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Non-null Surfaces with Constant Slope Ruling with Respect to Osculating Plane

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Abstract

The main purpose of this study is to investigate surface with a constant slope ruling with respect to osculating plane by using Frenet Frame according to casual characters in Minkowski space. In accordance with this purpose, surface with constant slope ruling with respect to osculating plane in Minkowski Space is defined and many features of this surface are investigated. In addition, examples of the given characterizations are obtained and the geometrical structures of these examples are be examined and visualized.

Keywords: Ruled surface; Surface with constant slope; Minkowski space.

Oskülatör Düzleme Göre Dayanak Eğrisi Sabit Eğimli Null Olmayan Yüzeyler

Öz

Bu çalışmanın amacı, Minkowski uzayında Frenet çatısını kullanarak oskülatör düzleme göre dayanak eğrisi sabit eğimli yüzeyleri incelemektir. Bu amaç doğrultusunda, Minkowski uzayında Frenet çatısını kullanarak oskülatör düzleme göre dayanak eğrisi sabit eğimli yüzeylerin

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tanımları elde edilmiştir ve bu yüzeylerin birçok özelliği ayrı ayrı ele alınmıştır. Elde edilen yüzeylerin örnekleri incelenerek görselleri elde edilmiştir.

Anahtar Kelimeler: Regle yüzeyler; Sabit eğimli yüzeyler; Minkowski uzayı.

1. Introduction

Curves and surfaces are geometric structures that are frequently encountered in daily life. The most known curve helix which is an important curve has a lot of applications. In the 3-dimensional Euclidean space, if the angle between a fixed direction and each tangent is constant, such a curve is called a helix curve. The path that a bean follows as it grows around the stick follows, the sequence of molecules in the structure of the DNA, the carbon nanotubes, the progression of the screw, the path that the flying creatures are heading towards the point source or the prey and the way the ants walk on a tree from point A to point B are all examples of helix curves.

Ruled surface in 3-dimensional Minkowski space is a special surface which is formed by moving a line given in 3-dimensional Minkowski space along a given curve. The line is the generating line of the ruled surface and the given curve is the base curve of the surface. The ruled surface is one of the most important topic of differential geometry.

Minkowski space is more interesting than the Euclidean space because curves and surfaces have different casual characters such as timelike, spacelike or null (lightlike). In the literature, studies involving the subject of curves and surfaces in Minkowski space are very common [1-4]. Therefore, ruled surfaces in Minkowski space can be classified according to the Lorentzian character of their ruling and surface normal.

Firstly, K. Malecek and others defined a surface with a constant slope with respect to the given plane in Euclidean Space in [5]. In this study, we give definition of surfaces with a constant slope ruling with respect to osculating plane by using Frenet frame Minkowski spaces. Many features of these surfaces are examined and characterized. In addition, examples of the given characterizations are given and the geometrical structures of these examples are visualized using the Mathematica program. It is shown that rotational surface and one sheet rotational hyperboloid have constant slopes. The conditions are given for being a torsal surface in Minkowski space.

2. Preliminaries

A vector $\boldsymbol{v}$ tangent to a semi-Riemanian manifold M is spacelike if $g(v,v) > 0$ or $v = 0$, null if $g(v,v) = 0$ and $v \neq 0$, timelike if $g(v,v) < 0$. The norm of a tangent vector v is given by $|v| = \sqrt{|g(v,v)|}$. A curve in a manifold M is a smooth mapping $\alpha: I \to M$, where I is an open interval in the real line R. A curve α in a semi-Riemannian manifold M is spacelike if all of its velocity vectors $\alpha'(s)$ are spacelike, null if all of its velocity vector $\alpha'(s)$ are null, timelike if all of its velocity vectors $\alpha'(s)$ are timelike.

In this study we give the Frenet frames and formulas in the Minkowski space ${E_1}^3$ with metric $g = -{dx_1}^2 + {dx_2}^2 + {dx_3}^2$.

A curve $\alpha(s)$ in ${E_1}^3$ has different causal characters. We consider that $\alpha(s)$ is spacelike or timelike separately and construct their Frenet frame $\{t, n, b\}$.

If α is spacelike, we take the arclength parameter s or α such that $g(\alpha'(s), \alpha'(s)) = 1$. $t(s)$ is the velocity or unit tangent vector field of $\alpha(s)$. If $\alpha''(s) \neq 0$, then $\alpha''(s)$ is perpendicular to $t(s)$, so we take $n(s) = \lambda\alpha''(s)$, $\lambda \in R$ and $\lambda > 0$. Depending on the causal character of $\alpha''(s)$ we have the different cases.

If $g(\alpha''(s), \alpha''(s)) > 0$, the principal normal vector field $n(s)$ is then the normalized vector field $\alpha''(s)$. The binormal vector field $b(s)$ is the unique timelike unit vector field perpendicular to the spacelike plane $\{t(s), n(s)\}$ at every point $\alpha(s)$ of α, such that $\{t, n, b\}$ has the same orientation as ${E_1}^3$. The Frenet formulas are, in matrix notation,

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}$$

If $g(\alpha''(s), \alpha''(s)) < 0$, the principal normal vector field $n(s)$ is then the normalized timelike vector field $\alpha''(s)$. The binormal vector field $b(s)$ is the unique spacelike unit vector field perpendicular to the timelike plane $\{t(s), n(s)\}$ at every point $\alpha(s)$ of a, such that $\{t, n, b\}$ has the same orientation as ${E_1}^3$. The Frenet formulas are, in matrix notation,

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}$$

If $g(\alpha''(s), \alpha''(s) = 0$, to rule out straight lines and points of inflexion on a, we shall suppose that $\alpha''(s) \neq 0$. The principal normal vector field $n(s)$ is then the vector field $\alpha''(s)$. The binormal vector field $b(s)$ is the unique null vector field perpendicular to $t(s)$ at every point $\alpha(s)$ of α, such that $g(n, b) = 1$. The Frenet formulas are, in matrix notation,

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & \tau(s) & 0 \\ -\kappa(s) & 0 & -\tau(s) \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix},$$

where the curvature κ can only take two values; 0 , when a is a straight line, or 1 in all other cases. If $\alpha(s)$ is a straight line, then $\alpha''(s) = 0 = t'(s)$ which means that $\kappa = 0$. If $\alpha(s)$ is not a straight line, then there exists an interval I on which $\alpha''(s) \neq 0$ [6].

Let x and y be future pointing (or past pointing) timelike vectors in ${R_1}^3$. Then there is a unique real number $\theta > 0$ such that $< x, y >= -\|x\|\|y\| cos\, h\theta$.

Let x and y be spacelike vectors in ${R_1}^3$ that span a timelike vector subspace. Then there is a unique real number $\theta > 0$ such that $< x, y >= \|x\|\|y\| cos\, h\theta$.

Let x and y be spacelike vectors in ${R_1}^3$ that span a spacelike vector subspace. Then there is a unique real number $\theta > 0$ such that $< x, y >= \|x\|\|y\| cos\theta$.

Let x be a spacelike vector and y be a timelike vector in ${R_1}^3$. Then there is a unique real number $\theta > 0$ such that $< x, y >= \|x\| y sinh\theta$. [7].

Surface of rotation or surface of revolution are formed from circles centered on one of the axes with variable radii and the ruled surfaces are formed from lines along some fixed curve, but in variable direction. A surface is called surface of rotation, if it is obtained by rotation of a regular curve $t \to (r(t), h(t))$ around the 2-axis, in other words if it admits parameterization form as following

$$f(t, \phi) = (r(t) cos\phi, r(t) sin\phi, h(t)).$$

Let $\alpha = \alpha(s)$ be a curve in Minkowski space and $X = X(s)$ be vector field along α, we have the parametrization for the ruled surface M in ${R_1}^3$

$$\varphi(s, v) = \alpha(s) + v.X(s),$$

where the curve $\alpha = \alpha(s)$ is called based curve and $X = X(s)$ is called a director curve of the ruled surface [8].

If the angle between $X(s)$ and (spacelike or timelike) osculating (rectifying, normal) plane is constant, then the surface $\varphi(s, v)$ is called a constant slope surface with respect to (spacelike or timelike) osculating (rectifying, normal) planes to the curve α [9].

3. Non-null Surfaces with Constant Slope Ruling with respect to Osculating Plane

Non-null surface with a constant slope ruling with respect to given plane is a term used of surfaces whose generating lines have the same $\boldsymbol{\sigma}$ deviation from the plane. $\boldsymbol{\sigma}$ is called the slope with respect to the plane. The ruled surface is defined by

$$\varphi(s,v) = \alpha(s) + v.X(s),$$

where $\alpha(s)$ is a base curve and $X(s)$ is a direction vector. Surface with a constant slope ruling with respect to given plane in Minkowski space is a surface whose generating lines have the same σ slope with respect to the given plane by direction vectors. The casual character of the direction vectors are changed according to character of the curve α given following cases.

Case 3.1. If $\alpha(s)$ is a spacelike base curve with the principal spacelike normal vector field $n(s)$, then the generating line of the surface is given as follows

$$X(s) = \sigma e_1 + cos\, w(s)t(s) + sin\, w(s)n(s),$$

and the surface with a constant slope ruling is parameterized by

$$\phi(s,v) = \alpha(s) + v(\sigma e_1 + cos\, w(s)t(s) + sin\, w(s)n(s)).$$

Case 3.2. If the curve $\alpha(s)$ is spacelike and its normal vector field $n(s)$ is timelike, then the generating line of the surface is given by

$$\bar{X}(s) = cosh\, w(s)t(s) + sinh\, w(s)n(s) + \sigma(-e_3),$$

and the surface with a constant slope ruling is parameterized by

$$\bar{\phi}(s,v) = \alpha(s) + v(cosh\, w(s)t(s) + sinh\, w(s)n(s) + \sigma(-e_3)).$$

Case 3.3. If the $\alpha(s)$ is timelike base curve with $n(s)$ spacelike, then the generating line of the surface is given as follows

$$\tilde{X}(s) = sinh\, w(s)t(s) + cosh\, w(s)n(s) + \sigma e_3,$$

and the surface with a constant slope ruling is parameterized by

$$\tilde{\phi}(s,v) = \alpha(s) + v(sinh\, w(s)t(s) + cosh\, w(s)n(s) + \sigma e_3).$$

3.1. Rotational non-null surface with a constant slope ruling with respect to osculating plane

In a surface with a constant slope ruling, if we take the base curve α is pseudo circle and the director vector is given by constant angle, then we obtain the rotational surface. The circle is given by the vector functions in Minkowski space. We have different cases as following:

Case 3.1.1. The $\alpha(s)$ is a spacelike circle and its normal vector field n is a spacelike vector. Then the curve $\alpha(s)$ is given by

$$\alpha(s) = \left(0, rcos\left(\frac{s}{r}\right), rsin\left(\frac{s}{r}\right)\right),$$

and its Frenet frame are obtained as follows

$$t(s) = \left(0, -sin\left(\frac{s}{r}\right), cos\left(\frac{s}{r}\right)\right),$$

$$n(s) = \left(0, -cos\left(\frac{s}{r}\right), -sin\left(\frac{s}{r}\right)\right),$$

$$b(s) = -e_1 = (-1,0,0).$$

The surface with a constant slope ruling $\varphi(s, v)$ with respect to osculating plane is obtained by

$$\varphi(s, v) = \alpha(s) + v.X(s),$$

where $X(s) = \sigma e^1 + cos\, w(s)t(s) + sin\, w(s)n(s)$, with $w(s)$ is a constant.

The surface has the parametric representation as follows

$$x = \sigma v,$$

$$y = rcos\left(\frac{s}{r}\right) - vsin\left(w(s) + \left(\frac{s}{r}\right)\right),$$

$$z = rsin\left(\frac{s}{r}\right) + vcos\left(w(s) + \left(\frac{s}{r}\right)\right),$$

$s \in [0,2\pi r], v \in R$. The Cartesian representation of the surface $\varphi(s, v)$ is obtained as follows

$$\sigma^2(y^2 + z^2) - (x - \sigma rsin\, w(s))^2 = \sigma^2 r^2 cos^2 w(s).$$

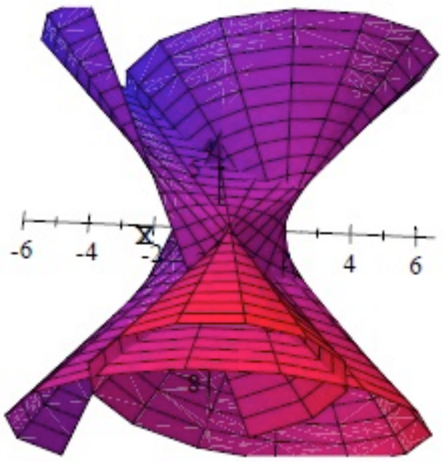

Figure 1: Example of the surface for $\sigma = \sqrt{3}, r = 2,\ u = [0, 10]$

Case 3.1.2. If the circle $\alpha(s)$ is a spacelike and its normal vector field $n(s)$ is a timelike vector then the curve $\alpha(s)$ is given by

$$\alpha(s) = \left(rcosh\left(\frac{s}{r}\right), rsinh\left(\frac{s}{r}\right), 0\right),$$

and its Frenet - Serret frame vectors are obtained as follows

$$t(s) = \left(sinh\left(\frac{s}{r}\right), cosh\left(\frac{s}{r}\right), 0\right),$$

$$n(s) = \left(cosh\left(\frac{s}{r}\right), sinh\left(\frac{s}{r}\right), 0\right),$$

$$b(s) = -e_3 = (0,0,-1).$$

The rotational surface with a constant slope ruling given as follows

$$\bar{\varphi}(s, v) = \alpha(s) + v\bar{X}(s),$$

with the directional vector

$$\bar{X}(s) = cosh\, w(s)t(s) + sinh\, w(s)n(s) - \sigma e_3,$$

and $w(s) = const., g(X, X) = 1 + \sigma^2 > 0$, then the vector $\bar{X}$ is a spacelike vector. The parametric representation of the spacelike surface $\bar{\varphi}(s, v)$ as follows;

$$\bar{\varphi}(s, v) = \left(rcosh\left(\frac{s}{r}\right) + vsinh\left(w + \left(\frac{s}{r}\right)\right), rsinh\left(\frac{s}{r}\right) + vcosh\left(w + \left(\frac{s}{r}\right)\right), -v\sigma\right).$$

In the Cartesian coordinate system the equation of the surface is written by

$$\sigma^2(x^2+y^2)+(z+\sigma r sinh\, w(s))^2=\sigma^2 r^2 cosh^2 w(s).$$

Case 3.1.3. The rotational surface with a constant slope ruling generated by timelike circle $\alpha(s)=(rsinh\left(\frac{s}{r}\right), rcosh\left(\frac{s}{r}\right), 0)$ in the timelike plane

$$\tilde{\varphi}(s,v)=\alpha(s)+v\tilde{X}(s),$$

where

$$\tilde{X}(s)=sinh\, w(s)t(s)+cosh\, w(s)n(s)+\sigma e_3,$$

$(w(s)=const.)$. After the similar calculations we have the surface as follows

$$(z+\sigma r cosh\, w(s))^2-\sigma^2(y^2-x^2)=\sigma^2 r^2 sinh^2 w(s).$$

Remark 3.1.1. In all cases, each rotational surfaces with the constant slope $\sigma, \sigma\in(0,\infty)$ is a rotational one sheet hyperboloid.

3.2. Torsal non-null surface with constant slope ruling with respect to osculating plane

A generatrix of a ruled surface is torsal, if for its each point there is one and the same tangent plane on the surface. A surface is developable if and only if it is a torsal ruled surface. In this section, we will investigate the conditions necessary for the surface to be a torsal surface.

Case 3.2.1. Frenet-Serret vectors of the spacelike circle $\alpha(s)$ can be expressed as follows

$$t(s)\ =\ (0, cos\beta(s), sin\beta(s)),$$

$$n(s)\ =\ (0,-sin\beta(s), cos\beta(s)),$$

Derivatives of the vector functions Frenet-Serret trihedron are given by

$$t'(s)=\ \beta'(s).n(s),$$

$$n'(s)\ =\ -\beta'(s).t(s),$$

The partial derivatives $\left(\frac{d\varphi(s,v)}{ds}\right)$ of the vector functions can be expressed as follows

$$\left(\frac{d\varphi(s,v)}{ds}\right)=\left(1-vw'(s)sin\, w(s)-v\beta'(s)sin\, w(s)\right)t(s)$$

$$+vcos\, w(s).\left(\beta'(s)+w'(s)\right)n(s).$$

In the last equation vector function defines the direction vectors of the tangents to the parametric curves for the constant v.

So $v = 0$, $(\frac{d\varphi(s,v)}{ds}) = t(s)$ and for $v = 1$,

$$(\frac{d\varphi(s,v)}{ds}) = (1 - w'(s)\sin\ w(s) - \beta'(s)\sin\ w(s))t(s) + (\cos\ w(s)\beta'(s)$$

$$+cos\ w(s)w'(s))n(s).$$

These vectors must be linearly dependent, so we obtained

$$cos\ w(s)\ =\ 0,$$

$$w(s) = \left(\frac{k\pi}{2}\right), k\varepsilon Z,$$

or

$$\beta'(s) + w'(s)\ =\ 0,$$

$$w(s) =\ -\beta(s) + c, c = const..$$

Remark 3.2.1. The surface $\varphi(s,v)$ is developable if and only if $det(t,X,X') = 0$. So, we calculate this equation as follows;

$$det(t,X,X')) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos w(s) & \sin w(s) \\ 0 & 1 - w'\sin w - \beta'\sin w & \cos w(\beta' + w') \end{vmatrix},$$

$$cos\ w(s)(\beta'(s) + w'(s)) = 0.$$

Case 3.2.2. If the circle $\alpha(s)$ is spacelike curve and its normal vector $n(s)$ is timelike, then the curve's Frenet-Serret vectors are

$$t(s)\ =\ (cosh\beta(s), sinh\beta(s),0),$$

$$n(s)\ =\ (sinh\beta(s), cosh\beta(s),0).$$

Derivatives of the these vectors are

$$t'(s)\ =\ \beta'(s).n(s),$$

$$n'(s)\ =\ \beta'(s).t(s).$$

The partial derivative $(\frac{d\bar{\varphi}(s,v)}{ds})$ of the vector functions are,

$$\left(\frac{d\bar{\varphi}(s,v)}{ds}\right) = [1 + v.w'(s).sinh\, w(s) + v\beta'(s)sinh\, w(s)]t(s) + vcosh\, w(s).\big(\beta'(s)w'(s)\big)n(s),$$

for $v = 0$, $\left(\frac{d\bar{\varphi}(s,v)}{ds}\right) = t(s)$, and for $v = 1$,

$$(\frac{d\bar{\varphi}(s,v)}{ds}) = [1 + w'(s)sinh\, w(s) + \beta'(s)sinh\, w(s)]t(s) + cosh\, w(s).(\beta'(s) + w'(s)).n(s)$$

If it is considered that the vector obtained for $v = 1$ and the vector obtained for $v = 0$ are linearly dependent the following equation is obtained as follows

$$cosh\, w(s).(w'(s) + \beta'(s)) = 0.$$

Thus

$$i) cosh\, w(s) = 0\ ,$$

or

$$ii) w'(s) + \beta'(s) = 0,$$

and $w(s) = -\beta(s) + c$.

Case 3.2.3. The rotational surface generated by timelike circle $\alpha(s)$ curve's Frenet-Serret from vectors in timelike plane are

$$t(s) = (sinh\beta(s), cosh\beta(s),0),$$

$$n(s) = (cos\beta(s), sinh\beta(s),0).$$

The partial derivative $(\frac{d\tilde{\varphi}(s,v)}{ds})$ of the vector functions are

$$(\frac{d\tilde{\varphi}(s,v)}{ds}) = [1 + v.w'(s).cosh\, w(s) + v\beta'(s)cosh\, w(s)]t(s) + vsinh\, w(s).(\beta'(s) + w'(s))n(s)\ ,$$

for $v = 0$,

$(\frac{d\tilde{\varphi}(s,v)}{ds}) = t(s),$

and for $v = 1$,

$$(\frac{d\tilde{\varphi}(s,v)}{ds}) = (1 + w'(s)\cosh w(s) + \beta'(s)\cosh w(s))t(s) + sinh\, w(s).(\beta'(s) + w'(s)).n(s).$$

Thus,

$i)\ sinh\, w(s) = 0 \Rightarrow w(s) = 0,$

or

$ii)\ w'(s) + \beta'(s) = 0,\ w(s) = -\beta(s) + c.$

4. Generalized Non-null Surface with Constant Slope Ruling with respect to Osculating Plane

Generating lines of the surface are given by points on the curve $X(s)$ and they have the constant slope with respect to the osculating planes to the curve at every point on the curve $X(s)$. The surface will be called generalized surface with constant slope ruling with respect to the osculating planes. The definitions of this type surface according to casual characters in Minkowski space are described in the following cases.

Case 4.1. The generalized surface with constant slope ruling with respect to osculating plane of the spacelike curve (whose normal vector is spacelike) is given as follows

$M(s,v) = \alpha(s) + v(u(s)).,$

such that

$u(s) = sin\, w(s)t(s) + cos\, w(s)n(s) + \sigma b(s).$

Case 4.2. The generalized surface with constant slope ruling with respect to osculating plane with spacelike curve and timelike normal vector is defined as follows

$M^{-}(s,v) = \alpha(s) + v(\bar{u}(s)),$

with following generating line

$\bar{u}(s) = cosh\, w(s)t(s) + sinh\, w(s)n(s) + \sigma b(s).$

Case 4.3. The generalized surface with constant slope ruling with respect to the osculating plane with timelike curve and spacelike normal vector is defined by

$$\widetilde{M}(s,v) = \alpha(s) + v(\tilde{u}(s)),$$

where

$$\tilde{u}(s) = sinh\, w(s)t(s) + cosh\, w(s)n(s) + \sigma b(s).$$

Example 4.1. Let the curve parameterized by vector function

$$\alpha(s) = \left(8cos\left(\frac{s}{2}\right), 8sin\left(\frac{s}{2}\right), \sqrt{6\left(\frac{s}{2}\right)}\right).$$

The generalized surface with constant slope ruling with respect to the osculating plane for $s \in [0,30]$, $w(s) = (\frac{2s}{10\sqrt{2}})$ and $\sigma = \left(\frac{1}{5}\right)$ is shown in following Fig. 2.

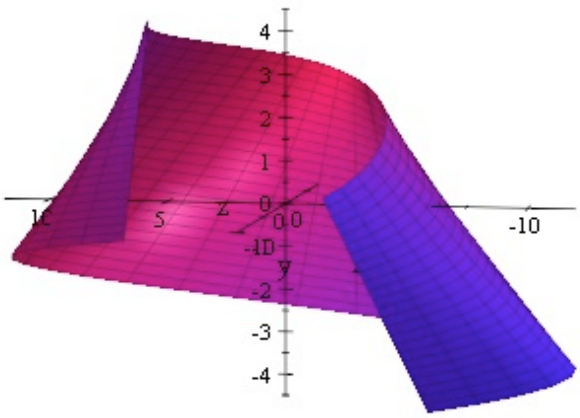

Figure 2: Example of the surface for $s \in [0,30]$, $w(s) = (\frac{2s}{10\sqrt{2}})$ and $\sigma = \left(\frac{1}{5}\right)$

4.1. Developable of generalized non-null surface with constant slope ruling with respect to the osculating plane

In this subsection, we investigate the developable condition for the generalized surface with constant slope ruling with respect to the osculating plane with Frenet Frame in Minkowski Space and give some relations and special cases about developable condition.

Theorem 4.1.1. The generalized surface $M(s,v)$ with constant slope ruling of the spacelike curve (whose normal vector is spacelike) is developable surface if and only if

$$\tau(cos^2 w(s) - \sigma^2) = \sigma sin\, w(s)(\kappa - w'(s)),$$

where the functions κ and τ are the first and second curvatures of the spacelike base curve, respectively.

Proof. If the surface $M(s,v)$ is developable surface, then it satisfies the following equality

$$det(t, u(s), u(s)') = 0.$$

It is calculated $det(t, u(s), u(s)')$ for the surface $M(s,v)$

$$u(s)' = (cos\, w(s)(w'(s) - \kappa))t(s) + (sin\, w(s)(\kappa - w'(s)) + \sigma\tau)n(s) + (\tau cos\, w(s) b(s)),$$

$$det(t, u(s), u(s)') = \tau(cos^2 w(s) - \sigma^2) - \sigma sin\, w(s)\big(\kappa - w'(s)\big) = 0.$$

So developable condition for generalized surface with constant slope ruling with respect to the osculating plane $M(s,v)$ is given by

Remark 4.1.1. If $cos\, w(s) = x^1 = const.$ and sin $w(s) = x_2 = const.$

$$\left(\frac{\tau}{\kappa}\right) = \left(\frac{\sigma x_2}{{x_1}^2 - \sigma^2}\right),$$

then the surface $M(s,v)$ is developable if and only if the base curve $\alpha(s)$ is a helix.

Theorem 4.1.2. The surface $\overline{M}(s,v)$ is developable surface if and only if

$$\tau(sin\, h^2 w(s) - \sigma^2) = \sigma cosh\, w(s)(\kappa + w'(s)).$$

Proof. If the surface $\overline{M}(s,v)$ is developable, then it satisfies the following equality

$$\bar{u}(s)' = \big(sinh\, w(s)(w'(s) - \kappa)\big)t(s) + \big(cosh\, w(s)\big(\kappa - w'(s)\big) + \sigma\tau\big)n(s) + \big(\tau sinh\, w(s)\big)b(s),$$

$$det(t, \bar{u}(s), \bar{u}(s)') = \tau(sinh^2 w(s) - \sigma^2) - \sigma cosh\, w(s)\big(\kappa - w'(s)\big) = 0.$$

So developable condition for the surface $\overline{M}\ (s,v)$ is obtained as follows

$$\tau(sinh^2 w(s) - \sigma^2) = \sigma cosh\, w(s)(\kappa - w'(s)).$$

Remark 4.1.2. The surface $\overline{M}(s, v)$ is developable surface if and only if the base curve is a helix with

$$(\frac{\tau}{\kappa}) = (\frac{\sigma x_1}{{x_2}^2 - \sigma^2}),$$

where $cos\ w(s) = x^1 = const.$ and $\sin w(s) = x_2 = const.$

Theorem 4.1.3. The surface $\widetilde{M}(s, v)$ is developable surface if and only if

$$\tau(\cos h^2 w(s) + \sigma^2) = \sigma \sinh w(s) \left(\kappa + w'(s)\right).$$

Proof. By taking derivative of $\tilde{u}(s)$ is obtained as follows

$$\tilde{u}(s)' = (sinh\ w(s)(w'(s) - \kappa))t(s) + (cosh\ w(s)(\kappa - w'(s)) + \sigma\tau)n(s) + (\tau sinh\ w(s))b(s).$$

Then

$$\tau(cosh^2 w(s) + \sigma^2) - \sigma sinh\ w(s)(\kappa + w'(s)) = 0.$$

Thus, the condition that ensures the surface can be developable is as following

$$\tau(cos\ h^2 w(s) + \sigma^2) = \sigma sinh\ w(s)(\kappa + w'(s)).$$

Remark 4.1.3. We can say that the surface $\widetilde{M}(s, v)$ is developable if and only if the base curve is a helix with

$$(\frac{\tau}{\kappa}) = (\frac{\sigma sinh\ b}{cosh^2 b + \sigma^2}),$$

where $cos\ w(s) = x_1 = const$ and $\sin w(s) = x_2 = const.$

4.2. Striction line of generalized non-null surface with constant slope ruling with respect to osculating plane

The striction point on a surface is the foot of the common normal between two consecutive generators (or ruling). The set of striction points defines the striction curve given by [10]

$$\beta = \alpha(s) - (\frac{<t, X>}{<X', X'>}).X(s).$$

Theorem 4.2.1. The striction line on $M(s,v)$ is given by

$$\beta=\alpha(s)-\left(\frac{\cos w(s)\left(\kappa-w'(s)\right)}{\left(\kappa-w'(s)\right)^2+\tau^2(\sigma^2-\cos^2 w(s))+2\sin^2 w(s)(\kappa-w'(s))\sigma\tau}\right)u(s).$$

Proof. By taking derivative of $u(s)$ is calculated as

$$u(s)'=(\cos w(s)(w'(s)-\kappa))t(s)+(\sin w(s)(\kappa-w'(s))+\sigma\tau)n(s)+(\tau\cos w(s)b(s))$$

and

$$<t,u(s)>=\cos w(s)(\kappa-w'(s)),$$

$$<u(s)',u(s)'>=(\kappa-w'(s))^2+\tau^2(\sigma^2-\cos^2 w(s))+2\sin^2 w(s)(\kappa-w'(s))\sigma\tau.$$

Theorem 4.2.2. The striction line on surface $\overline{M}(s,v)$ is given as follows

$$\beta=\alpha(s)-\left(\frac{-\sinh w(s)\left(\kappa+w'(s)\right)}{(\kappa+w'(s))\left((\kappa+w'(s))-2\cosh w(s)\sigma\tau\right)+\tau^2(\sigma^2+\sinh^2 w(s))}\right)u(s).$$

Proof. By taking derivative of $\bar{u}(s)$ is obtained as follows

$$\bar{u}(s)'=(\sinh w(s)(w'(s)-\kappa))t(s)+(\cosh w(s)(\kappa-w'(s))+\sigma\tau)n(s)$$
$$+(\tau\sinh w(s))b(s)$$

and

$$<t,\bar{u}(s)>=-\sinh w(s)(\kappa+w'(s)),$$

$$<\bar{u}(s)',\bar{u}(s)'>=(\kappa+w'(s))((\kappa+w'(s))-2\cosh w(s)\sigma\tau)+\tau^2(\sigma^2+\sinh^2 w(s)).$$

If the obtained values are written in place of the striction line equation, then the proof is completed.

Theorem 4.2.3. The striction line on surface $\widetilde{M}(s,v)$ is defined by

$$\beta=\alpha(s)-\left(\frac{\cosh w(s)\left(\kappa+w'(s)\right)}{\left(\cosh^2 w(s)(\kappa-w'(s))\right)^2+\left[\sinh w(s)(\kappa-w'(s))-\sigma\tau\right]^2}+\cosh^2 w(s)\tau^2\right)\tilde{u}(s).$$

Proof. Proof of the theorem can be obtained by making calculations similar to the proof of the previous two theorems.

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