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Approximate Analytical Solutions of Conformable Time Fractional Clannish Random Walker's Parabolic(CRWP) Equation and Modified Benjamin-Bona-Mahony(BBM) equation

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Abstract

In this paper, we propose the approximate analytical solutions of conformable time fractional Clannish Random Walker's Parabolic(CRWP) equation and Modified Benjamin-Bona-Mahony(BBM) equation with the aid of generalized homotopy analysis method (q-HAM). The h curves of approximate solutions for both equations are illustrated by graphics to determine the convergence interval. h values obtained from these graphics are used to compare approximate solutions with the analytical solutions. The results show that approximate solutions are consistent with the analytical solutions. Also it is understood that the method is reliable, applicable and efficient technique to get the exact solutions of fractional partial differential equations.

1. Introduction

Fractional derivative and fractional integration are challenging works in applied mathematics. They can also be used in different branches of science and engineering. Thus, there are many approaches in literature to define fractional derivative and integration. Among them, Riemann-Lioville definition and Caputo definition are the most used ones [1–3]. However, these definitions have some drawbacks in the following cases:

1. The Riemann-Lioville derivative does not satisfy $D_a^\alpha 1 = 0$ (Liouville-Caputo derivative satisfies), if α is not a natural number.
2. All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions.

$$D_a^\alpha (fg) = gD_a^\alpha (f) + fD_a^\alpha (g)$$

3. All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions.

$$D_a^\alpha \left(\frac{f}{g}\right) = \frac{fD_a^\alpha (f) - gD_a^\alpha (g)}{g^2}$$

4. All fractional derivatives do not satisfy the chain rule.

$$D_a^\alpha (fog)(t) = f^\alpha (g(t))g^\alpha (t)$$

5. All fractional derivatives do not satisfy $D^\alpha D^\beta = D^{\alpha+\beta}$ in general.
6. The Caputo definition assumes that the function f is differentiable.

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Because of these drawbacks of the existing definitions, researchers have been working on a better definition so that all the cases can be satisfied. In recent years, a new simple definition was introduced by Khalil *et.al* [4] called conformable fractional derivative. For the function $f:[0,\infty)\to\mathbb{R}$, the fractional derivative of f was defined by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

where $t>0$ and $\alpha\in(0,1)$.
If f is α-differentiable in some $(0,a), a>0$ and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists then define $f^{(\alpha)}(0)=\lim_{t\to 0^+} f^{(\alpha)}(t)$. The *"conformable fractional integral"* of a function f starting from $a\geq 0$ is defined as:

$$I_\alpha^a(f)(t)=\int_a^t \frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$. The following properties of conformable fractional derivative are given in [4].

Theorem 1.1. *Let $\alpha\in(0,1]$ and suppose f,g are α-differentiable at point $t>0$. Then*

1. $T_\alpha(cf+dg)=cT_\alpha(f)+cT_\alpha(g)$ *for all* $a,b\in\mathbb{R}$.
2. $T_\alpha(t^p)=pt^{p-\alpha}$ *for all* $p\in\mathbb{R}$.
3. $T_\alpha(\lambda)=0$ *for all constant functions* $f(t)=\lambda$.
4. $T_\alpha(fg)=fT_\alpha(g)+gT_\alpha(f)$.
5. $T_\alpha\left(\frac{f}{g}\right)=\frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
6. *If, in addition f is differentiable, then* $T_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}$.

Recently, Abdeljawad applied Khalil's definition on some fractional calculus functions, such as fractional versions of chain rule, exponential functions, Laplace transforms and so on [5]. Due to the simplicity and usefulness of this conformable version of fractional calculus, many researches applied this method in their field. Chung [6] applied conformable fractional derivatives on Newton's mechanics and constructed the fractional Euler-Lagrange equation. Neirameh [7] applied this method on a fractional order of extended biological population model. Benkhettou *et al.* [8] extended Khalil's definition to an arbitrary time scale. Similarly, Zhao and Li [9] introduced the conformable delta fractional derivative and delta fractional integral on time series with respect to Khalil's definition. Eslami and Rezazadeh [10] studied the first integral method for Wu-Zhang system with conformable time fractional derivative.
The q-Homotopy Analysis Method (q-HAM) was introduced by El-Tawil and Huseen [11] to solve non-linear differential equations. This is a more general method of Homotopy Analysis Method [17–20] (HAM) developed by Liao [12] and has one more parameter n. With the aid of this additional parameter, q-HAM provides more flexibility than HAM in controlling and adjusting the convergence region. q-HAM have been used to solve many mathematical problems in recent years. Iyiola *et.al.* [13] investigated an analytical solution of the time-fractional foam drainage equation using the advantages of q-HAM. In another study, q-HAM was used to find approximate series solutions of a fractional diffusion equation model [14].
The CRWP equation determines the behavior of two species A and B of random walkers who execute a concurrent one-dimensional random walk characterized by an intensification of the clannishness of the members of one species as the density of the other increases and the Benjamin-Bona-Mahony(BBM) equation is used for the analysis of the surface waves of long wavelength in liquids, hydromagnetic waves in cold plasma.

2. Description of q-Homotopy Analysis Method

We give an overview of the q-homotopy analysis method in this section and show how it is used in fractional differential equations.
Consider the differential equation below,

$$\mathscr{N}\left[D_t^\alpha u(x,t)\right]-g(x,t)=0 \tag{2.1}$$

where $\mathscr{N}$ is non-linear operator, D_t^α is the conformable fractional operator, g is the given function and $u(x,t)$ is the function which will be obtained after this solution procedure. The zeroth-order deformation equation for q-HAM is given as follows:

$$(1-nq)\mathscr{L}\left(\varphi(x,t;q)-u_0(x,t)\right)=qhH(x,t)\left(\mathscr{N}\left[D_t^\alpha\varphi(x,t;q)\right]-g(x,t)\right), \tag{2.2}$$

where $n\geqslant 1$, $q\in\left[0,\frac{1}{n}\right]$ is the embedded parameter, $h\neq 0$ is an auxiliary parameter and L is an auxiliary linear operator. $H(x,t)$ is a non-zero auxiliary function.
When $q=0$ and $q=\frac{1}{n}$, which are the boundary values, we have equation (2.2) to become

$$\varphi(x,t;0)=u_0(x,t) \quad and \quad \varphi\left(x,t;\frac{1}{n}\right)=u(x,t) \tag{2.3}$$

respectively. When we increment q from 0 to $\frac{1}{n}$, the solution $\varphi(x,t;q)$ changes from the initial guess $u_0(x,t)$ to the solution $u(x,t)$.
If $u_0(x,t)$, $\mathscr{L}$, h, $H(x,t)$ are chosen properly, then there always exists a solution $\varphi(x,t;q)$ of equation (2.2) for $q\in\left[0,\frac{1}{n}\right]$.
The Taylor series expansion of $\varphi(x,t;q)$ is

$$\varphi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m. \tag{2.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\Big|_{q=0}. \tag{2.5}$$

If the auxiliary linear operator L, the initial guess u_0, the auxiliary parameter h and $H(x,t)$ are properly chosen, then the Taylor series expansion of $\varphi(x,t;q)$ (2.4) converges at $q = \frac{1}{n}$. Hence, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)\left(\frac{1}{n}\right)^m. \tag{2.6}$$

Let the vector $\vec{u}_n$ is defined as follows:

$$\vec{u}_n = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}. \tag{2.7}$$

First, the equation (2.2) is differentiated m-times with respect to the embedding parameter q. Then, $q = 0$ is taken and placed in the differentiated equation. Finally, the whole equation is divided by $m!$, and the m^{th}-order deformation equation is obtained as follows:

$$\mathscr{L}\left[u_m(x,t) - \chi_m^* u_{m-1}(x,t)\right] = hH(x,t)\mathscr{R}_m\left(\vec{u}_{m-1}\right). \tag{2.8}$$

with initial conditions

$$u_m^{(k)}(x,0) = 0, \quad k = 0,1,2,...,m-1. \tag{2.9}$$

where

$$\mathscr{R}_m\left(\vec{u}_{m-1}\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\left(\mathscr{N}\left[Du_t^{\alpha}\varphi(x,t;q)\right] - g(x,t)\right)}{\partial q^{m-1}}\Big|_{q=0} \tag{2.10}$$

and

$$\chi_m^* = \begin{cases} 0 & m \leqslant 1 \\ n & otherwise. \end{cases} \tag{2.11}$$

3. Applications of the Method

3.1. Conformable Time-Fractional Clannish Random Walker's Parabolic(CRWP) Equation

Considering the conformable time-fractional CRWP equation as:

$$D_t^{\alpha} v + pv_x + svv_x + rv_{xx} = 0 \tag{3.1}$$

and initial condition

$$v(x,0) = \frac{r\ln(A) + \sqrt{r^2(\ln(A))^2 - 2sK}}{s} - \frac{2r\ln(A)}{s(1+dA^x)} \tag{3.2}$$

where $\alpha \in (0,1)$. The exact solution of CRWP was provided from ref. [15] as:

$$v(x,t) = \frac{r\ln(A) + \sqrt{r^2(\ln(A))^2 - 2sK}}{s} - \frac{2r\ln(A)}{s(1+dA^{x-(\sqrt{r^2(\ln(A))^2-2sK}+p)\frac{t^{\alpha}}{\alpha}})} \tag{3.3}$$

This analytical solution is used to compare with the numerical solutions obtained in this study. To obtain the approximate solution of Eq. (3.1) with initial condition (3.3), we can choose the linear operator is as

$$\mathscr{L}\left[\varphi(x,t;q)\right] = D_t^{\alpha}\varphi(x,t;q)$$

which satisfies the property

$$\mathscr{L}[m] = 0$$

where m is a constant. From Eq. (3.1), we define the nonlinear operator as follow:

$$\mathscr{N}\left[\varphi(x,t;q)\right] = \frac{\partial^{\alpha}\varphi(x,t;q)}{\partial t^{\alpha}} + p\frac{\partial\varphi(x,t;q)}{\partial x} + s\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} + r\frac{\partial^2\varphi(x,t;q)}{\partial x^2}.$$

From Theorem 1, the nonlinear operator can be re-written as follows:

$$\mathscr{N}\left[\varphi(x,t;q)\right] = t^{1-\alpha}\frac{\partial\varphi(x,t;q)}{\partial t} + p\frac{\partial\varphi(x,t;q)}{\partial x} + s\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x} + r\frac{\partial^2\varphi(x,t;q)}{\partial x^2}.$$

The zeroth-order deformation equation is designed as:

$$(1-nq)\mathscr{L}\left[\varphi(x,t;q)-v_0(x,t)\right]=qh\mathscr{N}\left[\varphi(x,t;q)\right].$$

Considering $H(x,t)=1$, the m^{th}-order deformation equation

$$\mathscr{L}\left[v_m(x,t)-\chi_m^* v_{m-1}(x,t)\right]=hR_m\left(\mathbf{v}_{m-1}\right) \tag{3.4}$$

where

$$R_m\left(\mathbf{v}_{m-1}\right)=t^{1-\alpha}\frac{\partial v_{m-1}(x,t)}{\partial t}+s\sum_{n=0}^{m-1}v_n(x,t)\frac{\partial v_{m-1-n}(x,t)}{\partial x}+p\frac{\partial v_{m-1}(x,t)}{\partial x}+r\frac{\partial^2 v_{m-1}(x,t)}{\partial x^2}.$$

The solutions of the m^{th}-order deformation Eq. (3.4) for $m\geq 1$ is below

$$v_m(x,t)=\chi_m^* v_{m-1}(x,t)+h\mathscr{L}^{-1}\left[R_m\left(\mathbf{v}_{m-1}\right)\right]. \tag{3.5}$$

By using Eq.(3.5) with initial condition given by (3.3) we calculated $v_m(x,t)$ for $m=0,1$ respectively.

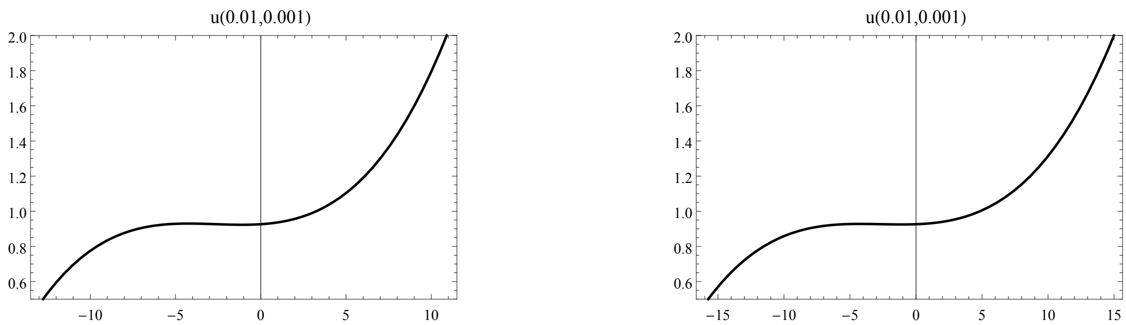
$$v_0(x,t)=\frac{r\ln(A)+\sqrt{r^2(\ln(A))^2-2sK}}{s}-\frac{2r\ln(A)}{s(1+dA^x)},$$

$$v_1(x,t)=\frac{2t^{\alpha}A^x dhr(\ln(A))^2(p+\sqrt{-2ks+r^2(\ln(A))^2})}{(1+A^x d)^2 s\alpha}.$$

One can obtain $v_m(x,t)$ for $m=2,3,\cdots$, following the same procedure using computer software such as Mathematica.
Finally, the series solution of equation (3.1) by applying q-HAM can be written in the form

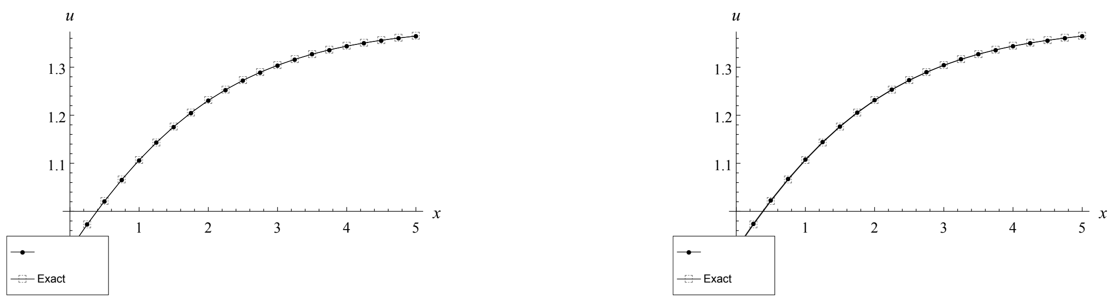
$$v(x,t,n,h)=v_0(x,t)+\sum_{n=1}^{\infty}v_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{3.6}$$

Equation (3.6) is an approximate solution to the problem (3.1) in terms of convergence parameter h and n. We take the parameters $p=2$, $q=2$, $r=2$, $d=2$, $A=2$ and $k=2$ are used for all calculations by 4^{th}-order q-HAM solution. Any other solutions can be calculated for different parameters.
The h-curves of the conformable time-fractional CRWP equation for $n=2$ and for $\alpha=0.8$ and $\alpha=0.9$ are given in Fig. 1. For $h=-1.11$, $t=0.001$ (fixed) and $\alpha=0.8$, $\alpha=0.9$, the approximate solution and the exact solution for the conformable time-fractional CRWP equation are compared in Fig. 2.



(a) $\alpha=0.8$ **(b)** $\alpha=0.9$

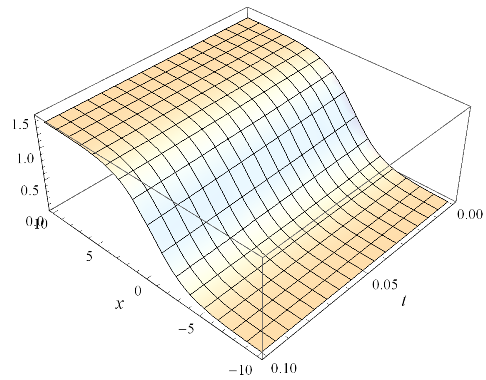
Figure 3.1: The h-curves of the conformable time-fractional CRWP equation for $n=2$ and different values of α.



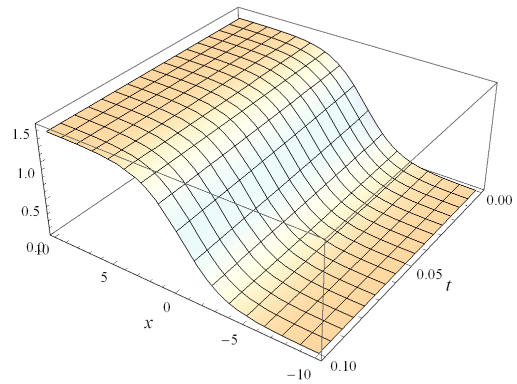
(a) Approximate solution for $\alpha=0.8$ **(b)** Approximate solution for $\alpha=0.9$

Figure 3.2: Comparison of the approximate solutions obtained for $h=-1.11$, $n=2$, $t=0.001$ and different values of α by the q-HAM with the exact solution for the conformable time-fractional CRWP equation

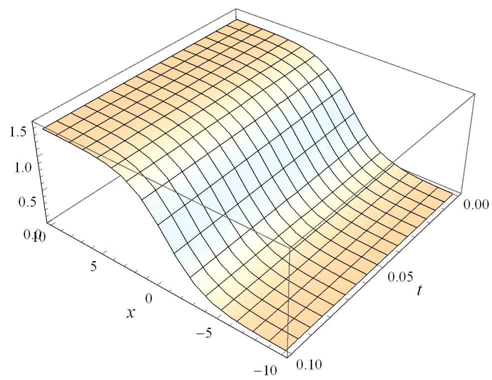
For $h=-1.11$ value which is selected from these graphics for both α values, the surfaces of the approximate solutions and the exact solutions are drawn in Fig. 3. As seen in Figure 3, the surfaces of the approximate solutions and the analytical solutions are so similar that cannot be distinguished.



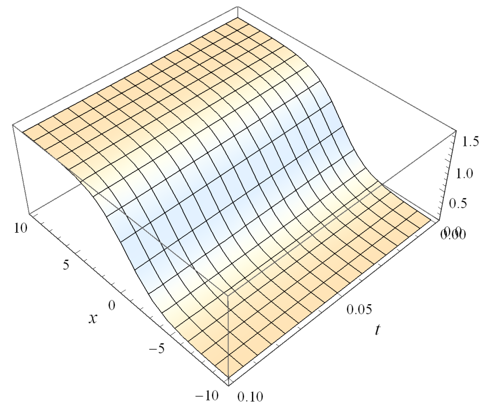
(a) Approximate solution for $\alpha = 0.8$



(b) Exact solution for $\alpha = 0.8$



(c) Approximate solution for $\alpha = 0.9$



(d) Exact solution for $\alpha = 0.9$

Figure 3.3: Comparison of the approximate solutions obtained for $h = -1.11$, $n = 2$ and different values of α by the q-HAM with the exact solution for the conformable time-fractional CRWP equation

3.2. Conformable Time-Fractional Modified Benjamin-Bona-Mahony(BBM) Equation

As the second application of q-HAM, we discuss the approximate solutions for BBM. The conformable time-fractional BBM equation is given as:

$$D_t^{\alpha} u + u_x - \nu u^2 u_x + u_{xxx} = 0 \tag{3.7}$$

and initial condition

$$u(x,0) = \frac{\sqrt{3}}{x+1} \tag{3.8}$$

where $\alpha \in (0,1)$. The exact solution was taken from [16]

$$u(x,t) = \frac{\sqrt{6}k}{\sqrt{\nu}(kx - k\frac{t^{\alpha}}{\alpha} + C)} \tag{3.9}$$

and $k = 1$, $C = 1$, $\nu = 2$ are chosen for all calculations by 4^{th}-order q-HAM solution. To find the series approximate solution of Eq. (3.7) with the initial condition (3.9), the linear operator is picked as

$$\mathscr{L}[\varphi(x,t;q)] = D_t^{\alpha} \varphi(x,t;q)$$

with the property

$$\mathscr{L}[m] = 0$$

where m is a constant. From Eq. (3.7), the nonlinear operator can be designed as following,

$$\mathscr{N}[\varphi(x,t;q)] = \frac{\partial^{\alpha} \varphi(x,t;q)}{\partial t^{\alpha}} + \frac{\partial \varphi(x,t;q)}{\partial x} - \nu \varphi(x,t;q)^2 \frac{\partial \varphi(x,t;q)}{\partial x} + \frac{\partial^3 \varphi(x,t;q)}{\partial x^3}.$$

Using the property of the conformable fractional derivative, the nonlinear operator can be re-written as follows,

$$\mathscr{N}[\varphi(x,t;q)] = t^{1-\alpha} \frac{\partial \varphi(x,t;q)}{\partial t} + \frac{\partial \varphi(x,t;q)}{\partial x} - \nu \varphi(x,t;q)^2 \frac{\partial \varphi(x,t;q)}{\partial x} + \frac{\partial^3 \varphi(x,t;q)}{\partial x^3}.$$

Hence, the zeroth-order deformation equation can be constructed as:

$$(1-nq)\mathscr{L}\left[\varphi(x,t;q)-u_0(x,t)\right]=qh\mathscr{N}\left[\varphi(x,t;q)\right].$$

For $H(x,t)=1$, the mth-order deformation equation turns into

$$\mathscr{L}\left[u_m(x,t)-\chi_m^* u_{m-1}(x,t)\right]=hR_m\left(\mathbf{u}_{m-1}\right) \tag{3.10}$$

where

$$R_m\left(\mathbf{u}_{m-1}\right)=t^{1-\alpha}\frac{\partial u_{m-1}(x,t)}{\partial t}+\frac{\partial u_{m-1}(x,t)}{\partial x}+\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}-\nu\sum_{n=0}^{m-1}\left(\sum_{k=0}^{n}u_k(x,t)u_{n-k}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}.$$

The solutions of the mth-order deformation Eq. (3.10) for $m\geq 1$ is obtained as

$$v_m(x,t)=\chi_m^* v_{m-1}(x,t)+h\mathscr{L}^{-1}\left[R_m\left(\mathbf{v}_{m-1}\right)\right]. \tag{3.11}$$

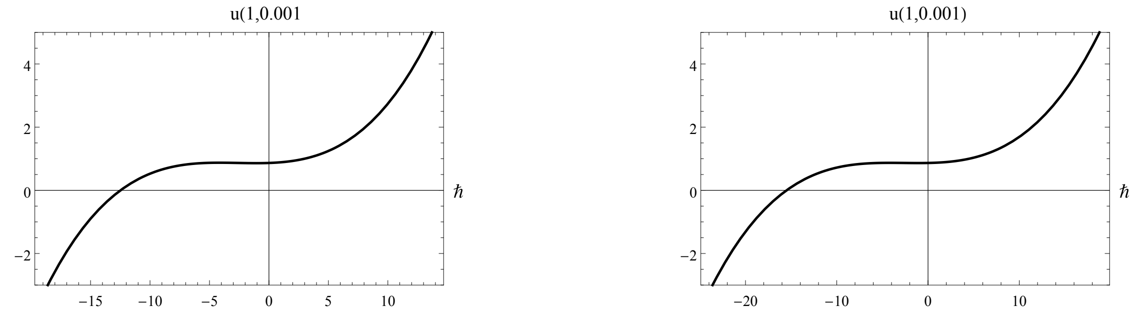
With the aid of Eq.(3.11) and the initial condition (3.9) we get the approximate solutions of $u_m(x,t)$ for $m=0,1$ as follows:

$$\begin{aligned}
u_0(x,t)&=\frac{\sqrt{3}}{x+1},\\
u_1(x,t)&=\frac{\sqrt{3}ht^{\alpha}(5x^2+10x-1)}{(x+1)^4\alpha}.
\end{aligned}$$

The series solution of (3.7) for $m=0,1,\cdots$ expression by q-HAM can be expressed as:

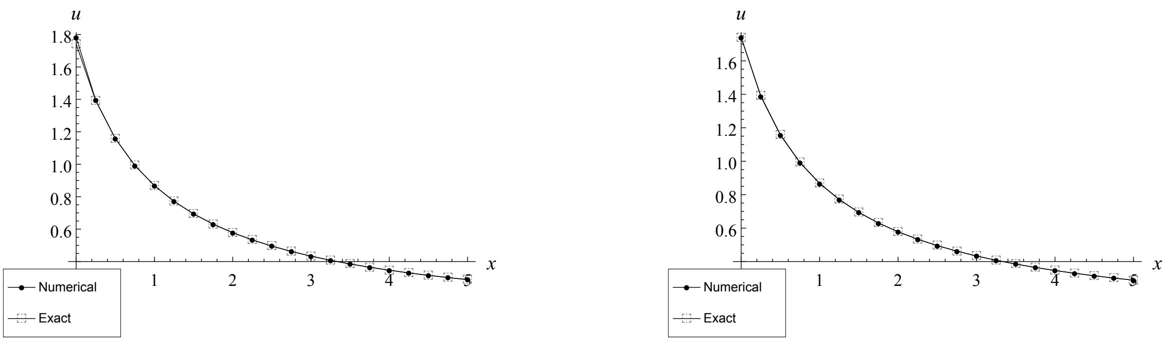
$$u(x,t,n,h)=\frac{\sqrt{3}}{x+1}+\sum_{n=1}^{\infty}u_i(x,t;n;h)\left(\frac{1}{n}\right)^i. \tag{3.12}$$

The h-curves of the conformable time-fractional BBM equation for $n=2$ and for $\alpha=0.8$ and $\alpha=0.9$ are given in Fig. 4. In Fig. 5, the approximate solution and the exact solution for the conformable time-fractional BBM equation are compared, where $h=-2.787$, $t=0.001$ (fixed) and $\alpha=0.8$, $\alpha=0.9$. In Fig. 6, for $h=-2.787$ value which is chosen from these graphics for both α values, the surfaces of the approximate solutions and the exact solutions are illustrated.



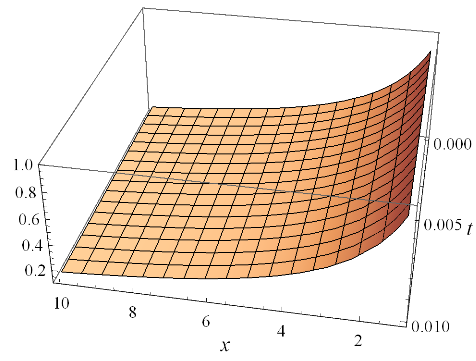
(a) $\alpha=0.8$ **(b)** $\alpha=0.9$

Figure 3.4: The h-curves of the conformable time-fractional BBM equation for $n=2$ and different values of α.

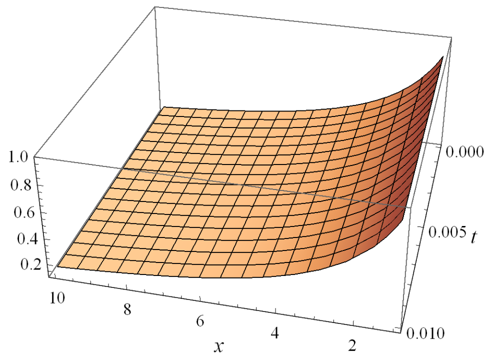


(a) Approximate solution for $\alpha=0.8$ **(b)** Approximate solution for $\alpha=0.9$

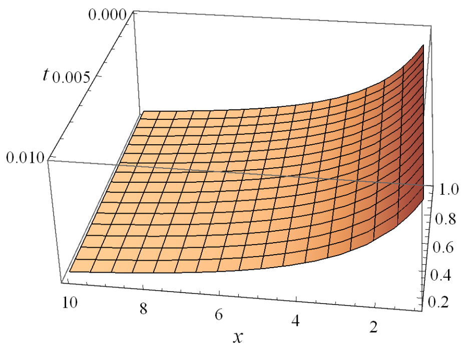
Figure 3.5: Approximate solutions obtained for $h=-2.787$, $n=2$, $t=0.001$ and different values of α by the q-HAM in comparison with exact solution for the conformable time-fractional BBM equation



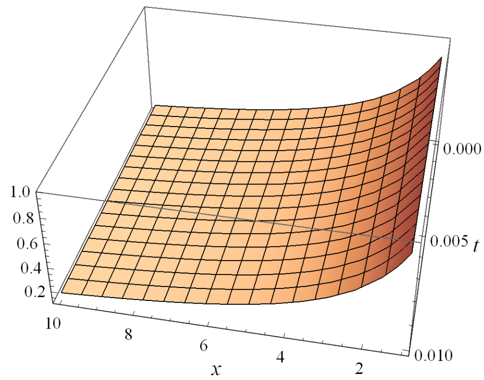
(a) Approximate solution for $\alpha = 0.8$



(b) Exact solution for $\alpha = 0.8$



(c) Approximate solution for $\alpha = 0.9$



(d) Exact solution for $\alpha = 0.9$

Figure 3.6: Approximate solutions obtained for $h = -2.787$, $n = 2$ and different values of α by the q-HAM in comparison with exact solution for the conformable time-fractional BBM equation

4. Conclusions

In this paper we employ the q-Ham to obtain approximate analytical solutions of Modified Benjamin-Bona-Mahony(BBM) and Clannish Random Walker's Parabolic(CRWP) equations. q-HAM contains the parameter h which is used for adjusting the suitable convergence interval. The results show that the obtained approximate solutions are compatible with the exact solutions. As a result, q-HAM is an efficient and reliable technique to obtain the approximate analytical solutions of nonlinear conformable fractional partial differential equations. When $\mu = 1$ is chosen, it is seen that the solutions are the same of the non-fractional cases. In fact, this is the nature of fractional calculus. When fractional operations are converted to integer order operations, the results are consistent.

References

[1] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[2] A. A. Kilbas , H. M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.
[3] K.S. Miller, B. Ross ,*An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, 1993.
[4] R. Khalil, M. A. Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math., **264** (2014), 65-70.
[5] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math., **279** (2015), 57-66.
[6] W.S. Chung, *Fractional Newton mechanics with conformable fractional derivative*, J. Comput. Appl. Math., **290** (2015), 150-158.
[7] A. Neirameh, *New fractional calculus and application to the fractional-order of extended biological population model*, Bol. da Soc. Parana. Matematica, **36**(3)(2018), 115.
[8] N. Benkhettou, S. Hassani, D. F. M. Torres, *A conformable fractional calculus on arbitrary time scales*, J. King Saud Univ. - Sci., **28**(1) (2016), 93-98.
[9] D. Zhao and T. Li, *On conformable delta fractional calculus on time scales*, J. Math. Comput. Sci., **16** (2016), 324-335.
[10] M. Eslami, H. Rezazadeh, *The First integral method for Wu-Zhang system with conformable time-fractional derivative*, Calcolo, **53**(3) (2016), 475-485.
[11] M. A. El-Tawil, S. N. Huseen, *The Q-homotopy analysis method (Q-HAM)*, Int. J. Appl. Math. Mech., **8**(15)(2012), 51-75.
[12] S. J. Liao, *The Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems*, Shanghai Jiao Tong University, 1992.
[13] O. S. Iyiola, M. E. Soh, C. D. Enyi, *Generalised homotopy analysis method (q-HAM) for solving foam drainage equation of time fractional type*, Math. Eng. Sci. Aerosp., **4**(4)(2013), 429-440.
[14] O. S. Iyiola, F. D. Zaman, *A fractional diffusion equation model for cancer tumor*, AIP Adv., **4**(10)(2014), 107121.
[15] A. Korkmaz, *Explicit exact solutions to some one-dimensional conformable time fractional equations*, Waves Random Complex Media (2017), 1-14.
[16] K. Hosseini, A. Bekir, M. Kaplan, O. Guner, *On a new technique for solving the nonlinear conformable time-fractional differential equations*, Opt. Quant. Electron. (2017) 49-343.
[17] P. A. Naik, J. Zu, J., M. Ghoreishi, *Estimating the approximate analytical solution of HIV viral dynamic model by using homotopy analysis method*, Chaos Solitons Fractals (2019), 109500.
[18] M. Ghoreishi, A. I. B. Md.Ismail, A. K. Alomari, A.S. Batainehc, *The comparison between homotopy analysis method and optimal homotopy asymptotic method for nonlinear age-structured population models*, Commun. Nonlinear Sci. Numer. Simul., **17**(3)(2012),1163-1177.
[19] M. Yavuz, *Novel solution methods for initial boundary value problems of fractional order with conformable differentiation*, Int. J. Optim. Control. Theor. Appl. IJOCTA, **8**(1) (2017), 1-7.

[20] M. Ghoreishi, A. M. Ismail, A. K. Alomari, *Comparison between homotopy analysis method and optimal homotopy asymptotic method for nth-order integro-differential equation*, Math. Methods Appl. Sci., **34**(15) (2011), 1833-1842.