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Review of the convex contractions of Istratescu's type in various generalized metric spaces

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Abstract

The main purpose of this paper is to consider convex contraction of Istratescu's type in various generalized metric spaces (partial metric spaces, cone metric spaces, cone b-metric spaces, partial b-metric spaces, and others). In it, among other things, we generalize, extend, correct and enrich the recent announced results in existing literature.

Keywords: Convex contraction; fixed point; b-metric space; partial metric space; partial b-metric space; cone metric space; cone b-metric space; G-metric space.
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1. Introduction and Preliminaries

Intellectual activity is may be divided into four basic categories: art, science, philosophy and mathematics. Each of these categories has its own sub-categories and further development has made progress in each of them in terms of generalization where it is possible. In other words, a so-called hierarchy of concepts and activities emerged. It can be clearly identified in each of four mentioned categories.

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In this paper we will discuss one small part of one of the categories - mathematics. More specifically, we will observe mathematical analysis as one of its many sub-categories. Further, nonlinear and functional analysis are its sub-categories, while the fixed point theory is maybe the main foundation of the latter.

Recall here that the fixed point theory was conceived in the late 19th century by introducing the notion of an iterative sequence of some mapping starting from a given point x_0. In other words, if X is nonempty set, $x_0 \in X$ is a given point and T is self-mapping on X, then the sequence $\{x_n\}$, $n \in \mathbb{N}$ defined as $x_n = Tx_{n-1}$, $n = 1, 2, ...$ is called Picard sequence. It is actually the sequence:

$$x_0,\, x_1 = Tx_0,\, x_2 = Tx_1 = T^2x_0,\, ..., x_n = Tx_{n-1} = T^nx_0, ...$$

where $T^n = \underbrace{T \circ T \circ \cdots \circ T}_{n}$. In the fixed point theory (metrical and topological) Picard sequence has a special significance.

We say that $u \in X$ is a fixed point of the mapping $T : X \to X$ if $Tu = u$. It is unique if beside it, there is no more fixed points in X.

In this paper we will examine a special type of mappings defined on metric space (X, d) introduced and studied by Istratescu at the beginning of the nineties in the last century. Istratescu's result is one of the topics in the fixed point theory that has the potential to be elegantly generalized and applied in wide class of metric spaces. In this paper we will use quite simple approach to improve and generalize some of results from the context of metric spaces. Beside that, we will consider this type of mappings in some generalized metric spaces such as:

- b-metric
- partial metric
- partial b-metric
- cone metric
- G-metric
- G_b-metric
- S-metric
- S_b-metric spaces.

Now, we recall the Istratescu's notion of the convex contraction mapping of order two from [16] in metric space (X, d). A mapping $T : X \to X$ is a convex contraction of order two if there are non-negative numbers $a, b \in \mathbb{R}$, $a + b < 1$ such that for all $x, y \in X$ the following is satisfied:

$$d(T^2x, T^2y) \leq a \cdot d(Tx, Ty) + b \cdot d(x, y) \tag{1}$$

where $T^2 = T \circ T$. In [16] is showed that if T is a continuous mapping and convex contraction of order two then it has a fixed point. Recently, Istratescu's results [16, 17, 18, 19] have again attracted the attention of many authors, see for example [5, 6, 9, 13, 14, 22, 28]. In [29] one can see more on contractive mappings summarized.

Here we state definition of k-continuity from [26] and Theorem 2.1. from [8]. Later in the paper we will show how the proof of this theorem can be simplified and shortened. The k-continuity is recently introduced weaker form of continuity but strong enough to keep Istratescu's type contractions valid.

Definition 1.1. *[26] A self-mapping T of a metric space X is k-continuous, $k = 1, 2, 3, ...$, if $T^kx_n \to T\bar{x}$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to \bar{x}$.*

Theorem 1.2. *[8] Let T be a self-mapping of a complete metric space (X, d) such that for each distinct $x, y \in X$,*

$$
\begin{aligned}
&0 \le p_0 + p_1 + q_1 + q_2 + r_1 + r_2 < 1 \\
&d(T^2x, T^2y) \le p_0 d(x, y) + p_1 d(Tx, Ty) + q_1 d(x, Tx) + q_2 d(Tx, T^2x) \\
&+r_1 d(y, Ty) + r_2 d(Ty, T^2y).
\end{aligned} \tag{2}
$$

Suppose that T is k-continuous for $k \in \mathbb{N}$. Then T has a unique fixed point.

In the next section we will also use the following well known Lemma (see [27]) :

Lemma 1.3. *Let (X, d) be a metric space and let $\{x_n\}$, $n \in \mathbb{N}$ be a sequence in X such that $\lim_{n\to\infty} d(x_n, x_{n+1}) \to 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist two sequences of integer numbers $\{n_k\}$ and $\{m_k\}$ such that the limit of all of the sequences $d(x_{n_k+a}, x_{m_k+b})$, where $a, b \in \{-1, 0, 1\}$, is $\varepsilon+$ when $k \to \infty$.*

2. Main results

The first new result we present in this paper is the following Lemma:

Lemma 2.1. *For a given point $x_0 \in X$ the Picard sequence of the convex contraction mapping $T : X \to X$ (satisfying (1)) is a Cauchy sequence.*

Proof. Suppose that $x_k \ne x_{k-1}$ for all $k \in \mathbb{N}$, since the case $x_k = x_{k-1}$ for some $k \in \mathbb{N}$ is trivial. Replacing $x = x_k, y = x_{k-1}$ into (1) we obtain:

$$d(x_{k+2}, x_{k+1}) \le a \cdot d(x_{k+1}, x_k) + b \cdot d(x_k, x_{k-1}). \tag{3}$$

To prove that $\{x_n\}$ is Cauchy, it is enough to prove that the following sequence

$$S_n = d_1 + d_2 + ... + d_{n-1} + d_n,\ d_n = d(x_{n+1}, x_n)$$

is convergent. Starting from (3) for $k = 1, 2, ..., n-1, n$, we get:

$$
\begin{aligned}
d(x_3, x_2) &\le a \cdot d(x_2, x_1) + b \cdot d(x_1, x_0) \\
d(x_4, x_3) &\le a \cdot d(x_3, x_2) + b \cdot d(x_2, x_1) \\
&\ \ \ ... \\
d(x_{n+1}, x_n) &\le a \cdot d(x_n, x_{n-1}) + b \cdot d(x_{n-1}, x_{n-2}) \\
d(x_{n+2}, x_{n+1}) &\le a \cdot d(x_{n+1}, x_n) + b \cdot d(x_n, x_{n-1}).
\end{aligned}
$$

Summing up the previous inequalities, we obtain:

$$S_n - d_1 + d_{n+1} \le a \cdot S_n + b \cdot (S_n - d_n + d_0)$$

and further

$$S_n(1 - a - b) \le d_1 + bd_0 - d_{n+1} - bd_n \le d_1 + bd_0.$$

Since $a + b < 1$ we conclude that S_n is bounded from above and also increasing, so there exists $\lim_{n\to\infty} S_n$. ☐

Remark 2.2. *The presented proof of the Lemma 2.1 is elementary, unlike the corresponding proof in [16].*

Remark 2.3. *If we release the assumption that T is continuous from [16] and replace it by the assumption that T is 2−continuous, we again obtain that Istratescu's theorem is valid. Really, from $x_n \to \bar{x}$, $n \to \infty$ we get also*

$$x_{n+1} = Tx_n \to \bar{x} \text{ and } T^2x_n = x_{n+2} \to \bar{x},\ n \to \infty.$$

But since T is 2−continuous we get that $x_{n+2} = T^2x_n \to T\bar{x}$, $n \to \infty$. Now, from the uniqueness of the limit, we finally conclude that $T\bar{x} = \bar{x}$. The uniqueness of the fixed point $\bar{x}$ follows from (1).

Remark 2.4. *Similar as in Remark 2.3 we can additionally release that assumption that T is k-continuous for $k > 2$ instead of it suppose that T is 2-continuous.*

Now, let us look back at Theorem 2.1. from [8]. The proof that the Picard sequence $\{x_n = T^n x_0\}$ is a Cauchy follows easily from the previous Lemma 2.1. Namely, replacing $x = x_n$, $y = x_{n-1}$ into (2) and denoting by $d_n = d(x_{n+1}, x_n)$, we get

$$(1 - q_2)d_{n+1} \le (p_1 + q_1 + r_2) \cdot d_n + (p_0 + r_1) \cdot d_{n-1}.$$

Hence,

$$d_{n+1} \le a \cdot d_n + b \cdot d_{n-1},$$

where $a = \dfrac{p_1 + q_1 + r_2}{1 - q_2}$, $b = \dfrac{p_0 + r_1}{1 - q_2}$. Since $a + b < 1$, from Lemma 2.1 we conclude that $\{x_n = T^n x_0\}$ is a Cauchy. Notice here that the proof we just carried is much simpler and shorter than the one in [8].

Remark 2.5. *The assumption "for each distinct $x, y \in X$" in Theorem 1.2 is not necessary and the theorem remains valid without it.*

We keep focusing our attention on [8], now the Theorem 2.2. which claims that the self-mapping $T : X \to X$ has a unique fixed point if next to (2) for all $x, y \in X$ it satisfies

$$d(Tx, Ty) \le \phi \left(\max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\} \right) \tag{4}$$

where $\phi : [0, +\infty) \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$.
We think that the assumption of continuity of function ϕ is missing in order the proof in [8] be correct. Then, let us also note that the condition (4) itself is sufficient to prove that Picard sequence $\{x_n = T^n x_0\}$ is a Cauchy, using the Lemma 1.3. Namely, since $d(x_n, x_{n+1}) \to 0$ when $n \to \infty$, if $\{x_n\}$ is not Cauchy then exist two sequences $\{n_k\}$ and $\{m_k\}$ such that all limits form the Lemma 1.3 are $\varepsilon+$. Then, we easily obtain contradiction by replacing $x = x_{m_k}$, $y = x_{n_k}$ into (4):

$$\begin{aligned} d(x_{m_k+1}, x_{n_k+1}) &= d(Tx_{m_k}, Tx_{n_k}) \\ &\le \phi \left(\max \left\{ d(x_{m_k}, x_{n_k}), d(x_{m_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), \right.\right. \\ &\quad \left.\left. \frac{d(x_{m_k}, Tx_{n_k}) + d(x_{n_k}, Tx_{m_k})}{2} \right\} \right). \end{aligned}$$

Remark 2.6. *Note that proof of Lemma 2.1 can not be applied in the context of b-metric spaces $(X, d, s > 1)$. Indeed, consider the complete b-metric space $X = \mathbb{R}$ with $d(x, y) = |x - y|^p$, $p > 1$. Then, for the sequence $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, the sum $\sum_{n=1}^{\infty} d(x_{n+1}, x_n)$ is convergent, but $\{x_n\}$ is not the Cauchy sequence. We easily obtain:*

$$\begin{aligned} d(x_{n+m}, x_n) &= \left(1 + \frac{1}{2} + \cdots + \frac{1}{n+m} - 1 - \frac{1}{2} - \cdots - \frac{1}{n} \right)^p \\ &= \left(\frac{1}{n+1} + \cdots + \frac{1}{n+m} \right)^p. \end{aligned}$$

If we put $m = n$ into the previous relation, we obtain

$$d(x_{2n}, x_n) = \left(\frac{1}{n+1} + \cdots + \frac{1}{2n} \right)^p \ge \frac{1}{2^p},$$

and it does not converge to 0 when $n \to \infty$. Therefore, in b-metric case Istratescu's theorem for convex contraction can not be proven using the Lemma 2.1.

But also note that, from condition (1) we obtain

$$d(x_{n+2}, x_{n+1}) \le \max\{d(x_1, x_0), d(x_2, x_1)\}(a + b)^{n/2},$$

for all $n \in \mathbb{N}$. So, from Lemma 2.2 in [24] we can conclude that $\{x_n\}$ is Cauchy. Further with the assumption of continuity of T immediately follows that in b-metric spaces convex contraction satisfying (1) has a unique fixed point. Note here that this is a more general result than [10], namely in b-metric and in metric spaces the contraction condition is the same: $a + b < 1$. Note also that the same result is obtained in [23] but by a different approach.

2.1. Partial case

In 1986 Matthews introduced the partial metric spaces as the generalization of usual metric ones, see [21]. The following Matthews' definitions and results will be needed in the sequel.

Definition 2.7. *A partial metric on a non-empty set X is a function $p : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:*

(p1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(p2) $p(x, x) \leq p(x, y)$,
(p3) $p(x, y) = p(y, x)$,
(p4) $p(x, y) \leq p(x, y) + p(y, z) - p(y, y)$.

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X.

Each partial metric p on X generates a T_0 topologfy τ_p on X with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 2.8. *([27], Definition 1.2). Let (X, p) be a partial metric space. Then*
(a) *A sequence $\{x_n\}$ in (X, p) converges to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x_n, x)$;*
(b) *A sequence $\{x_n\}$ in (X, p) is called a Cauchy if and only if there exist (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$;*
(c) *A partial metric space (X, p) is sad to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$;*
(d) *A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if and only if*

$$\lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

We say that (X, p) is 0-complete if every 0-Cauchy sequence in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = 0$;
(e) *A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B_p(x_0, \delta)) \subseteq B_p(fx_0, \varepsilon)$.*

Our first result on convex contraction in the frame of partial metric spaces is the following:

Theorem 2.9. *Let T be a continuous convex contraction of order two defined on a 0-complete partial metric space (X, p). Then T has a unique fixed point (say) $\overline{x}$ and $p(\overline{x}, \overline{x}) = 0$.*

Proof. If x_0 is an arbitrary point in X, then for the Picard's sequence $x_{n+1} = Tx_n$, $n \in \{0\} \cup \mathbb{N}$ initiated with x_0 we get

$$p(x_{n+2}, x_{n+1}) \leq a \cdot p(x_{n+1}, x_n) + b \cdot p(x_n, x_{n-1}), \text{ for each } n \in \mathbb{N}.$$

Denote $p_n = p(x_{n+1}, x_n)$, $n \in \mathbb{N}$, then analogous to Lemma 2.1 we obtain that the sequence

$$P_n = p_1 + p_2 + \cdots + p_n$$

converges. This means that the sequence $\{x_n\}$ is a 0-Cauchy. Since (X, p) is a 0-complete there exists $\overline{x} \in X$ such that

$$\lim_{n\to\infty} p(x_n, \overline{x}) = \lim_{n,m\to\infty} p(x_n, x_m) = 0 = p(\overline{x}, \overline{x}).$$

For more details see [20], [21], [33]. Further, if we suppose that $\overline{x} \neq T\overline{x}$, from the continuity of the mapping T, we get that $p(\overline{x}, T\overline{x}) = p(T\overline{x}, T\overline{x})$. Indeed, by (p4) we have

$$\begin{aligned} p(\overline{x}, T\overline{x}) &\leq p(\overline{x}, x_{n+1}) + p(Tx_n, T\overline{x}) - p(x_{n+1}, x_{n+1}) \\ &\leq p(\overline{x}, x_{n+1}) + p(Tx_n, T\overline{x}) \to p(\overline{x}, \overline{x}) + p(T\overline{x}, T\overline{x}) \\ &= p(T\overline{x}, T\overline{x}), \text{ as } n \to \infty. \end{aligned}$$

Now according (p2) follows the result, i.e. $p(\overline{x}, T\overline{x}) = p(T\overline{x}, T\overline{x})$.

Recall here an usual metric d_p introduced in [15] with

$$d_p(x, y) = \begin{cases} 0, \text{ if } x = y \\ p(x, y), \text{ if } x \neq y. \end{cases}$$

For the metric d_p see also [12] and references therein. Finally, the last equality implies that $d_p(\overline{x}, T\overline{x}) = 0$, i.e., $\overline{x} = T\overline{x}$ which is a contradiction. Hence, we proved that $\overline{x}$ is a unique fixed point of T. □

Remark 2.10. *From the condition*

$$p\left(T^2x, T^2y\right) \leq a \cdot p(Tx, Ty) + b \cdot p(x, y)$$

where $a, b \geq 0$ *and* $a + b < 1$, *for all* $x, y \in X$, *immediately follows the next contractive condition*:

$$d_p\left(T^2x, T^2y\right) \leq a \cdot d_p(Tx, Ty) + b \cdot d_p(x, y).$$

Proof. Indeed, for all $x, y \in X$ by the definition of the function d_p we have

$$d_p\left(T^2x, T^2y\right) \leq p\left(T^2x, T^2y\right) \leq a \cdot p(Tx, Ty) + b \cdot p(x, y).$$

Now, if $Tx \neq Ty$ then $x \neq y$ and we obtain that

$$d_p\left(T^2x, T^2y\right) \leq a \cdot p(Tx, Ty) + b \cdot p(x, y) = a \cdot d_p(Tx, Ty) + b \cdot d_p(x, y).$$

If $Tx = Ty$ then either $x = y$ or $x \neq y$. In both cases we have that

$$\begin{aligned} a \cdot p(Tx, Ty) + b \cdot p(x, y) &= a \cdot 0 + \begin{cases} b \cdot 0 = 0, \text{ if } x = y \\ b \cdot p(x, y), \text{ if } x \neq y \end{cases} \\ &= a \cdot d_p(Tx, Ty) + b \cdot d_p(x, y). \end{aligned}$$

Our claim is proved. □

Remark 2.11. ***a)*** *If in the definition of convex contraction of order two we put* $b = 0$ *and if* TX *is a closed subset in* (X, d), *then again the mapping* T *has a unique fixed point* $\overline{x} \in TX \subset X$.

b) Also, let (X, p) be a 0-complete partial metric space and $T : X \to X$ be a convex contraction of order two with $a \in [0, 1), b = 0$. If TX is a closed subset in (X, p), then T has a unique fixed point $z \in TX \subset X$ and $p(z, z) = 0$.

Proof. **a)** Indeed since $T : X \to X$ it follows that $T : TX \to TX$. By the assumption it is clear that (TX, d) is a complete metric space. Further, from the condition $d\left(T^2x, T^2y\right) \leq a \cdot d(Tx, Ty), 0 \leq a < 1, x, y \in X$ follows that T is a contraction on complete metric space (TX, d). This follows from the

$$d(Tu, Tv) = d(T(Tx), T(Ty)) = d\left(T^2x, T^2y\right) \leq a \cdot d(Tx, Ty) = a \cdot d(u, v),$$

where $u = Tx, v = Ty$ both belong to TX. Hence, T has a unique fixed point (say) $\overline{x} \in TX \subset X$.

b) First we see that (TX, p) is a 0-complete partial metric space and T is a contraction on it. Now, the result follows according Banach contraction principle in the frame of 0-complete partial metric space.□

In the case of complete partial b-metric space $(X, p_b, s \geq 1)$ we can also conclude that continuous convex contraction of order two has a unique fixed point. The proof goes directly according to Remark 2.6, since Theorem 2.4. from [12] proved that partail b-metric case comes down to b-metric case.

2.2. Cone metric

Let (X, d) be a cone metric space over a solid cone C. Our next aim is to prove that Theorem 1 from [16] and Theorem 3 from [4] are equivalent. Using that result, as an added benefit we additionally obtain simpler proof of the Theorem 3 from [4]. In order to make this paper self-readable, Theorem 1 from [16] is stated in Introduction, while now we paraphrase Theorem 3 from [4] which claims that cone convex contraction mapping of order two $T : X \to X$ has a unique fixed point when (X, d) is a complete cone metric space over a solid cone.

Theorem 2.12. *Theorem 1 from [16] and Theorem 3 from [4] are equivalent.*

Proof. First let us start from Theorem 3 in [4], claiming that in a cone metric space (X, d) over an ordered Banach space E with a solid cone C, a d-continuous mapping $T : X \to X$ satisfying $d(T^2x, T^2y) \preceq a \cdot d(Tx, Ty) + b \cdot d(x, y)$, for all $x, y \in X$ with $a,\ b \geq 0,\ a + b < 1$ has a unique fixed point. If now we suppose that $E = \mathbb{R}$, with solid cone $C = [0, \infty)$ and $|| \cdot || = | \cdot |$, then the previous contractive condition becomes Istratescu's contraction (1) on standard metric space (X, d). Since T already has a fixed point, by this reasoning we obtain that Theorem 1 from [16] is a special case of Theorem 3 from [4].

Note that previous consideration may be said in one sentence - namely, standard metric space is a special case of cone metric space, so Istratescu's Theorem 1 from [16] is a direct consequence of Theorem 3 form [4].

Now, let us prove other implication. Suppose that $T : X \to X$ is a d-continuous convex contraction satisfying $d(T^2x, T^2y) \preceq a \cdot d(Tx, Ty) + b \cdot d(x, y)$ for all $x, y \in X$ defined on cone metric space (X, d) over a solid cone. Recall that according to Proposition 2.2. from [2] we can suppose that (X, d) is a cone metric space over a normal solid cone with a normal constant $K = 1$. Then, from contraction condition we obtain

$$D(T^2x, T^2y) \leq a \cdot D(Tx, Ty) + b \cdot D(x, y)$$

where $a,\ b \geq 0,\ a + b < 1$, for all $x, y \in X$ and with $D = ||d||$. So, we get that T is a continuous convex contraction on complete metric space (X, D) and according to Theorem 1 from [16] it has a unique fixed point. □

Remark 2.13. *From the previous analysis we conclude that results from [4] are not generalization of the [16] in the case of cone metric spaces.*

Open question: Does continuous mapping $T : X \to X$ have a unique fixed point if (X, d) is a cone metric space over Banach algebra $\mathcal{A}$ and it satisfies

$$d(T^2x, T^2y) \preceq a \circ d(Tx, Ty) + b \circ d(x, y)$$

where $a, b \succeq \theta, r(a + b) < 1$, r is a spectral diameter and $\circ$ is a multiplication in algebra $\mathcal{A}$.

2.3. G-metric case

In 2004 Mustafa and Sims in [25] introduced a new generalization of metric spaces known as G-metric spaces. Afterwards, many new results in this kind of spaces were obtained, including various fixed point results. Here, we continue to consider convex contraction of order two, now in G-metric space. First, we give remind reader on definition and some properties of G-metric spaces and for more details we refer to [1], [7].

Definition 2.14. *Let X be a nonempty set and $G : X^3 \to [0, \infty)$ satisfies the following conditions:*

(G1) *for all $x, y, z \in X$, $x = y = z$ if and only if $G(x, y, z) = 0$*

(G2) $0 < G(x, x, y)$ *for all $x, y \in X$ with $x \neq y$*

(G3) $G(x, x, z) \leq G(x, y, z)$ *for all $x, y \in X$ with $z \neq y$*

(G4) $G(x,y,z) = G(x,z,y) = G(z,y,x) = G(y,x,z) = G(y,z,x) = G(z,x,y)$

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ *for all* $a,x,y,z \in X$.

Our first result in this subsection is next:

Theorem 2.15. *Let (X,G) be a complete G-metric space and $T : X \to X$ is a continuous convex contraction of order two, ie. a mapping that satisfies*

$$G(T^2x, T^2y, T^2z) \le aG(Tx,Ty,Tz) + bG(x,y,z) \tag{5}$$

for all $x,y,z \in X$ and $a,b \ge 0$ and $a+b<1$. Then T has a unique fixed point x^* and for all $x \in X$ Picard sequence $T^n x$ converges to x^*.

Proof. From contraction condition (5) we obtain for all $x,z \in X$:

$$G(T^2x, T^2z, T^2z) \le aG(Tx,Tz,Tz) + bG(x,z,z)$$

and

$$G(T^2x, T^2x, T^2z) \le aG(Tx,Tx,Tz) + bG(x,x,z).$$

By adding previous inequalities we get

$$\begin{aligned} G(T^2x, T^2z, T^2z) + G(T^2x, T^2x, T^2z) \le\ & a(G(Tx,Tz,Tz) + G(Tx,Tx,Tz)) \\ & +b(G(x,z,z) + G(x,x,z)), \end{aligned}$$

that is

$$d_G(T^2, T^2z) \le d_G(Tx,Tz) + d_G(x,z), \text{ for all } x,y \in X$$

where d_G is a metric obtained from G by $d_G(p,q) = G(p,p,q) + G(p,q,q)$. This way, we conclude that T is convex contraction on complete metric space (X, d_G) and from Theorem 1 in [16] follows the conclusion that T has a unique fixed point $x^* \in X$.□

Notice here that if we change (G5) into

$$(G_b5) \quad G_b(x,y,z) \le s\,(G_b(x,a,a) + G_b(a,y,z)) \text{ for all } a,x,y,z \in X$$

then we get recently introduced G$_b$-metric spaces and refer the reader to see more on that topic in [7]. As we can assign corresponding b-metric d_{G_b} to each G_b-metric, then we get following outcome for this class of generalized metric spaces from already known results for b-metric:

Theorem 2.16. *Let (X, G_b) be a complete G_b-metric space and $T : X \to X$ continuous convex contraction of order two. Then, T has a unique fixed point.*

We leave out the proof since with corresponding b-metric $d_{G_b}(x,y) = G_b(x,x,y) + G_b(x,y,y)$ the proof follows from Remark 2.6.

2.4. S-metric case

First we recall some notions, results, and examples needed in this subsection.

Definition 2.17. *Let X be a nonempty set. A function $S : X^3 \to [0,\infty)$ is said to be an S-metric on X, if for each $x,y,z,a \in X$,*

(S1) $S(x,y,z) = 0$ *if and only if* $x = y = z$,

(S2) $S(x,y,z) \le S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an $S-$metric space.

Example 2.18. *We can easily check that the following examples are S-metric spaces.*

1. Let $X=\mathbb{R}^n$ and $\|\cdot\|$ a norm on X, then $S(x,y,z)=\|y+z-2x\|+\|y-z\|$ is an S-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ a norm on X. Then it is easy to see that

$$S(x,y,z)=\|\alpha y+\beta z-\lambda x\|+\|y-z\|,$$

where $\alpha+\beta=\lambda$ for every $\alpha,\beta\geq 1$, is an $S-$metric on X.

2. Let X be a nonempty set and d_1,d_2 be two ordinary metrics on X. Then

$$S(x,y,z)=d_1(x,z)+d_2(y,z),$$

is an S-metric on X.

Next result is very useful in consideration some fixed point results in the context of $S-$metric spaces and for its' proof and more details on S-metric spaces see [11].

Proposition 2.19. *Let (X,S) be an $S-$metric space. Then $\left(X,d,s\geq\frac{3}{2}\right)$ is a b-metric space, where*

$$d(x,y)=S(x,x,y),$$

for all $x,y\in X$.

Now we can prove Istratescu's type result for $S-$metric space. Let us first formulate the next definition:

Definition 2.20. *Let (X,S) be a complete $S-$metric space. The mapping $T:X\to X$ is the convex contraction of order 2 if for all $x,y,z\in X$*

$$S\left(T^2x,T^2y,T^2z\right)\leq a\cdot S(Tx,Ty,Tz)+b\cdot S(x,y,z) \tag{6}$$

where $a,b\geq 0$ and $a+b<1$.

Theorem 2.21. *Each continuous convex contraction of order 2 defined on complete $S-$metric space (X,S) has a unique fixed point, say $u\in X$ and for all $x\in X$ the sequence $\{T^nx\}_{n\in\mathbb{N}}$ converges to the fixed point u.*

Proof. According to Proposition 2.1. there is $\left(X,d,s\geq\frac{3}{2}\right)$ b-metric space, where

$$d(x,y)=S(x,x,y) \tag{7}$$

for all $x,y\in X$. Further, (6) implies

$$S\left(T^2x,T^2x,T^2z\right)\leq a\cdot S(Tx,Tx,Tz)+b\cdot S(x,x,z) \tag{8}$$

that is

$$S\left(T^2x,T^2z,T^2z\right)\leq a\cdot S(Tx,Tz,Tz)+b\cdot S(x,z,z). \tag{9}$$

Now, by adding (8) and (9) we get

$$d\left(T^2x,T^2z\right)\leq a\cdot d(Tx,Tz)+b\cdot d(x,z) \tag{10}$$

The result further follows by Remark 2.6.□

In [32] authors introduced generalization of $S-$metric spaces as follows:

Definition 2.22. *Let X be a nonempty set and $b\geq 1$ a given real number. A S_b-metric on X is a function $S_b:X^3\to[0,\infty)$ that satisfies the following conditions for all $x,y,z,a\in X$.*

(Sb1) $0<S_b(x,y,z)$ *for all $x,y,z\in X$ with $x\neq y\neq z\neq x$,*

(Sb2) $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,

(Sb3) $S_b(x,y,z) \leq b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$ *for all* $x,y,z,a \in X$.

Then S_b *is called a* S_b-*metric and the pair* (X, S_b) *is called a* S_b-*metric space.*

Remark 2.23. *It is clear that each* $S-$*metric space is a* S_b-*metric space with* $b = 1$. *While, the conversely is not true (see [30], [31] and [32]). The next is specially important:*

Example 2.24. *Let* (X, S) *be a* $S-$*metric space, and* $S_b(x,y,z) := (S(x,y,z))^p$, *where* $p > 1$ *is a real number. Note that* S_b *is a* S_b-*metric with* $b = 2^{2(p-1)}$, *i.e.,* (X, S_b) *is a* S_b-*metric space associated to given* $S-$*metric space* (X, S).

Open problem: Does the continuous mapping $T : X \to X$ has a unique fixed point if (X, S_b) is a S_b-metric space and it satisfies

$$S_b\left(T^2x, T^2y, T^2z\right) \leq a \cdot S_b(Tx, Ty, Tz) + b \cdot S_b(x,y,z),$$

for all $x,y,z \in X$ where $a, b \geq 0$ and $a + b < 1$?

References

[1] R.P. Agarwal, E. Karapinar, D. O'Regan, A.F.R.L. -de Hierro, Fixed Point Theory in Metric Type Spaces, Springer International Publishing Switzerland 2015

[2] S. Aleksić, Z. Kadelburg, Z.D. Mitrović, S. Radenović, A new survey: Cone metric spaces, Journal of the International Mathematical Virtual Institute, 2019, 9, pp. 93–121.

[3] M.A. Alghamdi, S. H. Alnafei, S. Radenović, N. Shahzad, Fixed point theorems for mappings with convex diminishing diameters on cone metric spaces, Appl. Math. Let., 24 (2011) 2162–2166.

[4] M. A. Alghamdi, S. H. Alnafei, S. Radenović, N. Shahzad, *Fixed point theorems for convex contraction mappings on cone metric spaces,* Math. Comput. Model. 54 (2011) 2020-2026

[5] C.K. Ampadu, Some fixed point theory results for convex contraction mapping of order 2, JP Journal of Fixed Point Theory and Applications, Volume 12, Numbers (2-3), 2017, Pages 81–130.

[6] Sz. Andras, *Fiber Picard operators and convex contractions*, Fixed Point Theory, 2003, 4, pp. 121–129.

[7] H. Aydi, D. Rakić, A. Aghajani, T. Došenović, M.S. Noorani and H. Qawaqneh, On fixed point results in G_b- metric spaces, Mathematics **2019**, 7, 617.

[8] R.K. Bisht, V. Rakočević, Fixed points of convex and generalized convex contractions,. Rendiconti del Circolo Matematico di Palermo Series 2.

[9] R.K.Bisht, N. Hussain, A note on convex contraction mappings and discontinuity at fixed point, Journal of Mathematical Analysis, Volume 8, Issue 4 (2017), Pages 90–96.

[10] D.Dj. Dolićanin, B. B. Mohsin, Some new fixed point results for convex contractions in b-metric spaces, University Thought, Publication in Nature Sciences, Vol. 9, No. 1, 2019.

[11] T. Došenović, S. Radenović, S. Sedghi, Generalized Metric Spaces: Survey, TWMS J. Pure Appl. Math. V. 9, N.1, 2018, pp. 3–17.

[12] N.V. Dung and V.T.L. Hanh, Remarks on partial b-metric spaces and fixed point theorems, Matematički Vesnik 22, 2, 2016, 151–164.

[13] K.S. Eke, V.O. Olisama and S.A. Bishop, Some fixed point theorems for convex contractive mappings in complete metric spaces with applications, Cogent Mathematics & Statistics (2019)6: 1655870.

[14] F. Georgescu, IFSs consisting of generalized convex contractions, An. St. Univ. Ovidius Constanta, vol. 25 (1), 2017, 77–86.

[15] R.H. Haghi, S. Rezapour, N. Shahzad, Be careful on partial metric fixed point results, Topology Appl. 160 (3), (2013), 450–454.

[16] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and convex non-expansive mapping, Libertas Mathematica, 1981 (1), pp. 151–163.

[17] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters-I, Annali di Mat. Pura Appl., 1982, 130, pp. 89–104.

[18] V.I. Istratescu, Some fixed point theorems for convex contraction mappings and mappings with convex diminishing diameters-II, Annali di Mat. Pura Appl., 1983, 130, pp. 327–362.

[19] E. Karapınar, A. Fulga, A. Petruşel, On Istratescu Type Contractions in b-Metric Spaces. Mathematics 2020, 8, 388.

[20] W.A. Kirk, and N. Shahzad, Fixed Point Theory in Distance Spaces, 2014, Springer International Publishing Switzerland 2014.

[21] S.G. Matthews, Metric domains for completeness technical report 76, PhD Thesis, Department of Computer Science, Univertsity of Warwick, Conventry, (1986).

[22] M. A. Miandaragh, M. Postolache, S. Rezapour, Aproximate fixed points of generalized convex contractions, Fixed Point Theory Appl. 2013, 2013:255.

[23] R. Miculescu, A. Mihail, A generalization of Istratescu's fixed point theorem for convex contractions, Fixed Point Theory 18(2), (2017), 689-702.

[24] R. Miculescu, A. Mihail, New fixed point theorems for set-valued contractions in b-metric spaces, J. Fixed Point Theory Appl. 19, (2017), 2153-2163.

[25] Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, International Conference on Fixed Point Theory and Applications, Yokohama, Japan, 10 (2004)

[26] A. Pant, R.P. Pant, Fixed points and continuity of contractive maps, Filomat 31(11), 3501-3506 (2017).

[27] S. Radenović, Classical fixed point results in 0-complete partial metric spaces via cyclic-type extension, Bull. Allahabad Math. Soc. 31, 1, 2016, 39–55.

[28] M. Ramezani, Orthogonal metric space and convex contractions, Int. J. Nonlinear Anal. Appl. 6 (2015) No. 2, 127-132.

[29] B.E. Rhoades, Comparison of Various Definitions of Contractive Mappings, Transactions of the American Mathematical Society, 1977, 226, pp. 257-290

[30] Y. Rohen, T. Došenović, S. Radenović, A note on paper " A fixed point theorem in S_b-metric spaces", Filomat, 31:11, (2017), 3335-3346.

[31] N. Saleem, A. H. Ansari, M. Pavlović, S. Radenović, Some new results in the framework of S_b-metric spaces, Scientific Publications of The State University of Novi Pazar, Ser. A. Appl. Math. Inform. and Mech. vol. 9, 2 (2017), 151–165

[32] S. Sedghi, A. Gholidahnen, T. Došenović, J. Esfahani, S. Radenović, Common fixed point of four maps in S_b- metric spaces, Journal of Linear and Topological Algebra, Vol. 05, No. 02, (2016), 93-104

[33] J. Vujaković, H. Aydi, S. Radenović, A. Mukheimer, Some remarks and new results in ordered partial b-metric spaces, Mathematics, **2019,** 7, 334, doi: 10.3390/math7040334