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# Hahn-Banach Theorem for Operators on Lattice Normed Riesz Algebras

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**ABSTRACT**

Let $X$ and $E$ be Riesz algebras and $p: X \to E_+$ be a monotone vector norm. Then the triple $(X, p, E)$ is called lattice normed Riesz algebra. In this paper, we prove a generalization of the extension of the Hahn-Banach theorem for operators on the lattice normed Riesz algebras, in which the extension of one-step of that is not similar to the other Hahn-Banach theorems. In addition, we give some applications and results.

**Keywords**: Hahn-Banach theorem, Lattice normed space Riesz algebra, Riesz apaces

# Kafes Normlu Riesz Cebirleri Üzerindeki Operatörler İçin Hahn-Banach Teoremi

**ÖZ**

$X$ ve $E$ Riesz cebirleri ve $p: X \to E_+$ monoton bir vektör normu olsun. Böylece $(X, p, E)$ üçlüsü kafes normlu Riesz cebiri olarak adlandırılır. Bu çalışmada, Hahn-Banach teoreminin kafes normlu Riesz cebirlerindeki operatörler için genişletilmesini vereceğiz. Fakat bu çalışmadaki genişleme diğer Hahn-Banach teoremlerinden farlı olmaktadır. Ayrıca bu genişlemenin bazı sonuçlarının olduğunu göstermeketeyiz.

**Anahtar Kelimeler**: Hahn-Banach theoremi, Kafes normlu uzayı, Riesz cebiri, Riesz uzayı

## INTRODUCTION and PRELIMINARITIES

The Hahn-Banach theorem has a lot of applications in different fields of analysis, which attracted the attention of several authors such as Vincent-Smith [11] and Turan [10]. In this present paper, we give an extension of the Hahn-Banach theorem on lattice normed Riesz algebras and some applications. The extension of one step in our theorem is not similar to the other Hahn-Banach theorems.

Vector lattices (i.e., Riesz spaces) are ordered vector spaces that have many applications in measure theory, operator theory, and applications in economics. We suppose that the reader to be familiar with the elementary theory of vector lattices, and we refer the reader for information on vector lattices [1,8,12] as sources of unexplained terminology. Besides, all vector lattices are assumed to be real and Archimedean. A vector lattice $E$ is a lattice-ordered algebra (briefly, $l$-algebra) if $E$ is an associative algebra whose positive cone $E_+$ is closed under the algebra multiplication. A Riesz algebra $E$ is called $f$-algebra if $E$ has additional property that $x \wedge y = 0$ implies $(x \cdot z) \wedge y = (z \cdot x) \wedge y = 0$ for all $z \in E_+$. For an order complete vector lattice (i.e., Dedekind complete), the set $L_b(E)$ of all order bounded operators on $E$ and the set $C(X)$ of all real valued continuous function on a topological space $X$ are examples of lattice-ordered algebra. However, $L_b(E)$ is not $f$-algebra because it is Archimedean vector lattice, but not commutative because every Archimedean $f$-algebra is commutative; see for example Theorem 140.10 in [12]. Consider $Orth(E) := \{T \in L_b(E) : x \perp y \implies Tx \perp y\}$ the set of orthomorphisms on a vector lattice $E$. Then, space $Orth(E)$ is not only vector lattice but also an $f$-algebra. On the other hand, a sublattice $A$ of an $l$-algebra $E$ is called $l$-subalgebra of $E$ whenever it is also an $l$-algebra under the multiplication operation in $E$. In this paper, we assume that if a positive element has an inverse then the inverse also positive. We refer the reader for much more information on Riesz algebras [1-3, 6, 9, 12]. In addition, for more details information on the following example, we refer the reader to [4].

**Example 1.** Let $E$ be a vector lattice. An order bounded band preserving operator $T: D \to E$ on an order dense ideal $D \subseteq E$ is an extended orthomorphism. $Orth^\infty(E)$ denote the set of all extended orthomorphisms: denote by $\mathcal{M}$ the collection of all pairs $(D, \pi)$, where $D$ is order dense ideal in $E$ and $\pi \in Orth(D, E)$. Then space $Orth^\infty(E)$ is an $f$-algebra. Moreover, $Orth(E)$ is an $f$-subalgebra of $Orth^\infty(E)$. On the other hand, $L(E)$ stands for the order ideal generated by the identity operator $I_E$ in $Orth(E)$. Then $L(E)$ is an $f$-subalgebra of $Orth(E)$.

Recall that a net $(x_\alpha)_{\alpha \in A}$ in a vector lattice $X$ is called order convergent (or shortly, $o$-convergent) to $x \in X$, if there exists another net $(y_\beta)_{\beta \in B}$ satisfying $y_\beta \downarrow 0$ (i.e. $y_\beta \downarrow$ and $\inf(y_\beta) = 0$), and for any $\beta \in B$ there exists

$\alpha_\beta \in A$ such that $|x_\alpha - x| \leq y_\beta$ for every $\alpha \geq \alpha_\beta$. In this case, we write $x_\alpha \xrightarrow{o} x$. On the other hand, for a given positive element $u$ in a vector lattice $E$, a net $(x_\alpha)_{\alpha \in A}$ in $E$ is said to converge $u$-uniformly to the element $x \in E$ whenever, for every $\varepsilon > 0$, there exists an index $\alpha_0$, such that $|x_\alpha - x| < \varepsilon u$ for every $\alpha \geq \alpha_0$. Moreover, $E$ is said to be $u$-uniformly complete if every $u$-uniform Cauchy net has an $u$-uniform limit [8].

Let $X$ be a vector space, $E$ be a vector lattice, and $p: X \to E_+$ be a vector norm (i.e., $p(x) = 0 \Leftrightarrow x = 0$; $p(\omega x) = |\omega| p(x)$ for all $\omega \in \mathbb{R}$ and every $x \in X$; $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$) then the triple $(X, p, E)$ is called a lattice normed space, abbreviated as $LNS$. A subset $Y$ of $X$ is called $p$-closed whenever every net $(y_\alpha)_{\alpha \in A}$ in $Y$ with $p(y_\alpha - y) \xrightarrow{o} 0$ implies $y \in Y$. Let $(X, p, E)$ and $(Y, q, F)$ be two $LNS$s. Then an operator $T : X \to Y$ is called dominated operator if there is a positive operator $S : E \to F$ such that $q(T(x)) \leq S(p(x))$ for all $x \in X$. In this case, $T$ is called a dominated operator and $S$ is called dominant of $T$. Take $maj(T)$ as the set of all dominants of the operator $T$. If there is at least element in $maj(T)$ then it is called the exact dominant of $T$ and denoted by $[T]$; see for much more details information [4,7]. If $X$ is decomposable space and $F$ is order complete then exact dominant exists; see Theorem 4.1.2 in [7].

**Definition 2.** Consider an $LNS$ $(X, p, E)$. Assume $X$ and $E$ are Riesz algebras, and the vector norm $p$ is monotone (i.e. $x \leq y$ implies $p(x) \leq p(y)$) then the triple $(X, p, E)$ is said to be lattice normed Riesz algebra (or lattice normed $l$-algebra, for short) and abbreviated as $LNFA$.

**Definition 3.** Let $(X, p, E)$ be an $LNFA$ and $Y$ be an $l$-subalgebra of $X$. If $p(x \cdot y) = y \cdot p(x)$ holds for all $y \in Y$ and $x \in X$ then $p$ is said to be $l$-subalgebra linear. In addition, we said that $(X, p, E)$ has the $l$-subalgebra linear property.
Recall that an element $x$ in Riesz algebra is called nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. Moreover, an algebra $E$ is called semiprime if the only nilpotent element in $E$ is zero.

**Lemma 4.** Let $E$ be a semiprime $f$-algebra. Then $x \leq y$ and $x \leq z$ imply $x^2 \leq y \cdot z$ for all $x, y, z \in E_+$.

*Proof:* Suppose $x, y, z$ are positive elements in $E$ such that $x \leq y$ and $x \leq z$. It follows from Theorem 3.2.(ii) in [8] that $x^2 \leq y \cdot z$.

**Example 5.** Let $E$ be a vector lattice such that $x^2 = x$ for all $x \in E_+$ and $p: L(E) \to Orth(E)$ be a map denoted by $T \to p(T)$ such that $P(T)(x) := |T(x)|$ for each $x \in E$. Then one can see that $p$ is vector norm and $(L(E), p, Orth(E))$ is an $LNS$. Moreover, since $L(E)$ and $Orth(E)$ are $f$-algebras and $|\cdot|$ is monotone, and so, $(L(E), p, Orth(E))$ is an $LNFA$. Take arbitrary $T, S \in L(E)$. Then there exist some positive scalars $\lambda_T$ and $\lambda_S$ such that $|T| \leq \lambda_T I$ and $|S| \leq \lambda_S I$ because $L(E)$ is an order ideal generated by the identity operator $I_E$. So, by using the fact [1, p.12], we have

$$p(S(T))(x) = |S(Tx)| \leq |S|(|Tx|) \leq \lambda_S I(|Tx|) \leq \lambda_S |Tx|$$

for each $x \in E$, and similarly, we have

$$p(S(T))(x) = |S(Tx)| \leq |S|(|Tx|) \leq |S|(|T|(|x|)) = |S|(\lambda_T I)(|x|) \leq \lambda_T |S|(|x|)$$

for all $x \in E$. Hence, it follows from Lemma 4. and assumption that $p(S(T))(x) = [p(S(T))(x)]^2 \leq \lambda_S \lambda_T |S|(|x|) \cdot |Tx| = \lambda_S \lambda_T |S|(x) \cdot p(T)(x)$ holds because $Orth(E)$ is a semiprime; see Theorem 142.5 in [12]. Next, consider a new $LNFA$ $(L(E)_+, q, Orth(E))$, where $q(T) = \frac{1}{\lambda_T} p(T)$ for all $T \in L(E)_+$. Then it follows from the above observation that the $LNFA$ space $(L(E)_+, q, Orth(E))$ has the $f$-subalgebra linear property.
For the following example, we consider Theorem 2.62 in [1].

**Example 6.** Let $E$ be an $l$-algebra. Then we define a map $p$ from $E$ to $Orth(E)$ by $u \to p(u) = p_u$ such that $p_u(x) = |x \cdot u|$ for each $x \in E$. So, by using the inequality in [6, p.1], it is easy to see that $p$ is a vector norm and $(E, p, Orth(E))$ is an $LNFA$ with the $l$-subalgebra linear property.
In this paper, unless otherwise, all lattice normed Riesz algebras are assumed to be with the $l$-subalgebra linear property.

**BASIC RESULTS**

We begin the section with the following de notion.

**Definition 7.** Let $(X, p, E)$ be an $LNS$. Then an operator $T: X \to E$ is said to be $E$-dominated if it is dominated by $p$ on $E$. It means that

$$|Tx| \leq p(x)$$

for all $x \in X$.

It can be seen that every dominated operator on $LNS$s is $E$-dominated because the identity operator is dominant of it.

**Lemma 8.** Let $X$ be an $f$-algebra and $Y$ be an $l$-subalgebra of $X$. Then, for any $w \in X_+$, the set $A = \{u + v \cdot w^n : u, v \in Y \text{ and } u \perp v \text{ and } n \in \mathbb{N}\}$ is an $f$-subalgebra of $X$.

*Proof:* Firstly, we show that $A$ is a sublattice of $X$. Take an arbitrary $u + v \cdot w^n \in A$. Then we have $u \perp v$, and so, $u \perp v \cdot w^n$ for all $n \in \mathbb{N}$ because of $w^n \geq 0$ for each $n \in \mathbb{N}$ and $X$ is $f$-algebra. Then, by applying Exercise 2.$(b)$ p.21 in [1], we have $|u + v \cdot w^n| = |u| + |v| \cdot w^n \in A$ because of $|u|, |v| \in Y$ and $|u| \perp |v|$. Then we get the desired result.

Next, we show that A is an $f$-subalgebra of $X$. For any positive elements $u_1 + v_1 \cdot w^n, u_2 + v_2 \cdot w^m \in A_+$, we have

$$(u_1 + v_1 \cdot w^n) \cdot (u_2 + v_2 \cdot w^m) = u_1 \cdot u_2 + u_1 \cdot v_2 \cdot w^m + v_1 \cdot u_2 \cdot w^n + v_1 \cdot v_2 \cdot w^{m+n} \in A_+.$$

Thus, $A$ is an $l$-algebra. On the other hand, assume $(u_1 + v_1 \cdot w^n) \wedge (u_2 + v_2 \cdot w^m) = 0$ for arbitrary $u_1 + v_1 \cdot w^n, u_2 + v_2 \cdot w^m \in A$. Then we have $[(u + v \cdot w^k)\,(u_1 + v_1 \cdot w^n)] \wedge (u_2 + v_2 \cdot w^m) = 0$ for all $u + v \cdot w^k \in A_+$ because of $A_+ \subseteq X_+$ and $X$ is $f$-algebra. Therefore, we obtain that A is a $f$-subalgebra of $X$.

**Proposition 9.** Let $X$ be an $f$-algebra and $Y$ be an $u$-uniformly complete $l$-subalgebra of $X$. Then, for any $w \in X_+$, the set $A = \{u + v \cdot w^n : u, v \in Y \text{ and } u \perp v \text{ and } n \in \mathbb{N}\}$ is also an $u$-uniformly complete $f$-subalgebra of $X$.

*Proof:* Suppose $Y$ is $u$-uniformly complete $l$-subalgebra of $X$. Then, by applying Lemma 8., we see that $A$ is $f$-subalgebra of $X$. On the other hand, WLOG, take an $u$-uniform Cauchy net $(x_\alpha)_{\alpha \in A}$ in $A_+$ with disjoint each other nets $(y_\alpha)_{\alpha \in A}$ and $(z_\alpha)_{\alpha \in A}$ such that $x_\alpha = y_\alpha + z_\alpha \cdot w^n$ with $y_\alpha \perp z_\beta$ for all $\alpha, \beta \in A$. Thus, there exists $u \in E_+$ such that, for every $\varepsilon > 0$, there exists an index $\alpha_0$, such that $|x_\alpha - x_{\alpha_0}| < \varepsilon u$ for all $\alpha \geq \alpha_0$. Then, by using the following the equality

$$|x_\alpha - x_{\alpha'}| = |(y_\alpha + z_\alpha \cdot w^n) - (y_{\alpha'} + z_{\alpha'} \cdot w^n)| = |y_\alpha - y_{\alpha'}| + |z_\alpha - z_{\alpha'}| \cdot w^n,$$

one can obtain that $(y_\alpha)_{\alpha \in A}$ and $(z_\alpha)_{\alpha \in A}$ are $u$-uniform Cauchy nets in $Y$. So, there exist $y, z \in Y$ such that $y_\alpha \overset{u}{\rightarrow} y$ and $z_\alpha \overset{u}{\rightarrow} z$ because Y is $u$-uniformly complete. Therefore, we get $x_\alpha = y_\alpha + z_\alpha \cdot w^n \overset{u}{\rightarrow} y + z \cdot w^n$. As a result, $A$ is also $u$-uniformly complete.

Now, by considering some results in [10], we give the main result of this paper.

**Theorem 10.** Let $(X, p, E)$ be an $LNFA$ with $X$ being $f$-subalgebra of order complete $f$-algebra $E$, and $G$ be an unital $f$-subalgebra of $X$. If $T: G \rightarrow E$ is an $E$-dominated operator and $G$ is $e$-uniform complete then there exists another $E$-dominated operator $\hat{T}: X \rightarrow E$ such that $\hat{T}(g) = T(g)$ for all $g \in G$.

*Proof:* First of all, if we take $T = 0$ or X = G then the poof is obvious.

Suppose, $G$ is a proper subspace of $X$ and $T \neq 0$. So, there is a vector $w$ in $X$ so that it is not in $G$. WLOG, we assume $w \in X_+$ . Then we consider the set $G_1 := \{u + v \cdot w^n : u, v \in G \text{ and } u \perp v \text{ and } n \in \mathbb{N}\}$ which is like Proposition 9. Thus, by Lemma 4., we get that $G_1$ is also an $f$-subalgebra of $X$. Also, by using this extension, we can arrive at $X$ because $G$ is $f$-subalgebra with the multiplicative unit.

The extension of one step is not similar to the other Hahn-Banach theorems. It can be observed that $v \cdot w^n$ can be in $G$ for some $v \in G$. Thus, we have that the representation $G_1$ may not be unique. So, it causes difficulties getting an extension of one step. Whenever it is done, by using Zorn's lemma and applying Proposition 9., we can get the extension of $T$ to $X$.

Now, consider elements $u, v \in G$. Since $T$ is an $E$-dominated operator. Then, for every $n$, we have

$$T(u) + T(v) = T(u + v) \leq p(u - w^n + w^n + v) \leq p(u - w^n) + p(w^n + v)$$

Hence, we get $T(u) - p(u - w^n) \leq p(w^n + v) - T(v)$. From there, by applying order completeness of $E$, both

$$s = sup\{T(u) - p(u - w^n) : u \in G\}$$

and

$$r = inf\{p(w^n + v) - T(v) : v \in G\}$$

exist in $E$ for each $n$. So, it is also clear $s \leq r$. Now, we define a map

$$\hat{T}: G_1 \rightarrow E$$
$$(u + v \cdot w^n) \rightarrow \hat{T}(u + v \cdot w^n) = T\,(u) + v \cdot z^n,$$

where we take the element $z \in E$ such that $s \leq z^n \leq r$ for each $n$. We need to show that $T$ is a well-defined operator. To prove that, we first prove the $E$-dominatedness of $\hat{T}$. Let's apply the $e$-uniformly completeness of $G$. Then we have that $(v + e)^{-1}$ exits for any positive element $v \in G_+$; see Theorem 146.3 in [12] and Theorem 11.1 in [9], and also, the inverse element $(v + \frac{1}{k}e)^{-1}$ exists in $G_+$ for all $k \in \mathbb{N}_+$. Then, for each $u \in G_+$ and $k, n \in \mathbb{N}_+$, we have

$$z^n \leq r \leq p\left(u \cdot (v + \frac{1}{k}e)^{-1} + w^n\right) - T\left(u \cdot (v + \frac{1}{k}e)^{-1}\right)$$

and so, by using the $f$-subalgebra linear property of $p$, we get

$$T(u) + \left(v + \tfrac{1}{k}e\right) \cdot z^n \leq p\left(u + w^n \cdot \left(v + \tfrac{1}{k}e\right)\right) \leq p(u + w^n \cdot v) + \tfrac{1}{k}p(w^n).$$

Thus, we have $\hat{T}(u + v \cdot w^n) = T(u) + v \cdot z^n \leq p(u + w^n \cdot v)$ for any $u, v \in G_+$ because $F$ is an Archimedean vector lattice. Thus, we get the $E$-

dominatedness of $\hat{T}$ for arbitrary $u, v \in G_+$. Now, we show that for arbitrary $v \in G$. We can write $= v^+ - v^-$. By using the first observation, we can write

$$\hat{T}(u + v^+ \cdot w^n) = T(u) + v^+ \cdot z^n \leq p(u + w^n \cdot v^+)(1)$$

For the band $B_{v^+}$ generated by $v^+$, we consider the band projection $q: G \to B_{v^+}$. Then $q$ holds $q(v) = v^+$ and $q = q^2$, and it is a positive orthomorphism on $G$ because every order projection is a positive orthomorphism on vector lattices. By using Theorem 141.1 in [12], we can choose a positive element $t \in G_+$ such that $q(x) = x \cdot t$ for all $x \in G$. Thus, we have a positive vector $t \in G_+$ so that $v^+ = q(v) = v \cdot t$, and $t = e \cdot t = q(e) = q(q(e)) = t^2$, and $v^+ = q(v^+) = v^+ \cdot t$, and $0 = q(v^-) = v^- \cdot t$. Also, the equality $v^+ = q(v) = v \cdot t$ implies $v^- + v = v^+ = v \cdot t$, and so, we vet $v^- = v \cdot (t - e)$. Thus, we obtain the following both equalities

$$t \cdot (v^+ \cdot z^n) = (t \cdot v^+) \cdot z^n = v^+ \cdot z^n \ (2)$$

and

$$t \cdot (v^+ \cdot w^n) = t \cdot v^+ \cdot w^n = t \cdot (v \cdot t) \cdot w^n = t^2 \cdot v \cdot w^n = t \cdot v \cdot v^+ \cdot w^n \ \ (3)$$

It follows from $(1), (2)$ and $(3)$ and the $f$-subalgebra linear property of $p$ that we get

$$t \cdot (T(u) + v^+ \cdot z^n) \leq t \cdot p(u + v^+ \cdot w^n) = p(t \cdot u + t \cdot v^+ \cdot w^n) = t \cdot p(u + v \cdot w^n) \quad (4)$$

As one repeat the same way and use $r \leq z^n$, it can be seen the following inequality

$$(e - t) \cdot (T(u) - v^- \cdot z^n) \leq (e - t) \cdot p(u + v \cdot w^n)(5)$$

Therefore, by summing up the inequalities (4) and (5), we can get the following result

$$T(u) + v \cdot z^n \leq p(u + v \cdot w^n) \quad (6)$$

for arbitrary $v \in G$ and $u \in G_+$. Lastly, one can also show that for arbitrary element $u \in G$. Therefore, we get that $T$ is $E$-dominated. Now, we show well defined of $T$. Let's take arbitrary elements $u_1, u_2, v_1, v_2 \in G$ such that $u_1 + v_1 \cdot w^n = u_2 + v_2 \cdot w^n$. It follows from (6) that

$T(u_1 - u_2) + (v_1 - v_2) \cdot z^n \leq p((u_1 - u_2) + (v_1 - v_2) \cdot w^n) = p(0) = 0$ and $T(u_2 - u_1) + (v_2 - v_1) \cdot z^n \leq p((u_2 - u_1) + (v_2 - v_1) \cdot w^n) = p(0) = 0$. As a result, we get $\hat{T}(u_1 + v_1 \cdot w^n) = \hat{T}(u_2 + v_2 \cdot w^n)$.

Therefore, we have obtained that the map $\hat{T}$ is well defined. On the other hand, by using the linearity of $\hat{T}$, one can show that $\hat{T}$ is a linear map (or, operator) from $G_1$ to $F$ . Expressly, $\hat{T}$ is $E$-dominated operator by $f$-subalgebra linear map $p$. By applying Zorn's lemma under the desired conditions, we provide the extension of $\hat{T}$ to all of $X$.

Under the condition of Theorem 1, we have the following results.

**Corollary 11.** If $(X, p, E)$ is a decomposable $LNFA$ then we have $[T] = [\hat{T}]$.

*Proof:* Since $T$ is $E$-dominated operator, it is dominated. Indeed, Since $|T\ (g)| \leq p(g)$, we have $p(T(g)) \leq p(p(g))$ (for example we can take a dominant S = p). It follows from Theorem 4.1.2. in [7] that $T$ has the exact dominant [T]. Now, consider the $f$-subalgebra $G_1$ of $X$ in the proof of Theorem 1. For $v = 0$ the addition unit and $u \in G$, we have

$$\hat{T}(u) = T(u) \leq |T(u)| \leq S(p(u))$$

and also

$$-\hat{T}(u) = -T(u) \leq |T(u)| \leq S(p(u)):$$

Therefore, we get $|\hat{T}(u)| \leq S(p(u))$ for each $u \in G$. Hence, $\hat{T}$ is also dominated by $S$, and so, we get $[\hat{T}] \leq [T]$. On the other hand, by considering the $maj(T)$ and $maj(\hat{T})$, we have $[T] \leq [\hat{T}]$. As a result, we get the desired result.

For the next result, we consider the $f$-algebraic spaces $L(E) \subseteq Orth(E) \subseteq Orth^\infty(E)$ in Example 1.

**Corollary 2.7.** Let $E$ be an order complete vector lattice. $(Orth(E), |\cdot|, Orth^\infty(E))$ is an $LNFA$. Moreover, If $T: L(E) \to Orth^\infty(E)$ an $E$-dominated operator then it has an extension to $Orth(E)$.

*Proof*: Since $E$ is order complete vector lattice, $Orth^\infty(E)$ is order complete $f$-algebra; see [4, p.14]. Moreover, one can say that $(Orth(E), |\cdot|, Orth^\infty(E))$ is an $LNFA$ because $Orth(E)$ is $f$-subalgebra of $Orth^\infty(E)$ and $|\cdot|$ has the $l$-subalgebra linear property. By applying Theorem 3.1 in [5], we can see that $L(E)$ is order complete because $E$ is order complete. Moreover, by using Theorem 42.6 in [8], we also get that $L(E)$ is $e$-uniform complete because $L(E)$ has unit $I_E$. Then, we have an $E$-dominated extension of $T$ to $Orth(E)$.

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