w such that

$$1 + \frac{1}{b}\left[\frac{D_{p,q}\left(zD_{p,q}f(z)\right)}{D_{p,q}f(z)} - 1\right] = \phi\left(w(z)\right). \tag{2.8}$$

Let's define the function $p(z)$ as

$$p(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots \tag{2.9}$$

Since $w(z)$ is a Schwarz function, we see that Re $p(z) > 0$ and $p(0) = 1$. Therefore, we have

$$\phi\left(w(z)\right) = \phi\left(\frac{p(z)-1}{p(z)+1}\right)$$

$$= \phi\left(\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3\right] + \cdots\right)$$

$$= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots. \tag{2.10}$$

Now using (2.8), (2.9) and (2.10), we have

$$1 + \frac{1}{b}\left[\frac{D_{p,q}\left(zD_{p,q}f(z)\right)}{D_{p,q}\left(f(z)\right)} - 1\right] = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots$$

or

$$1 + \frac{[2]_{p,q}\left([2]_{p,q} - 1\right)}{b} a_2 z + \frac{[3]_{p,q}\left([3]_{p,q} - 1\right)a_3 - {[2]_{p,q}}^2\left([2]_{p,q} - 1\right)a_2^2}{b} z^2 + \cdots$$
$$= 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right] z^2 + \cdots.$$

From this equation, we can write

$$\frac{[2]_{p,q}\left([2]_{p,q} - 1\right)}{b} a_2 = \frac{B_1 c_1}{2}$$

$$\frac{[3]_{p,q}\left([3]_{p,q} - 1\right)a_3 - {[2]_{p,q}}^2\left([2]_{p,q} - 1\right)a_2^2}{b} = \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}$$

or

$$a_2 = \frac{B_1 c_1 b}{2[2]_{p,q}\left([2]_{p,q} - 1\right)}$$

$$a_3 = \frac{B_1 b}{2[3]_{p,q}\left([3]_{p,q} - 1\right)}\left\{c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q} - 1}\right)c_1^2\right\}.$$

Considering the complex number μ we have

$$a_3 - \mu a_2^2 = \frac{B_1 b}{2[3]_{p,q}\left([3]_{p,q} - 1\right)}\left\{c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q} - 1}\right)c_1^2\right\} - \mu \frac{B_1^2 c_1^2 b^2}{4{[2]_{p,q}}^2\left([2]_{p,q} - 1\right)^2}$$

$$= \frac{B_1 b}{2[3]_{p,q}\left([3]_{p,q} - 1\right)}\left\{c_2 - \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q} - 1}\left[1 - \mu \frac{[3]_{p,q}\left([3]_{p,q} - 1\right)}{{[2]_{p,q}}^2\left([2]_{p,q} - 1\right)}\right]\right)c_1^2\right\}.$$

By application of Lemma 3, we get

$$|a_3 - \mu a_2^2| = \frac{|B_1 b|}{2[3]_{p,q}\left([3]_{p,q} - 1\right)}|c_2 - v c_1^2|$$

$$\leq \frac{|B_1 b|}{2[3]_{p,q}\left([3]_{p,q}-1\right)} 2\max\{1; |2v-1|\}$$

$$= \frac{|B_1 b|}{[3]_{p,q}\left([3]_{p,q}-1\right)} \max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1 b}{[2]_{p,q}-1}\left(1 - \frac{[3]_{p,q}\left([3]_{p,q}-1\right)}{[2]_{p,q}{}^2\left([2]_{p,q}-1\right)}\mu\right)\right|\right\}$$

where

$$v = \frac{1}{2}\left(1 - \frac{B_2}{B_1} - \frac{B_1 b}{[2]_{p,q}-1}\left[1 - \mu \frac{[3]_{p,q}\left([3]_{p,q}-1\right)}{[2]_{p,q}{}^2\left([2]_{p,q}-1\right)}\right]\right).$$

Remark 1. If we take $b = 1$ in Theorem 4, then we have Theorem 2.1 given by Srivastava at al. (Srivastava at al., 2019).

Remark 2. If we take $p = 1$ in Theorem 4, then we have Theorem 1 given by Seoudy and Aouf (Seoudy and Aouf, 2016).

Remark 3. If we take $b = 1$ in Theorem 5, then we have Theorem 2.2 given by Srivastava at al. (Srivastava at al., 2019).

Remark 4. If we take $p = 1$ in Theorem 5, then we have Theorem 2 given by Seoudy and Aouf (Seoudy and Aouf, 2016).

CONCLUSION

For (p,q)-starlike and (p,q)-convex functions of complex order the Fekete-Szegö inequality investigated. The results obtained in this study generalize some of the previously obtained results.

REFERENCES

Acar T, Aral A, Mohiuddine SA, 2016. On Kantorovich modification of *(p, q)*-Baskakov operators. J. Inequal. Appl. 13 pages. DOI: http://dx.doi.org/10.1186/s13660-016-1045-9.

Cetinkaya A, Kahramaner Y, Polatoglu Y, 2018. Fekete-Szegö inequalities for q-starlike and q-convex functions. Acta Univ. Apulensis (53): 55-64.

Chakrabarti R, Jagannathan R, 1991. A (p,q)-oscillator realization of two-parameter quantum algebras. J. Phys. A: Math. Gen. 24(13): 711-718.

Jackson FH, 1908. On q-functions and a certain difference operator. Trans. Royal Soc. Edinburgh. (46): 253-281.

Jagannathan R, Rao KS, 2006. Two-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series. arXiv:math/0602613v1: 1-16.

Ma WC, Minda D, 1992. A unified treatment of some special classes of univalent functions. Proc. Conf. On Complex Analysis, 157-169.

Miller SS, Mocanu PT, 2000. Differential Subordinations: Theory and Applications, Marcel Dekker, New York, USA.

Seoudy TM, Aouf MK, 2016. Coefficient estimates of new classes of q-starlike and q-convex functions of complex order. J. Math. Inequal. 10(1): 135-145.

Srivastava HM, Raza N, Abujarad ESA, Srivastava G, Abujarad MH, 2019. Fekete-Szegö inequality for classes of (p,q)-Starlike and (p,q)-Convex functions. RASCAM (113): 3563-3584.

Uçar HEÖ, 2016. Coefficient inequality for q-starlike functions. Appl. Math. Comput. (276): 122-126.