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Conformally Flat Minimal C-totally Real Submanifolds of (κ,μ)-Nullity Space Forms

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Abstract

In this paper we study conformally flat minimal C-totally real submanifolds of (κ,μ)-nullity space forms.

***Keywords*:** Contact metric manifold; (κ,μ)-space form; Conformally flat manifold; Second fundamental form; Totally geodesic.

(κ,μ)-Nullity Uzay Formlarının Konformal Flat Minimal C-total Reel Altmanifoldları

Özet

Bu çalışmada (κ,μ)-nullity uzay formlarının konformal flat minimal C-total reel altmanifoldlarını çalıştık.

Anahtar Kelimeler: Değme metrik manifold; (κ,μ)-uzay formu; Konformal flat manifold; İkinci temel form; Total jeodezik.

1. Introduction

Let M^m be a minimal C-totally real submanifold of dimension m, having constant $\breve{\varphi}$-sectional curvature c in a $(2m+1)$-dimensional Sasakian space form $\breve{M}$ of constant $\breve{\varphi}$-sectional

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curvature $\breve{c}$. B.Y. Chen and K. Ogiue [1] studied totally real submanifolds and proved that if a such a submanifold is totally geodesic, then it is of constant curvature $c = \frac{1}{4}\breve{c}$. Then D. Blair [2] showed that such a submanifold is totally geodesic if and only if it is of constant curvature $c = \frac{1}{4}(\breve{c} + 3)$. Also S. Yamaguchi, M. Kon and T. Ikawa [3] stated that if such a submanifold is compact and has constant scalar curvature, then it is totally geodesic and has constant sectional curvature c satifying $c = \frac{1}{4}(\breve{c} + 3)$ or $c \leq 0$. Later D. E. Blair and K. Ogiue [4] proved that if M is compact and $c > \frac{m-2}{4(2m-1)}(\breve{c} + 3)$, then M is totally geodesic. Also P. Verheyen and L. Verstraelen [5] obtained that if M^m $(m \geq 4)$ is a compact conformally flat submanifold admitting constant scalar curvature $scal > \frac{(m-1)^3(m+2)}{4(m^2+m-4)}(\breve{c} + 3)$ and $\breve{\varphi}$-sectional curvature c satisfying $c > \frac{(m-1)^2}{4m(m^2+m-4)}(\breve{c} + 3)$, then it is totally geodesic.

In the present paper, we study the results indicated above for a conformally flat minimal C-totally real submanifold M in a (κ, μ)-nullity space form $\breve{M}^{2m+1}$ with constant $\breve{\varphi}$-sectional curvature $\breve{c}$. We prove the followings:

Theorem 1. Let $\breve{M}^{2m+1}$ be a (κ, μ)-nullity space form of constant $\breve{\varphi}$-sectional curvature $\breve{c}$ and M^m be an $m \geq 4$-dimensional compact conformally flat minimal C-totally real submanifold of a $\breve{M}^{2m+1}$. Then

$$scal > \frac{(m-1)^3(m+2)}{4(m^2+m-4)}(\breve{c} + 3) + \frac{2(m-1)\left[m(m^2-2)\lambda(\lambda+2)+(m-2)\lambda\right]}{4(m^2+m-4)},$$

implies that M^m is totally geodesic, where $\lambda = \sqrt{1-\kappa}$.

Theorem 2. Let M^m be a minimal C-totally real submanifold of a (κ, μ)-nullity space form $\breve{M}^{2m+1}$. If M^m has constant curvature c, then either

$$c = \frac{1}{4}[(\breve{c} + 3) + 2\lambda^2 + 8\lambda],$$

in which case M^m is totally geodesic, or $c \leq 0$.

2. Preliminiaries

Let $\breve{M}^{2m+1}$ be a contact metric manifold with the $(\breve{\varphi}, \xi, \breve{\eta}, \breve{g})$ satisfying

$$\breve{\varphi}^2 = -I + \breve{\eta} \otimes \xi,$$

$$\breve{\eta}(\xi) = 1, \breve{\varphi}\xi = 0, \breve{\eta}(U) = \breve{g}(U, \xi), \tag{1}$$

$$\breve{g}(\breve{\varphi}U, \breve{\varphi}V) = \breve{g}(U,V) - \breve{\eta}(U)\breve{\eta}(V), \quad \breve{g}(\breve{\varphi}U, V) = d\breve{\eta}(U,V),$$

for vector fields U and V on $\breve{M}$. The operator h defined by $h = -\frac{1}{2}L_{\xi}\breve{\varphi}$, vanishes iff ξ is Killing. Also we have

$$\breve{\varphi}h + h\breve{\varphi} = 0,\ h\xi = 0,\ \breve{\eta}oh = 0,\ tr\, h = tr\, \breve{\varphi}h = 0. \tag{2}$$

Due to anti-commuting h with $\breve{\varphi}$, if U is an eigenvector of h with the eigenvalue λ then $\breve{\varphi}U$ is also an eigenvector of h with the eigenvalue $-\lambda$ [6]. Moreover, for the Riemannian connection $\breve{\nabla}$ of $\breve{g}$, we have

$$\breve{\nabla}_U \xi = -\breve{\varphi}U - \breve{\varphi}hU. \tag{3}$$

If ξ is Killing then contact metric manifold $\breve{M}$ is said to be a *K-contact Riemannian manifold*. On a K-contact Riemannian manifold, we have

$$\breve{R}(U,\xi)\xi = U - \breve{\eta}(U)\xi.$$

A Sasakian manifold is known as a normal contact metric manifold. A contact metric manifold to be Sasakian if and only if $\breve{R}(U,V)\xi = \breve{\eta}(V)U - \breve{\eta}(U)V$, where $\breve{R}$ is the curvature tensor on $\breve{M}$. Moreover, *every Sasakian manifold is a K-contact manifold* [2].

The (κ,μ)-*nullity distribution* for a contact metric manifold $\breve{M}$ is a distribution

$$Null(\kappa,\mu): p \longrightarrow Null_p(\kappa,\mu) = \left\{ \begin{array}{c} W \in T_pM | \breve{R}(U,V)W = \kappa[\breve{g}(V,W)U - \breve{g}(U,W)V] \\ +\mu[\breve{g}(V,W)hU - \breve{g}(U,W)hV] \end{array} \right\},$$

for any $U, V \in T_p(\breve{M})$, where $\kappa, \mu \in \mathbb{R}$ and $\kappa \leq 1$. We consider that $\breve{M}$ is a contact metric manifold with ξ concerning to the (κ,μ)-nullity distribution, i.e.,

$$R(U,V)\xi = \kappa[\breve{\eta}(V)U - \breve{\eta}(U)V] + \mu[\breve{\eta}(V)hU - \breve{\eta}(U)hV]. \tag{4}$$

The necessary and sufficient condition for the manifold $\breve{M}$ to be a Sasakian manifold is that $\kappa = 1$ and $\mu = 0$ [7]. Also, for more details, one can see [8] and [9]. For $\kappa < 1$, (κ,μ)-contact metric manifolds have constant scalar curvature. Also, the sectional curvature $\breve{K}(U,\breve{\varphi}U)$ according to a $\breve{\varphi}$-section determined by a vector U is called a *$\breve{\varphi}$-sectional curvature*. A (κ,μ)-contact metric manifold with constant $\breve{\varphi}$-sectional curvature $\breve{c}$ is a *(κ,μ)-nullity space form*. The *curvature tensor* of a (κ,μ)-nullity space form $\breve{M}$ is given by [10]

$$\breve{R}(U,V)W = \frac{1}{4}(\breve{c}+3)\{g(V,W)U - g(U,W)V\}$$

$$+\frac{\breve{c}+3-4\kappa}{4}\begin{Bmatrix}\breve{\eta}(U)\breve{\eta}(W)V-\breve{\eta}(V)\breve{\eta}(W)U\\+g(U,W)\breve{\eta}(V)\xi-g(V,W)\breve{\eta}(U)\xi\end{Bmatrix}$$

$$+\frac{\breve{c}-1}{4}\begin{Bmatrix}2g(U,\breve{\varphi}V)\breve{\varphi}W+g(U,\breve{\varphi}W)\breve{\varphi}V\\-g(V,\breve{\varphi}W)\breve{\varphi}U\end{Bmatrix} \tag{5}$$

$$+\frac{1}{2}\begin{Bmatrix}g(hV,W)hU-g(hU,W)hV\\+g(\breve{\varphi}hU,W)\breve{\varphi}hV-g(\breve{\varphi}hV,W)\breve{\varphi}hU\\+g(\breve{\varphi}V,\breve{\varphi}W)hU-g(\breve{\varphi}U,\breve{\varphi}W)hV\\+g(hU,W)\breve{\varphi}^2V-g(hV,W)\breve{\varphi}^2U\end{Bmatrix}$$

$$+\frac{1}{2}\begin{Bmatrix}g(hV,W)hU-g(hU,W)hV\\+g(\breve{\varphi}hU,W)\breve{\varphi}hV-g(\breve{\varphi}hV,W)\breve{\varphi}hU\\+g(\breve{\varphi}V,\breve{\varphi}W)hU-g(\breve{\varphi}U,\breve{\varphi}W)hV\\+g(hU,W)\breve{\varphi}^2V-g(hV,W)\breve{\varphi}^2U\end{Bmatrix}$$

$$+\mu\begin{Bmatrix}\breve{\eta}(V)\breve{\eta}(W)hU-\breve{\eta}(U)\breve{\eta}(W)hV\\+g(hV,W)\breve{\eta}(U)\xi-g(hU,W)\breve{\eta}(V)\xi\end{Bmatrix},$$

where $\breve{c}$ is constant $\breve{\varphi}$-sectional curvature.

3. C-totally Real Submanifolds

Let M be an m-dimensional submanifold in a $(2m+1)$-dimensional manifold $\breve{M}$ equipped with a Riemannian metric g. We denote by ∇ (resp. $\breve{\nabla}$) the covariant derivation with respect to g (resp.$\breve{g}$). Then the *second fundamental form* B is given by

$$B(U,V)=\breve{\nabla}_UV-\nabla_UV. \tag{6}$$

For a normal vector field ξ on M, we write $\breve{\nabla}_U\xi=-A_\xi U+D_U\xi$, where $-A_\xi U$ (resp. $D_U\xi$) denotes the tangential (resp. normal) component of $\breve{\nabla}_U\xi$. Then, we have

$$\breve{g}(B(U,V),\xi)=g(A_\xi U,V). \tag{7}$$

A normal vector field ξ on M is said to be *parallel* if $D_U\xi=0$ for any tangent vector U. For any orthonormal basis $\{w_1,\dots,w_m\}$ of the tangent space T_pM, the *mean curvature vector* $H(p)$ is given by

$$H(p)=\frac{1}{m}\sum_{i=1}^{m}B(w_i,w_i). \tag{8}$$

The submanifold M is *totally geodesic* in $\breve{M}$ if $B=0$, and *minimal* if $H=0$. If $B(U,V)=g(U,V)H$ for all $U,V\in TM$, then M is *totally umbilical*. For the second fundamental form B, with respect to the covariant derivation $\overline{\nabla}$ is defined by

$$(\overline{\nabla}_U B)(V,W)=D_U(B(V,W))-B(\nabla_U V,W)-B(V,\nabla_U W), \tag{9}$$

for all U,V and W on M [11], where $\overline{\nabla}$ is the covariant differentiation operator of *van der Waerden-Bortolotti*.

Also the equations of *Gauss, Codazzi* and *Ricci* are given by

$$g(R(U,V)W,T)=g(\breve{R}(U,V)W,T) \tag{10}$$
$$+g(B(U,W),B(V,T))-g(B(V,W),B(U,T)),$$

$$(\breve{R}(U,V)W)^{\perp}=(\overline{\nabla}_U B)(V,W)-(\overline{\nabla}_V B)(U,W), \tag{11}$$

$$g(\breve{R}(U,V)W,N)=g(R^{\perp}(U,V)W,N)+g([A_N,A_W]U,V), \tag{12}$$

where R and $\breve{R}$ are the Riemannian curvature tensor of M and $\breve{M}$ and $(\breve{R}(U,V)W)^{\perp}$ denotes the normal component of $\breve{R}(U,V)W$ [11]. The second covariant derivative $\overline{\nabla}^2 B$ of B is defined by

$$(\overline{\nabla}^2 B)(W,T,U,V)=(\overline{\nabla}_U\overline{\nabla}_V B)(W,T)$$
$$=\nabla_U^{\perp}((\overline{\nabla}_V B)(W,T))-(\overline{\nabla}_V B)(\nabla_U W,T) \tag{13}$$
$$-(\overline{\nabla}_U B)(W,\nabla_V T)-(\overline{\nabla}_{\nabla_U V} B)(W,T).$$

Then, we have

$$(\overline{\nabla}_U\overline{\nabla}_V B)(W,T)-(\overline{\nabla}_V\overline{\nabla}_U B)(W,T)=(\bar{R}(U,V)B)(W,T)$$
$$=R^{\perp}(U,V)B(W,T)-B(R(U,V)W,T)-B(W,R(U,V)T), \tag{14}$$

where $\bar{R}$ is the curvature tensor belonging to the connection $\overline{\nabla}$. The *Laplacian of the square of the lenght of the second fundamental form* is defined

$$\frac{1}{2}\Delta\|B\|^2=g(\overline{\nabla}^2 B,B)+\left\|\overline{\nabla}B\right\|^2, \tag{15}$$

where $\|B\|$ is the length of the second fundemental form B, so that

$$\|B\|^2 = \sum_{i,j} g(B(w_i, w_j), B(w_i, w_j)), \tag{16}$$

and using (3.8), we can write

$$\|\overline{\nabla}B\|^2 = \sum_{i,j,k} g((\overline{\nabla}_{w_i}\overline{\nabla}_{w_i}B)(w_j, w_k), (\overline{\nabla}_{w_i}\overline{\nabla}_{w_i}B)(w_j, w_k)), \tag{17}$$

and

$$g(\overline{\nabla}^2 B, B) = \sum_{i,j,k} g((\overline{\nabla}_{w_i}\overline{\nabla}_{w_i}B)(w_j, w_k), B(w_j, w_k)). \tag{18}$$

A submanifold M in a contact metric manifold is called a C-*totally real submanifold* [12] if every tangent vector of M belongs to the contact distribution. Hence, a submanifold M in a contact metric manifold is a C-*totally real submanifold* if ξ is normal to M. A submanifold M in an almost contact metric manifold is called a C-*totally real submanifold* if $\breve{\varphi}(TM) \subset T^{\perp}(M)$ [13].

4. Conformally Flat Minimal C-totally Real Submanifolds of (κ, μ)-Nullity Space Forms

Let M^m be a C-totally real submanifold of a (κ, μ)-nullity space form $\breve{M}^{2m+1}$ with $\breve{\varphi}$-sectional curvature $\breve{c}$ and structure tensors $(\breve{\varphi}, \xi, \breve{\eta}, \breve{g})$, with ξ normal to M. The *conformal curvature tensor field* of M^m is defined by

$$C(U,V)W = R(U,V)W + \frac{1}{m-2}\begin{bmatrix} Ric(U,W)V - Ric(V,W)U \\ +g(U,W)QV - g(V,W)QU \end{bmatrix}$$

$$-\frac{scal}{(m-1)(m-2)}[g(U,W)V - g(V,W)U], \tag{19}$$

for all vector fields U, V, and W, where Q denotes the *Ricci operator* defined by $g(QU,V) = Ric(U,V)$. For $m \geq 4$, the manifold M is *conformally flat manifold* if and only if $C = 0$ [11].

Lemma 3. Let M be an m-dimensional C-totally real submanifold on (κ, μ)-contact metric manifold $\breve{M}^{2m+1}$. Then, we have

$i)$ $A_{\breve{\varphi}w_i}w_j = A_{\breve{\varphi}w_j}w_i$,

$ii)$ $tr(\sum_i A_i^2)^2 = \sum_{i,j}(trA_iA_j)^2$.

Lemma 4. A C-totally real submanifold M of dimension $m \geq 4$ *in a* (κ, μ)-nullity space form $\breve{M}^{2m+1}$ conformally flat if and only if

$$(m-1)(m-2)\begin{Bmatrix}\sum\limits_{\alpha}\{g(A_\alpha w_j,w_k)g(A_\alpha w_i,w_l)\\ -g(A_\alpha w_i,w_k)g(A_\alpha w_j,w_l)\}\end{Bmatrix}$$

$$+\left\{\sum_{\alpha}(tr(A_\alpha)^2-\|B\|^2\right\}\{g(w_j,w_k)g(w_i,w_l)-g(w_i,w_k)g(w_j,w_l)\} \quad (20)$$

$$+(m-1)\begin{Bmatrix}\sum\limits_{\alpha} tr(A_\alpha)\{g(A_\alpha w_i,w_k)g(w_j,w_l)-g(A_\alpha w_j,w_k)g(w_i,w_l)\\ +g(A_\alpha w_j,w_l)g(w_i,w_k)-g(A_\alpha w_i,w_l)g(w_j,w_k)\}\end{Bmatrix}$$

$$-(m-1)\begin{Bmatrix}\sum\limits_{\alpha,t}\{g(A_\alpha w_i,w_t)g(A_\alpha w_k,w_t)g(w_j,w_l)\\ -g(A_\alpha w_j,w_t)g(A_\alpha w_k,w_t)g(w_i,w_l)\\ +g(A_\alpha w_j,w_t)g(A_\alpha w_l,w_t)g(w_i,w_k)\\ -g(A_\alpha w_i,w_t)g(A_\alpha w_l,w_t)g(w_j,w_k)\}\end{Bmatrix}=0,$$

where

$$\|B\|^2=\sum_{\alpha,i,j} g(A_\alpha w_i,w_j)^2=trA^*, \quad (21)$$

and

$$A^*=\sum_{\alpha}(A_\alpha)^2. \quad (22)$$

Proof. Let M be a conformally flat manifold. Then, from Eqn. (5) and Eqn. (19), we have

$$(m-1)(m-2)g(R(w_i,w_j)w_k,w_l)$$

$$+(m-1)\begin{Bmatrix}Ric(w_i,w_k)g(w_j,w_l)-Ric(w_j,w_k)g(w_i,w_l)\\ +Ric(w_j,w_l)g(w_i,w_k)-Ric(w_i,w_l)g(w_j,w_k)\end{Bmatrix} \quad (23)$$

$$-scal\{g(w_i,w_k)g(w_j,w_l)-g(w_j,w_k)g(w_i,w_l)\}=0.$$

Using Eqn. (10) in Eqn. (23), we get

$$(m-1)(m-2)\sum_{\alpha}\{g(A_\alpha w_j,w_k)g(A_\alpha w_i,w_l)-g(A_\alpha w_i,w_k)g(A_\alpha w_j,w_l)\}$$

$$+\frac{(m-1)(m-2)}{4}\{(c+3)+2\lambda^2+8\lambda\}+scal\}\begin{Bmatrix}g(w_j,w_k)g(w_i,w_l)\\ -g(w_i,w_k)g(w_j,w_l)\end{Bmatrix} \quad (24)$$

$$+(m-1)\begin{Bmatrix}Ric(w_i,w_k)g(w_j,w_l)-Ric(w_j,w_k)g(w_i,w_l)\\ +Ric(w_j,w_l)g(w_i,w_k)-Ric(w_i,w_l)g(w_j,w_k)\end{Bmatrix}=0,$$

where Ric and $scal$, respectively, the Ricci tensor and scalar curvature of M, defined by

$$Ric(w_j, w_k) = \frac{(m-1)}{4}\{(c+3) + 2\lambda^2 + 8\lambda\}g(w_j, w_k) \tag{25}$$

$$+\sum_{\alpha} tr(A_\alpha)g(A_\alpha w_j, w_k) - g(A_\alpha w_j, A_\alpha w_k),$$

and

$$scal = \frac{m(m-1)}{4}\{(c+3) + 2\lambda^2 + 8\lambda\} + \sum_{\alpha} (tr(A_\alpha))^2 - \|B\|^2. \tag{26}$$

From Eqn. (24)-Eqn. (26), we have Eqn. (20).

Lemma 5. Let M be an m-dimensional C-totally real submanifold on (κ, μ)-contact metric manifold $\breve{M}^{2m+1}$. If M is minimal, then Eqn. (20) becomes

$$(m-1)(m-2)g([A_i, A_j]w_k, w_l)$$

$$-\|B\|^2\{g(w_j, w_k)g(w_i, w_l) - g(w_i, w_k)g(w_j, w_l)\} \tag{27}$$

$$-(m-1)\{g(w_j, w_l)tr(A_iA_k) - g(w_i, w_l)tr(A_jA_k)$$

$$+g(w_i, w_k)tr(A_jA_l) - g(w_j, w_k)tr(A_iA_l)\} = 0.$$

Lemma 6. Let M be a conformally flat minimal C-totally real submanifold of dimension $m \geq 4$ in a (κ, μ)-nullity space form $\breve{M}^{2m+1}$, then

$$(m-1)(m-2)\sum_{i,j} tr(A_iA_j)^2 = \|B\|^4 + (m-1)(m-4)tr(A^*)^2. \tag{28}$$

Also we have the following:

Lemma 7. In any (κ, μ)-contact metric manifold, we have

$$i)\left\|\overline{\nabla}B\right\|^2 \geq \|B\|^2, \tag{29}$$

$$ii)\ tr(A^*)^2 \leq \|B\|^4. \tag{30}$$

Now using Lemma 7, we get the following:

Lemma 8. Let $\breve{M}^{2m+1}$ be a (κ, μ)-nullity space form of constant $\breve{\varphi}$-sectional curvature $\breve{c}$ and M be an $m \geq 4$-dimensional minimal C-totally real submanifold of $\breve{M}$. The Laplacian of the square of the length of the second fundamental form B of M

$$\frac{1}{2}\Delta\|B\|^2 = \left\|\overline{\nabla}B\right\|^2 + \left(\frac{(\check{c}-1)+m(\check{c}+3)}{4} + \frac{\lambda}{2}(m(\lambda+4)-\lambda)\right)\|B\|^2$$

$$+2\sum_{\alpha,\beta} tr(A_\alpha A_\beta)^2 - 3tr(A^*)^2, \tag{31}$$

where $\lambda = \sqrt{1-\kappa}$.

Proof. If M is minimal then, from [11], we have

$$(\overline{\nabla}^2 B)(U,V) = \sum_i (R(w_i,U)B)(w_i,V). \tag{32}$$

For an orthonormal base w_i, from Eqn. (12), we have

$$(R(w_k,w_i)B)(w_k,w_j) = R^\perp(w_k,w_i)B(w_k,w_j) - B(R(w_k,w_i)w_k,w_j) \tag{33}$$

$$-B(w_k,R(w_k,w_i)w_j).$$

Using Eqn. (10) in Eqn. (33), we get

$$g((R(w_k,w_i)B)(w_k,w_j),B(w_i,w_j)) = g(R^\perp(w_k,w_i)B(w_k,w_j),B(w_i,w_j))$$

$$-g(B(\bar{R}(w_k,w_i)w_k,w_j),B(w_i,w_j)) - \sum_{\alpha,\beta} g(A_\beta A_\alpha w_k, A_\beta A_\alpha w_k) \tag{34}$$

$$+\sum_{\alpha,\beta} tr(A_\alpha)tr(A_\beta^2 A_\alpha) - g(B(w_k,\bar{R}(w_k,w_i)w_j),B(w_i,w_j))$$

$$-\sum_{\alpha,\beta}(tr(A_\alpha A_\beta))^2 + \sum_{\alpha,\beta} tr(A_\beta A_\alpha)^2.$$

Again using Eqn. (11) in Eqn. (34), we have

$$g((R(w_k,w_i)B)(w_k,w_j),B(w_i,w_j)) = g(\bar{R}(w_k,w_i)B(w_k,w_j),B(w_i,w_j))$$

$$-g(B(\bar{R}(w_k,w_i)w_k,w_j),B(w_i,w_j)) - g(B(w_k,\bar{R}(w_k,w_i)w_j),B(w_i,w_j))$$

$$+\sum_{\alpha,\beta}\begin{bmatrix} tr(A_\alpha A_\beta - A_\beta A_\alpha)^2 - (tr(A_\beta A_\alpha))^2 \\ tr(A_\alpha)trA_\beta^2 A_\alpha) \end{bmatrix}. \tag{35}$$

After some calculations, we have

$$g(\bar{R}(w_k,w_i)B(w_k,w_j),B(w_i,w_j)) = \left(\frac{\check{c}-1}{4} - \frac{\lambda^2}{2}\right)\|B\|^2, \tag{36}$$

$$g(B(\bar{R}(w_k,w_i)w_k,w_j),B(w_i,w_j)) = \frac{(1-m)((\check{c}+3)+2\lambda(\lambda+4))}{4}\|B\|^2, \tag{37}$$

$$g(B(w_k, \bar{R}(w_k, w_i)w_j), B(w_i, w_j)) = \left(\frac{-(\check{c}+3)-2\lambda(\lambda+4))}{4}\right)\|B\|^2, \tag{38}$$

$$\sum_{\alpha,\beta}\left[tr(A_\alpha A_\beta - A_\beta A_\alpha)^2 - (tr(A_\beta A_\alpha))^2\right] = \sum_{\alpha,\beta} 2tr(A_\beta A_\alpha) - 3tr(A^*)^2. \tag{39}$$

Thus, using Eqn. (36)-(39) in Eqn. (35), we get Eqn. (31).

5. Proofs of the Main Results

For a conformally flat submanifold M of dimension $m \geq 4$ we use equation Eqn. (28) to replace $\sum_{\alpha,\beta} tr(A_\alpha A_\beta)^2$ in Eqn. (31), we have

$$\begin{aligned} \tfrac{1}{2}(m-1)(m-2)\Delta\|B\|^2 &= (m-1)(m-2)\left\|\bar{\nabla}B\right\|^2 \\ &+(m-1)(m-2)\left(\frac{(\check{c}-1)+m(\check{c}+3)}{4}+\frac{\lambda}{2}(m(\lambda+4)-\lambda)\right)\|B\|^2 \\ &-(m-1)(m+2)tr(A^*)^2 + 2\|B\|^4. \end{aligned} \tag{40}$$

So from Lemma 7, we get

$$\begin{aligned} &\tfrac{1}{2}(m-1)(m-2)\Delta\|B\|^2 \\ &\geq (m-1)(m-2)\|B\|^2 + 2\|B\|^4 - (m-1)(m+2)\|B\|^4 \\ &+\tfrac{1}{4}(m-1)(m-2)[(\check{c}-1)+m(\check{c}+3)+2\lambda(m(\lambda+4)-\lambda)]\|B\|^2 \\ &= \|B\|^2\begin{bmatrix}\frac{(m-1)(m-2)(m+1)(\check{c}+3)}{4}+(m-1)(m-2)\frac{\lambda(m(\lambda+4)-\lambda)}{2} \\ -(m^2+m-4)\|B\|^2\end{bmatrix}, \end{aligned} \tag{41}$$

If $\check{c} > -3$, then

$$\|B\|^2 \leq \frac{(m^2-1)(m-2)(\check{c}+3)}{4(m^2+m-4)} + (m-1)(m-2)\frac{\lambda(m(\lambda+4)-\lambda)}{2(m^2+m-4)}, \tag{42}$$

which implies that $\Delta\|B\|^2 \geq 0$. For a compact submanifold M, Hopf's lemma states that $\Delta\|B\|^2 = 0$ and from Eqn. (41) and Eqn. (42), we conclude that $\|B\|^2 = 0$. Hence, we have

$$scal = \frac{m(m-1)}{4}\{(\check{c}+3)+2\lambda^2+8\lambda\} - \|B\|^2, \tag{43}$$

for every compact minimal C-totally real submanifold in a (κ,μ)-nullity space form $\breve{M}$. Thus, the proof of Theorem 1 is completed.

On the other hand, since M^m has constant curvature c and $scal = m(m-1)\tilde{c}$, from Eqn. (26), we have

$$\|B\|^2 = m(m-1)\left(\frac{(\check{c}+3)+2\lambda^2+8\lambda}{4} - c\right),$$

and

$$c \leq \frac{(\check{c}+3)+2\lambda^2+8\lambda}{4}.$$

Also, Eqn. (10) becomes

$$\left(c - \frac{1}{4}\{(\check{c}+3) + 2\lambda^2 + 8\lambda\}\right)\{g(w_j, w_k)g(w_i, w_l) - g(w_i, w_k)g(w_j, w_l)\}$$

$$= g([A_i, A_j]w_k, w_l). \tag{44}$$

Multiplying this equation by $\sum_N g(A_N w_l, w_i)g(A_N w_j, w_k)$, we obtain

$$\left(c - \frac{1}{4}\{(\check{c}+3) + 2\lambda^2 + 8\lambda\}\right)\|B\|^2 = \sum_{i,j} tr(A_iA_j)^2 - \sum_{i,j} (tr(A_iA_j))^2. \tag{45}$$

Since $Ric = \frac{scal}{m} g$, from Eqn. (25) and Lemma 3, we have

$$tr(A_jA_l) = g(A_\alpha w_j, A_\alpha w_l) = \frac{scal}{m} g(w_j, w_l) = \frac{\|B\|^2}{m} g(w_j, w_l),$$

and

$$tr(A_iA_j)^2 = \left(c - \frac{1}{4}\{(\check{c}+3) + 2\lambda^2 + 8\lambda\}\right)\|B\|^2 + \frac{\|B\|^4}{m}.$$

Substituting the last equation into Eqn. (31), we obtain

$$\|\overline{\nabla}B\|^2 = \left[\frac{(m+1)}{m(m-1)}\|B\|^2 - \frac{m(\check{c}+3) + (\check{c}-1)}{4} + \frac{\lambda(m(\lambda+4) - \lambda)}{2}\right]\|B\|^2.$$

Now using

$$\|B\|^2 = m(m-1)\{\tfrac{1}{4}\{(\check{c}+3) + 2\lambda^2 + 8\lambda\}\},$$

and Lemma 7, we get

$$\|\overline{\nabla}B\|^2 = m(m^2-1)\left(c - \left\{\frac{(\check{c}+3)+2\lambda^2+8\lambda}{4}\right\}\right)\left(c - \frac{1}{m+1}\right)$$

$$\geq m(m-1)\left\{\frac{(\check{c}+3)+2\lambda^2+8\lambda}{4}-c\right\}.$$

Thus, the proof of Theorem 2 is completed.

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