JOURNAL OF SCIENCE
SAKARYA UNIVERSITY

Sakarya University Journal of Science
ISSN 1301-4048 | e-ISSN 2147-835X | Period Bimonthly | Founded: 1997 | Publisher Sakarya University |
http://www.saujs.sakarya.edu.tr/en/

Title: Spectral Analysis of Non-selfadjoint Second Order Difference Equation with Operator Coefficient

Authors: Gökhan MUTLU, Esra KIR ARPAT
Recieved: 2019-10-01 12:41:15

Accepted: 2020-03-20 12:52:18

Article Type: Research Article
Volume: 24
Issue: 3
Month: June
Year: 2020
Pages: 494-500

How to cite
Gökhan MUTLU, Esra KIR ARPAT; (2020), Spectral Analysis of Non-selfadjoint Second Order Difference Equation with Operator Coefficient. Sakarya University Journal of Science, 24(3), 494-500, DOI:
https://doi.org/10.16984/saufenbilder.627496
Access link
http://www.saujs.sakarya.edu.tr/en/issue/52472/627496

Sakarya University Journal of Science 24(3), 494-500, 2020

Spectral Analysis of Non-selfadjoint Second Order Difference Equation with Operator Coefficient

Gökhan MUTLU*1, Esra KIR ARPAT2

Abstract

In this paper, we consider the discrete Sturm-Liouville operator generated by second order difference equation with non-selfadjoint operator coefficient. This operator is the discrete analogue of the Sturm-Liouville differential operator generated by Sturm-Liouville operator equation which has been studied in detail. We find the Jost solution of this operator and examine its asymptotic and analytical properties. Then, we find the continuous spectrum, the point spectrum and the set of spectral singularities of this discrete operator. We finally prove that this operator has a finite number of eigenvalues and spectral singularities under a specific condition.

Keywords: Sturm-Liouville's operator equation, Non-selfadjoint operators, Discrete operators, Continuous spectrum, Operator coefficients.

1. INTRODUCTION

Difference equations are very important for modelling certain problems in physics, biology, economics, engineering, control theory etc. Spectral analysis of certain difference equations gives us useful information about these problems.

Let us give some literature on the spectral analysis of non-selfadjoint operators and the concept of spectral singularities. Spectral analysis of non-selfadjoint Sturm-Liouville operator has begun and the spectral singularities was discovered by Naimark [1-2]. Spectral singularities of differential operators [3-4] and certain classes of abstract operators [5] are studied.

Recently, non-Hermitian Hamiltonians and complex extension of quantum mechanics have been studied extensively (see review papers [6-7]). Moreover, the spectral singularities are identified for some concrete complex scattering potentials and some physical interpretations are suggested [8-9]. In [9], the authors identify the spectral singularities of complex scattering potentials with the real energies at which the reflection and transmission coefficients tend to

*Corresponding Author
1Gazi University, Faculty of Science, Department of Mathematics, Ankara-Turkey. gmutlu@gazi.edu.tr ORCID: 0000-0002-0674- 2908
2Gazi University, Faculty of Science, Department of Mathematics, Ankara-Turkey. esrakir@gazi.edu.tr ORCID: 0000-0002-6322-5130

infinity, i.e., they correspond to resonances having a zero width.

Spectral analysis of selfadjoint difference equations has been studied by many authors (see [10-11] for review and references). Further, spectral analysis of the selfadjoint differential and difference equations with matrix coefficients has been investigated [12-16]. In [17-19], the authors investigated the difference equation

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N}, \quad (1)$$

where $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are complex sequences such that an $a_n \neq 0$ for all $n \in \mathbb{N}$ and the condition

$$\sum_{n=1}^{\infty} n(|1-a_n| + |b_n|) < \infty \quad (2)$$

holds. We can refer to Equation (1) as the Sturm-Liouville difference equation since it can be rewritten

$$\Delta(a_{n-1}\Delta y_{n-1}) + q_n y_n = \lambda y_n,\ n \in \mathbb{N},$$

where $q_n = a_{n-1} + a_n + b_n$ and Δ denotes the forward difference operator.

In [20-21], the authors considered the case $n \in \mathbb{Z}$ with the analogous condition to (2). Further, the spectral analysis of the Sturm-Liouville difference equation with finite dimensional non-Hermitian matrix coefficients has been done [22].

Although there are many studies on spectral properties of the Sturm-Liouville difference equation with scalar or finite dimensional matrix coefficients, there isn't any study when the coefficients are infinite dimensional operators. In scalar or matrix coefficient cases, the discrete spectrum and spectral singularities are obtained as zeros of Jost function by using the results about analytic scalar functions. The infinite dimensional case requires a different treatment and new methods since the Jost function is an operator function on the contrary to finite dimensional case. The new method is due to Keldysh [23] which gives the fundamental tools to examine the singular points of analytic operator functions.

We consider the following difference operator defined in the Hilbert space $H_1 := l_2(\mathbb{N}, H)$ of vector sequences $y = (y_n)_{n\in\mathbb{N}}$ $(y_n \epsilon H)$ such that

$$\sum_{n=1}^{\infty} \|y_n\|_H^2 < \infty,$$

where H is a separable Hilbert space $(dimH \leq \infty)$.

Let us denote the difference operator L defined in H_1;

$$l(y)_n := A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1},\ n \in \mathbb{N}, \quad (3)$$

$$y_0 = 0, \quad (4)$$

where A_n $(n \in \mathbb{N} \cup \{0\})$ and B_n $(n \in \mathbb{N})$ are non-selfadjoint, $A_n - I$ $(n \in \mathbb{N} \cup \{0\})$ and B_n $(n \in \mathbb{N})$ are completely continuous operators in H such that A_n is invertible for $n \in \mathbb{N} \cup \{0\}$.

In this paper, we investigate the spectral properties of the non-selfadjoint difference operator L which is generated by the Sturm-Liouville difference equation with non-selfadjoint operator coefficients. In particular, we find the Jost solution, continuous spectrum, discrete spectrum and spectral singularities of L. Finally, we prove the finiteness of eigenvalues and spectral singularities.

2. THE JOST SOLUTION AND CONTINUOUS SPECTRUM OF L

Let us consider the eigenvalue equation of L

$$A_{n-1}y_{n-1} + B_n y_n + A_n y_{n+1} = \lambda y_n,\ n \in \mathbb{N}, \quad (5)$$

where λ is the spectral parameter. Equivalently, we might consider the corresponding operator equation

$$A_{n-1}Y_{n-1} + B_n Y_n + A_n Y_{n+1} = \lambda Y_n,\ n \in \mathbb{N},$$

where (Y_n) is an operator sequence i.e. Y_n is an operator in H for each $n \in \mathbb{N}$.

Assumption 1. We assume that the coefficients of Equation (5) satisfy
$\sum_{n=1}^{\infty}(\|I - A_n\| + \|B_n\|) < \infty.$

Definition 1. Let $E(z) := E_n(z)$ $(n \in \mathbb{N} \cup \{0\})$ denote the operator solution of the equation

$$A_{n-1}Y_{n-1} + B_nY_n + A_nY_{n+1} = (z + z^{-1})Y_n,\ n \in \mathbb{N},$$

satisfying the condition

$$\lim_{n\to\infty} Y_n(z)\, z^{-n} = I,$$

for $z \in D_0 := \{z \in \mathbb{C}: |z| = 1\}$. $E(z)$ is called the Jost solution of Equation (5).

Remark 1. The remaining results of this section will be given without proofs since they are similar to the matrix coefficient case which have been obtained in [12].

Assumption 2. Let us assume

$$\sum_{n=1}^{\infty} n(\|I - A_n\| + \|B_n\|) < \infty.$$

Theorem 1. The Jost solution can be represented

$$E_n(z) = T_n z^n\left[I + \sum_{m=1}^{\infty} K_{n,m} z^m\right], n \in \mathbb{N} \cup \{0\}, \quad (6)$$

where

$$T_n = \prod_{p=n}^{\infty} A_p^{-1},$$

$$K_{n,1} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p,$$

$$K_{n,2} = -\sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=n+1}^{\infty} T_p^{-1}\left(I - A_p^2\right)T_p,$$

$$K_{n,m+2} = \sum_{p=n+1}^{\infty} T_p^{-1}\left(I - A_p^2\right)T_p K_{p+1,m} - \sum_{p=n+1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{n+1,m},$$

where $n \in \mathbb{N} \cup \{0\}$, $m \in \mathbb{N}$. Further,

$$\left\|K_{n,m}\right\| \le c \sum_{p=n+\left[\!\left[\frac{m}{2}\right]\!\right]}^{\infty}\left(\left\|I - A_p\right\| + \left\|B_p\right\|\right)$$

holds where $c > 0$ is a constant. Moreover, $E_n(z)$ $(n \in \mathbb{N} \cup \{0\})$ has an analytic continuation from D_0 to $D_1 := \{z \in \mathbb{C}: |z| < 1\}\backslash\{0\}$.

Theorem 2. The Jost solution satisfies the asymptotic relation

$$E_n(z) = z^n[I + o(1)], \quad n \to \infty,$$

for $z \in D := \{z \in \mathbb{C}: |z| \le 1\}\backslash\{0\}$.

Theorem 3. The continuous spectrum of L is $\sigma_c(L) = [-2, 2]$.

Proof. Let L_0 and L_1 denote the operators defined in H_1

$$L_0(y)_n = y_{n-1} + y_{n+1}, n \in \mathbb{N},$$

$$L_1(y)_n = (A_{n-1} - I)y_{n-1} + B_n y_n + (A_n - I)y_{n+1}, n \in \mathbb{N},$$

with the boundary condition

$$y_0 = 0,$$

respectively. It easily follows $L_0 = {L_0}^*$ and also $\sigma_c(L_0) = [-2, 2]$ (see [24]).

It is well known that L_1 is a compact operator iff L_1 is bounded and the set

$$R = \{L_1 y:\ \|y\|_1 \le 1\}$$

is compact in H_1. It is obvious that L_1 is bounded. Moreover, if we use the compactness criteria in l_p spaces (see [25], p. 167), we obtain the compactness of R. Indeed, let $y \in H_1$ such that $\|y\|_1 \le 1$. Then, Assumption 2 implies that for $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$\sum_{i=n+1}^{\infty}(\|A_i - I\| + \|B_i\|) < \frac{\varepsilon}{C}$$

holds and also

$$\sum_{i=n+1}^{\infty}\|(L_1 y)_i\|_H^2 = \sum_{i=n+1}^{\infty}\|(A_{i-1} - I)y_{i-1} + B_i y_i + (A_i - I)y_{i+1}\|_H^2$$

$$\le \sum_{i=n+1}^{\infty}\|A_{i-1} - I\|^2\|y_{i-1}\|_H^2 + \|B_i\|^2\|y_i\|_H^2 + \|A_i - I\|^2\|y_{i+1}\|_H^2$$

$$\le \|y\|_1^2\left(\sum_{i=n+1}^{\infty}\|A_{i-1} - I\|^2 + \|B_i\|^2 + \|A_i - I\|^2\right)$$

$$\le \sum_{i=n+1}^{\infty} 2\|A_i - I\|^2 + \|B_i\|^2$$

$$\le \sum_{i=n+1}^{\infty} C_1\|A_i - I\| + C_2\|B_i\|$$

$$\le \sum_{i=n+1}^{\infty} C(\|A_i - I\| + \|B_i\|) < \varepsilon\ ,$$

where

$C_1 = \frac{1}{2} sup_{i\in\mathbb{N}} \|A_i - I\|,\ C_2 = sup_{i\in\mathbb{N}} \|B_i\|,\ C = C_1 + C_2.$

Therefore, we proved that L_1 is a compact operator in H_1. Weyl's theorem of compact perturbation [26] implies

$\sigma_c(L) = \sigma_c(L_0) = [-2, 2].$

3. EIGENVALUES AND SPECTRAL SINGULARITIES OF L

It is easy to show that the discrete spectrum and the set of spectral singularities of L are

$\sigma_d(L) = \{\lambda: \lambda = z + z^{-1}, z\epsilon D_1, E_0(z) \text{ is not invertible}\},$

$\sigma_{ss}(L) = \{\lambda: \lambda = z + z^{-1}, z\epsilon D_0, E_0(z) \text{ is not invertible}\},$

respectively. $E_0(z)$ is called the Jost function of L. Note that, this function is an operator function on the contrary to finite dimensional case. Hence, the methods need to be changed in our case. We will use Keldysh [23] in order to analyze the singular points of $E_0(z)$. Let us define the sets

$M_1 = \{z\epsilon D_1: E_0(z) \text{ is not invertible}\},$

$M_2 = \{z\epsilon D_0: E_0(z) \text{ is not invertible}\}.$

Then,

$\sigma_d(L) = \{\lambda: \lambda = z + z^{-1}, z\epsilon M_1\},$

$\sigma_{ss}(L) = \{\lambda: \lambda = z + z^{-1}, z\epsilon M_2\}.$

From the representation (6) we have

$E_0(z) = T_0\left[I + \sum_{m=1}^{\infty} K_{0,m} z^m\right],$

where

$T_0 = \prod_{p=0}^{\infty} A_p^{-1}$

is invertible. This implies $E_0(z)$ is invertible iff

$F(z) := I + \sum_{m=1}^{\infty} K_{0,m} z^m$

is invertible. Hence

$M_1 = \{z\epsilon D_1: F(z) \text{ is not invertible}\}.$

From Theorem 1 we have

$K_{0,1} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p,$

$K_{0,2} = -\sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p,1} + \sum_{p=1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p,$

$K_{0,m+2} = \sum_{p=1}^{\infty} T_p^{-1} \left(I - A_p^2\right) T_p K_{p+1,m} - \sum_{p=1}^{\infty} T_p^{-1} B_p T_p K_{p,m+1} + K_{1,m},$

where $m \in \mathbb{N}$. These equations together with the conditions that $A_n - I$ $(n \in \mathbb{N} \cup \{0\})$ and B_n $(n \in \mathbb{N})$ are completely continuous operators imply that $K_{0,m}$ is completely continuous for every $m \in \mathbb{N}$. As a result,

$K(z) := \sum_{m=1}^{\infty} K_{0,m} z^m$

is a completely continuous operator for every $z \in D_1$. Moreover, since $E_0(z)$ is analytic on D_1 (see Theorem 1) $F(z)$ is also analytic on D_1. Hence, we can use [23] to find the singular points of the operator valued function $F(z)$ on D_1 and these singular points give us the eigenvalues.

Definition 2. [23] Let A and R be operators such that

$(I + R)(I - A) = I$

holds. Then, R is called the resolvent of A.

Theorem 4. Let $R(z)$ denote the resolvent of $-K(z)$. Then,

$M_1 = \{z\epsilon D_1: z \textit{ is a pole of } I + R(z)\}.$

Proof. Since $R(z)$ is the resolvent of $-K(z)$, we have

$I + R(z) = (I + K(z))^{-1} = (F(z))^{-1}.$

Since $M_1 \neq D_1$ there exists $z\epsilon D_1$ such that $I + R(z)$ exists. Therefore, $I + R(z)$ exists on D_1 except for a set of isolated points and $I + R(z)$ is a meromorphic function of z on D_1 [23]. These isolated points are clearly the eigenvalues of L. Hence we have

$$(F(z))^{-1} = \frac{U(z)}{v(z)},\ z\epsilon D_1, \quad (7)$$

where $U(z)$ is an operator function and $v(z)$ is a scalar function which are both analytic on D_1. Since

$M_1 = \{z\epsilon D_1 : F(z) \text{ is not invertible}\}$,

it follows from (7) that

$M_1 = \{z\epsilon D_1 : z\ is\ a\ pole\ of\ I + R(z)\} = \{z\epsilon D_1 : v(z) = 0\}$.

Corollary 1. $\sigma_d(L) = \{\lambda : \lambda = z + z^{-1}, z\epsilon D_1, v(z) = 0\}$.

Proof. The proof is obvious from Theorem 4 since $\sigma_d(L) = \{\lambda : \lambda = z + z^{-1}, z\epsilon M_1\}$.

Theorem 5. $\sigma_d(L)$ is bounded and countable. Moreover, its limit points can only lie in the circle $|z| \leq 2$.

Proof. From (6) it follows

$$E_0(z) = T_0\left[I + \sum_{m=1}^{\infty} K_{0,m} z^m\right]$$

and also

$$E_0(z) = T_0 I + o(1),\ |z| \to 0,$$

which implies $E_0(z)$ is invertible for sufficiently small z and hence $\lambda = z + z^{-1}$ is not an eigenvalue for $|z| \to 0$. Thus, $\sigma_d(L)$ is bounded. Since $v(z)$ is analytic on D_1, its zeros are isolated. From Corollary 5, it follows $\sigma_d(L)$ is countable. Further, the limit points (if exist) of the zeros of $v(z)$ lie on the boundary of D_1 i.e. on D_0 [27].

If z is a limit point of the set

$M_1 = \{z\epsilon D_1 : v(z) = 0\}$,

then $|z| = 1$ and $|\lambda| = |z + z^{-1}| \leq 2$. Hence, the limit points (if exist) lie in the circle $|z| \leq 2$.

Theorem 6. Let $R(z)$ denote the resolvent of $-K(z)$. Then,

$$\begin{aligned} M_2 &= \{z\epsilon D_0 : z\ is\ a\ pole\ of\ I + R(z)\} \\ &= \{z\epsilon D_0 : v(z) = 0\}. \end{aligned}$$

Proof. From (7) it follows

$$F(z)U(z) = v(z)I,$$

which implies $U(z)$ is invertible whenever $v(z) \neq 0$. Moreover, if $z\epsilon D_1$ such that $v(z) \neq 0$ then

$$F(z) = v(z)(U(z))^{-1}.$$

Since $F(z)$ is continuous on D, we can extend this representation continuously to D;

$$(F(z))^{-1} = \frac{U(z)}{v(z)},\ z\epsilon D. \quad (8)$$

Recall that

$M_2 = \{z\epsilon D_0 : F(z) \text{ is not invertible}\}$.

The representation (8) implies that

$M_2 = \{z\epsilon D_0 : z\ is\ a\ pole\ of\ I + R(z)\} = \{z\epsilon D_0 : v(z) = 0\}$.

Theorem 7. The set of spectral singularities $\sigma_{ss}(L)$ of L is compact and has zero Lebesgue measure.

Proof. It is well known that $\sigma_{ss}(L) \subset \sigma_c(L) = [-2,2]$. This gives us the boundedness of $\sigma_{ss}(L)$. We only have to show $\sigma_{ss}(L)$ or equivalently M_2 is closed for the compactness. Let $(\lambda_n) \subset M_2$ such that $\lambda_n \to \lambda$. $(\lambda_n) \subset M_2$ implies $\lambda_n \in D_0$ and $v(\lambda_n) = 0$ for every $n \in \mathbb{N}$. Since D_0 is closed $\lambda_n \to \lambda$ implies $\lambda \in D_0$. Moreover, since $v(z)$ is continuous on D, $\lambda_n \to \lambda$ implies $v(\lambda_n) \to v(\lambda)$ and hence $v(\lambda) = 0, \lambda \in M_2$ as required. Finally, since

$M_2 = \{z\epsilon D_0 : v(z) = 0\}$,

it follows from Privalov's Theorem [27] that M_2 has zero Lebesgue measure.

Assumption 3. Let us assume

$\sum_{n=0}^{\infty} e^{\varepsilon n}(\|I - A_n\| + \|B_n\|) < \infty,\ \varepsilon > 0.$

Theorem 8. L has a finite number of eigenvalues and spectral singularities.

Proof. Recall that

$$\|K_{n,m}\| \leq c \sum_{p=n+\left[\!\left[\frac{m}{2}\right]\!\right]}^{\infty} \left(\|I - A_p\| + \|B_p\|\right),$$

where c is a constant. This implies together with the Assumption 3 that

$$\|K_{0,m}\| \leq C e^{\frac{-\varepsilon}{6}m}, m \epsilon \mathbb{N},$$

where $C > 0$. The series

$$\sum_{m=0}^{\infty} K_{0,m} z^m , \sum_{m=0}^{\infty} m K_{0,m} z^{m-1}$$

are uniformly convergent iff $ln|z| < \frac{\varepsilon}{6}$. Hence, $E_n(z)$ $(n \in \mathbb{N} \cup \{0\})$ has an analytic continuation to $D_2 := \left\{ z \in \mathbb{C} : |z| < e^{\frac{\varepsilon}{6}} \right\}$. Thus, we can write

$$(F(z))^{-1} = \frac{U(z)}{v(z)},\ z \epsilon D_2.$$

Let us suppose that M_1 and M_2 are not finite. Since M_1 and M_2 are bounded (see Theorem 5 and 7), they have limit points by Bolzano-Weierstrass Theorem. Since $v(z)$ is analytic on D_2, the limit points of its zeros must lie on the boundary of the domain D_2 [27]. This gives a contradiction since $e^{\frac{\varepsilon}{6}} > 1$. Therefore, M_1 and M_2 must be finite.

REFERENCES

[1] M.A. Naimark, "Investigation of the spectrum and the expansion in eigenfunctions of a non-selfadjoint operator of second order on a semi-axis," AMS Transl. 2 (16), pp. 103-193, 1960.

[2] M.A. Naimark, "Linear Differential Operators, II," Ungar, New York, 1968.

[3] V.E. Lyance, "A differential operator with spectral singularities, I-II," AMS Transl. 2 (60), pp. 185-225, 227-283, 1967.

[4] B.S. Pavlov, "The non-selfadjoint Schrödinger operator in Spectral Theory and Wave Processes, Topics in Mathematical Physics, Vol.1, Birman M.S. (eds), Springer, Boston, MA, pp. 87-114, 1967.

[5] J.T. Schwartz, "Some non-selfadjoint operators," Comm. Pure Appl. Math., 13, pp. 609-639, 1960.

[6] C.M. Bender, "Making sense of non-Hermitian Hamiltonians," Rep. Prog. Phys., 70, pp. 947-1018, 2007.

[7] A. Mostafazadeh, "Pseudo-Hermitian representation of quantum mechanics," arXiv:0810.5643, 2008.

[8] B.F. Samsonov, "SUSY transformations between digonalizable and non-diagonalizable Hamiltonians," arXiv:quant-ph/0503075, 2005.

[9] A. Mostafazadeh, "Spectral singularities of complex scattering potentials and infinite reaction and transmission coefficients at real energies," arXiv:0901.4472, 2009.

[10] R.P. Agarwal, "Difference Equations and Inequalities: Theory, Methods and Applications," Marcel Dekker, New York, 2000.

[11] R.P. Agarwal and P.J.Y. Wong, "Advanced Topics in Difference Equations," Kluwer, Dordrecht, 1997.

[12] Y. Aygar and E. Bairamov, "Jost solution and the spectral properties of the matrix-valued difference operators," Appl. Math. Comput., 218, pp. 9676-9681, 2012.

[13] E. Bairamov, Y. Aygar and S. Cebesoy, "Spectral analysis of a selfadjoint matrix

valued discrete operator on the whole axis," J. Nonlinear Sci. Appl., 9, pp. 4257-4262, 2016.

[14] R. Carlson, "An inverse problem for the matrix Schrödinger equation," J. Math. Anal. Appl., 267, pp. 564-575, 2002.

[15] S. Clark, F. Gesztesy and W. Renger, "Trace formulas and Borg-type theorems for matrix valued Jacobi and Dirac finite difference operators," J. Differential Equations, 219, pp. 144-182, 2005.

[16] F. Gesztesy, A. Kiselev and K.A. Makarov, "Uniqueness results for matrix-valued Schrödinger, Jacobi and Dirac-type operators," Math. Nachr., 239, pp. 103-145, 2002.

[17] E. Bairamov, Ö. Çakar and A.M. Krall, "Non-selfadjoint difference operators and Jacobi matrices with spectral singularities," Math. Nachr., 229, pp. 5-14, 2001.

[18] A.M. Krall, E.Bairamov and Ö. Çakar, "Spectral analysis of a non-selfadjoint discrete Schrödinger operators with spectral singularities," Math. Nachr., 231, pp. 89-104, 2001.

[19] T. Koprubasi and R.N. Mohapatra, "Spectral properties of generalized eigenparameter dependent discrete Sturm-Liouville type equation," Quaestiones Mathematicae, 40 (4), pp. 491-505, 2017.

[20] M. Adivar and E. Bairamov, "Spectral properties of non-selfadjoint difference operators," J. Math. Anal. Appl., 261, pp. 461-478, 2001.

[21] M. Adivar and E. Bairamov, "Difference equations of second order with spectral singularities," J. Math. Anal. Appl., 277, pp. 714-721, 2003.

[22] S. Cebesoy, Y. Aygar and E. Bairamov, "Matrix valued difference equations with spectral singularities," International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering, 9 (11), pp. 579-582, 2015.

[23] M.V. Keldysh, "On the completeness of the eigenfunctions of some classes of nonselfadjoint linear operators," Russ. Math. Surv., 26 (4), pp. 15-44, 1971.

[24] V.P. Serebrjakov, "An inverse problem of the scattering theory for difference equations with matrix coeffcients," Dokl. Akad. Nauk SSSR, 250 (3), pp. 562-565, 1980 (in Russian).

[25] L.A. Lusternik and V.I. Sobolev, "Elements of Functional Analysis," Halsted Press, New York, 1974.

[26] I.M. Glazman, "Direct Methods of Qualitative Spectral Analysis of Singular Differential Operators," Israel Program for Scientific Translations, Jerusalem, 1965.

[27] E.P. Dolzhenko, "Boundary value uniqueness theorems for analytic functions," Math. Notes, 26 (6), pp. 437-442, 1979.