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Asymptotically Lacunary $\mathcal{J}_\sigma$-Equivalence of Sequences of Sets

Küme Dizilerinin Asimptotik Lacunary $\mathcal{J}_\sigma$-Denkliği

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Abstract

In this study, we introduce the notions of Wijsman asymptotically strongly p-lacunary invariant equivalence $([W^L_{N_{\sigma\theta}}]_p)$, Wijsman asymptotically lacunary $\mathcal{J}$-invariant equivalence $(W^L_{\mathcal{J}_{\sigma\theta}})$ and Wijsman asymptotically lacunary $\mathcal{J}^*$-invariant equivalence $(W^L_{\mathcal{J}^*_{\sigma\theta}})$ for sequences of sets. Also, the relationships among the notions of Wijsman asymptotically lacunary invariant equivalence, Wijsman asymptotically lacunary invariant statistical equivalence, $[W^L_{N_{\sigma\theta}}]_p$, $W^L_{\mathcal{J}_{\sigma\theta}}$ and $W^L_{\mathcal{J}^*_{\sigma\theta}}$ are investigated.

Keywords: Asymptotically equivalence, Lacunary sequence, $\mathcal{J}$-convergence, Invariant convergence, Sequences of sets, Wijsman convergence

Öz

Bu çalışmada, küme dizileri için Wijsman asimptotik kuvvetli p-lacunary invaryant denklik $([W^L_{N_{\sigma\theta}}]_p)$, Wijsman asimptotik lacunary $\mathcal{J}$-invaryant denklik $(W^L_{\mathcal{J}_{\sigma\theta}})$ ve Wijsman asimptotik lacunary $\mathcal{J}^*$-invaryant denklik $(W^L_{\mathcal{J}^*_{\sigma\theta}})$ kavramları tanıtıldı. Ayrıca, Wijsman asimptotik lacunary invaryant denklik, Wijsman asimptotik lacunary invaryant istatistiksel denklik, $[W^L_{N_{\sigma\theta}}]_p$, $W^L_{\mathcal{J}_{\sigma\theta}}$ ve $W^L_{\mathcal{J}^*_{\sigma\theta}}$ kavramları arasındaki ilişkiler araştırıldı.

Anahtar Kelimeler: Asimptotik denklik, Lacunary dizi, $\mathcal{J}$-yakınsaklık, İnvaryant yakınsaklık, Küme dizisi, Wijsman yakınsaklık

1. Introduction

The notion of statistical convergence was firstly introduced by Fast (1951) and this notion has been studied by Šalát (1980), Fridy (1985) and many others. Then Fridy and Orhan (1993), using the notion of lacunary sequence $\theta=\{k_r\}$, defined the notion of lacunary statistical convergent sequence.

Let σ be a mapping such that $\sigma:\mathbb{N}^+\to\mathbb{N}^+$ (the set of positive integers). A continuous linear functional ψ on ℓ_∞, the space of bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies following conditions:

1. $\psi(x_n)\geq 0$, when the sequence (x_n) has $x_n\geq 0$ for all n,
2. $\psi(e)=1$, where $e=(1,1,1,\ldots)$ and
3. $\psi(x_{\sigma(n)})=\psi(x_n)$ for all $(x_n)\in\ell_\infty$.

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The mappings σ are assumed to be one-to-one and such that $\sigma^m(n)\neq n$ for all $m,n\in\mathbb{N}^+$, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus ψ extends the limit functional on c, the space of convergent sequences, in the sense that $\psi(x_n)=\lim x_n$ for all $(x_n)\in c$.

Many authors have studied on the notions of invariant mean and invariant convergent sequences (for examples, see Mursaleen 1979, Mursaleen and Edely 2009, Nuray et al. 2011, Raimi 1963, Savaş 1989, Schaefer 1972).

The notion of lacunary strong σ-convergent sequence was defined by Savaş (1990). Then, Savaş and Nuray (1993) introduced the notion of lacunary σ-statistically convergence. Recently, the notions of lacunary invariant summability and the space $[V_{\sigma\theta}]_q$ have been studied by Pancaroğlu and Nuray (2013).

The idea of $\mathcal{J}$-convergence which is a generalization of statistical convergence was introduced by Kostyrko et al. (2000).

A family of sets $\mathcal{J}\subseteq 2^{\mathbb{N}}$ is called an ideal if

i) $\emptyset \in \mathcal{J}$, ii) $E,F \in \mathcal{J} \Rightarrow E \cup F \in \mathcal{J}$,
iii) $E \in \mathcal{J} \wedge F \subseteq E \Rightarrow F \in \mathcal{J}$.

An ideal $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{J}$ and a non-trivial ideal $\mathcal{J} \subseteq 2^{\mathbb{N}}$ is called admissible if $\{n\} \in \mathcal{J}$ for each $n \in \mathbb{N}$ (the set of natural numbers).

All ideals in this study will be assumed to be admissible in $2^{\mathbb{N}}$ (the power set of $\mathbb{N}$).

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter if

i) $\emptyset \notin \mathcal{F}$, ii) $E,F \in \mathcal{F} \Rightarrow E \cap F \in \mathcal{F}$,
iii) $E \in \mathcal{F} \wedge F \supseteq E \Rightarrow F \in \mathcal{F}$.

For any ideal $\mathcal{J} \subseteq 2^{\mathbb{N}}$, there is a filter $\mathcal{F}(\mathcal{J})$ corresponding with $\mathcal{J}$ such that

$$\mathcal{F}(\mathcal{J}) = \{M \subset \mathbb{N} : (\exists E \in \mathcal{J})(M = \mathbb{N} \backslash E)\}.$$

A sequence (x_n) is said to be $\mathcal{J}$-convergent to L if for every $\varepsilon > 0$, the set

$$E_\varepsilon = \{k : |x_n - L| \geq \varepsilon\}$$

belongs to $\mathcal{J}$. It is denoted by $\mathcal{J} - \lim x_n = L$.

A sequence (x_n) is said to be $\mathcal{J}^*$-convergent to L if there exists a set $M = \{m_1 < m_2 < \ldots\} \in \mathcal{F}(\mathcal{J})$ such that

$$\lim_{n \to \infty} x_{m_n} = L.$$

It is denoted by $\mathcal{J}^* - \lim x_n = L$.

An admissible ideal $\mathcal{J} \subset 2^{\mathbb{N}}$ is said to satisfy the property (*AP*) if for every countable family of mutually disjoint sets $\{E_1, E_2, \ldots\}$ belonging to $\mathcal{J}$ there exists a countable family of sets $\{F_1, F_2, \ldots\}$ such that the symmetric difference $E_j \Delta F_j$ is a finite set for $j \in \mathbb{N}$ and $F = (U_{j=1}^{\infty} F_j) \in \mathcal{J}$.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$.

Recently, the notions of $\sigma\theta$-uniform density of a subset E of $\mathbb{N}$ and corresponding $\mathcal{J}_{\sigma\theta}$-convergence for real sequences were introduced by Ulusu and Nuray (2016) as follows:

Let $\theta = \{k_r\}$ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r = \min_n |E \cap \{\sigma^m(n) : m \in I_r\}| \text{ and}$$
$$S_r = \max_n |E \cap \{\sigma^m(n) : m \in I_r\}|.$$

If the following limits exists

$$\underline{V_\theta}(E) = \lim_{r \to \infty} \frac{s_r}{h_r} \text{ and } \overline{V_\theta}(E) = \lim_{r \to \infty} \frac{S_r}{h_r},$$

then $\underline{V_\theta}(E)$ and $\overline{V_\theta}(E)$ are called a lower $\sigma\theta$-uniform density and an upper $\sigma\theta$-uniform density of the set E, respectively. If $\underline{V_\theta}(E) = \overline{V_\theta}(E)$, then $V_\theta(E) = \underline{V_\theta}(E) = \overline{V_\theta}(E)$ is called the $\sigma\theta$-uniform density of E.

The class of all $E \subseteq \mathbb{N}$ with $V_\theta(E) = 0$ is denoted by $\mathcal{J}_{\sigma\theta}$. Obviously $\mathcal{J}_{\sigma\theta}$ is admissible ideal in $2^{\mathbb{N}}$.

A sequence (x_n) is said to be lacunary $\mathcal{J}_\sigma$-convergent or $\mathcal{J}_{\sigma\theta}$-convergent to L if for every $\varepsilon > 0$, the set

$$E_\varepsilon = \{k : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{J}_{\sigma\theta}$, i.e., $V_\theta(E_\varepsilon) = 0$. It is denoted by $\mathcal{J}_{\sigma\theta} - \lim x_n = L$.

The notion of convergence for real sequences has been extended by many authors to notions of convergence for sequences of sets. One of these extensions considered in this study is the notion of Wijsman convergence (for examples, see Baronti and Papini 1986, Beer 1985, 1994, Kişi and Nuray 2013, Nuray and Rhoades 2012, Pancaroğlu and Nuray 2013, Pancaroğlu Akın et al. 2019, Sever et al. 2014, Ulusu and Nuray 2012, Ulusu and Dündar 2014, 2018, Wijsman 1964).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Throughout the study, (X, ρ) will be taken as a metric space and A, A_k, B_k will be taken as any non-empty closed subsets of X.

A sequence $\{A_k\}$ is said to be Wijsman convergent to A if for each $x \in X$,

$$\lim_{k \to \infty} d(x, A_k) = d(x, A).$$

A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} d(x, A_{\sigma^k(m)}) = d(x, A),$$

uniformly in $m = 1, 2, \ldots$.

Let $0 < p < \infty$. A sequence $\{A_k\}$ is said to be Wijsman strongly p-lacunary invariant convergent to A if for each $x \in X$,

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \left| d(x, A_{\sigma^k(m)}) - d(x, A) \right|^p = 0,$$

uniformly in m.

A sequence $\{A_k\}$ is said to be Wijsman lacunary invariant statistical convergent to A if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r: |d(x,A_{\sigma^k(m)})-d(x,A)|\geq\varepsilon\}|=0,$$
uniformly in m.

Marouf (1993) presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Then, the notion of asymptotically equivalence has been developed by many others (for examples, see Hazarika 2015, Patterson and Savaş 2006, Savaş and Patterson 2006, Savaş 2013).

Two non-negative sequences (x_n) and (y_n) are said to be asymptotically equivalent if

$$\lim_{n\to\infty}\frac{x_n}{y_n}=1.$$
It is denoted by $x_n\sim y_n$.

The notion of asymptotically equivalence for real sequences which was defined by Marouf (1993) has been firstly extended by Ulusu and Nuray (2013) to notion of asymptotically equivalence for sequences of sets. For more detail, see Pancaroğlu et al. (2013).

For any non-empty closed subsets $A_k,B_k\subseteq X$, where $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$, two sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman asymptotically equivalent if for each $x\in X$,

$$\lim_{k\to\infty}\frac{d(x,A_k)}{d(x,B_k)}=1.$$
It is denoted by $A_k\sim B_k$.

As an example, let's consider the following sequences of circles in the (x,y)-plane:

$A_k=\{(x,y):x^2+y^2+2kx=0\}$ and
$B_k=\{(x,y):x^2+y^2-2kx=0\}$.

Since
$$\lim_{k\to\infty}\frac{d(x,A_k)}{d(x,B_k)}=1,$$

the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically equivalent, i.e., $A_k\sim B_k$.

The term $d(x;A_k,B_k)$ is defined as follows:

$$d(x;A_k,B_k)=\begin{cases}\dfrac{d(x,A_k)}{d(x,B_k)}, & x\notin A_k\cup B_k\\ L, & x\in A_k\cup B_k.\end{cases}$$

Two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically lacunary invariant equivalent of multiple L if for each $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}d(x;A_{\sigma^k(m)},B_{\sigma^k(m)})=L,$$
uniformly in m. It is denoted by $A_k\overset{WN_{\sigma\theta}^L}{\sim}B_k$.

Two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically lacunary invariant statistical equivalent of multiple L if for every $\varepsilon>0$ and each $x\in X$,

$$\lim_{r\to\infty}\frac{1}{h_r}|\{k\in I_r: |d(x;A_{\sigma^k(m)},B_{\sigma^k(m)})-L|\geq\varepsilon\}|=0,$$
uniformly in m. It is denoted by $A_k\overset{WS_{\sigma\theta}^L}{\sim}B_k$.

From now on, for short, we use $d_x(A), d_x(A_k)$, and $d_x(A_k,B_k)$ instead of $d(x,A), d(x,A_k)$ and $d(x;A_k,B_k)$, respectively.

2. Results

In this section, we introduce the notions of Wijsman asymptotically strongly p-lacunary invariant equivalence $([W_{N_{\sigma\theta}}^L]_p)$, Wijsman asymptotically lacunary $\mathcal{J}$-invariant equivalence $(W_{\mathcal{J}_{\sigma\theta}}^L)$ and Wijsman asymptotically lacunary $\mathcal{J}^*$-invariant equivalence $(W_{\mathcal{J}_{\sigma\theta}^*}^L)$ for sequences of sets. Also, the relationships among the notions of Wijsman asymptotically lacunary invariant equivalence, Wijsman asymptotically lacunary invariant statistical equivalence, $[W_{N_{\sigma\theta}}^L]_p, W_{\mathcal{J}_{\sigma\theta}}^L$ and $W_{\mathcal{J}_{\sigma\theta}^*}^L$ are investigated.

Definition 2.1 Two sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman asymptotically lacunary $\mathcal{J}_\sigma$-equivalent or Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$-equivalent of multiple L if for every $\varepsilon>0$ and each $x\in X$, the set

$$E_{\varepsilon,x}^{\sim}:=\{k:|d_x(A_k,B_k)-L|\geq\varepsilon\}$$

belongs to $\mathcal{J}_{\sigma\theta}$, i.e., $V_\theta(E_{\varepsilon,x}^{\sim})=0$. In this case, we write $A_k\overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim}B_k$ and simply Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$-equivalent if $L=1$.

The class of all Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$-equivalent sequences will be denoted by $W_{\mathcal{J}_{\sigma\theta}}^L$.

Theorem 2.1 Let $d_x(A_k)=O(d_x(B_k))$. If $A_k\overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim}B_k$, then $A_k\overset{WN_{\sigma\theta}^L}{\sim}B_k$.

Proof. Let $m\in\mathbb{N}$ be an arbitrary and $\varepsilon>0$ is given. Now, we calculate

$$\mathcal{T}(\theta,m):=\left|\frac{1}{h_r}\sum_{k\in I_r}d_x(A_{\sigma^k(m)},B_{\sigma^k(m)})-L\right|.$$

For each $x\in X$, we have

$$\mathcal{T}(\theta,m)\leq\mathcal{T}_1(\theta,m)+\mathcal{T}_2(\theta,m),$$

where

$$\mathcal{T}_1(\theta,m):=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d_x(A_{\sigma^k(m)},B_{\sigma^k(m)})-L|\geq\varepsilon}}|d_x(A_{\sigma^k(m)},B_{\sigma^k(m)})-L|$$
and
$$\mathcal{T}_2(\theta,m):=\frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d_x(A_{\sigma^k(m)},B_{\sigma^k(m)})-L|<\varepsilon}}|d_x(A_{\sigma^k(m)},B_{\sigma^k(m)})-L|$$

For each $x \in X$ and every $m = 1,2,...,$ it is obvious that $\mathcal{T}_2(\theta,m) < \varepsilon$. Since $d_x(A_k) = O(d_x(B_k))$, there exists a $K > 0$ such that

$$|d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \leq K,$$

for each $x \in X (k \in I_r, m = 1,2,...)$. So, this implies that

$$\begin{aligned}\mathcal{T}_1(\theta,m) &\leq \frac{K}{h_r} |\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}| \\ &\leq K \frac{\max_m |\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}|}{h_r} \\ &= K \frac{S_r}{h_r}.\end{aligned}$$

Then, due to our hypothesis, $A_k \overset{WN_{\sigma\theta}^L}{\sim} B_k$.

Definition 2.2 Let $0 < p < \infty$. Two sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman asymptotically strongly p -lacunary invariant equivalent of multiple L if for each $x \in X$

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p = 0,$$

uniformly in m. In this case, we write $A_k \overset{[W_{N_{\sigma\theta}}^L]_p}{\sim} B_k$ and simply Wijsman asymptotically strongly p-lacunary invariant equivalent if $L = 1$.

Theorem 2.2 If $A_k \overset{[W_{N_{\sigma\theta}}^L]_p}{\sim} B_k$, then $A_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim} B_k$.

Proof. Let $A_k \overset{[W_{N_{\sigma\theta}}^L]_p}{\sim} B_k$ and $\varepsilon > 0$ be given. For each $x \in X$, we can write

$$\begin{aligned}&\sum_{k \in I_r} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &\geq \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon}} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &\geq \varepsilon^p \left|\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}\right| \\ &\geq \varepsilon^p \max_m \left|\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}\right|\end{aligned}$$

and so

$$\begin{aligned}&\frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &\geq \varepsilon^p \frac{\max_m \left|\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}\right|}{h_r} \\ &= \varepsilon^p \frac{S_r}{h_r},\end{aligned}$$

for all m. By our assuption, this implies that $\lim_{r \to \infty} \frac{S_r}{h_r} = 0$ and consequently $A_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim} B_k$.

Theorem 2.3 Let $d_x(A_k) = O(d_x(B_k))$. If $A_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim} B_k$, then $A_k \overset{[W_{N_{\sigma\theta}}^L]_p}{\sim} B_k$.

Proof. Let $d_x(A_k) = O(d_x(B_k))$ and $\varepsilon > 0$ is given. Also, we suppose that $A_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim} B_k$. By assumption, we have $V_\theta(E_{\varepsilon,x}^{\sim}) = 0$. Since $d_x(A_k) = O(d_x(B_k))$, there exists a $K > 0$ such that

$$|d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \leq K,$$

for each $x \in X (k \in I_r, m = 1,2,...)$. Then, for each $x \in X$ we get

$$\begin{aligned}&\frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &= \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon}} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &+ \frac{1}{h_r} \sum_{\substack{k \in I_r \\ |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| < \varepsilon}} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p \\ &\leq K \frac{\max_m \left|\{k \in I_r : |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L| \geq \varepsilon\}\right|}{h_r} + \varepsilon^p \\ &\leq K \frac{S_r}{h_r} + \varepsilon^p,\end{aligned}$$

for all m. Hence, for each $x \in X$ we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |d_x(A_{\sigma^k(m)}, B_{\sigma^k(m)}) - L|^p = 0.$$

This completes the proof.

Theorem 2.4 Let $d_x(A_k) = O(d_x(B_k))$. Then, $A_k \overset{W_{\mathcal{J}_{\sigma\theta}}^L}{\sim} B_k$ if and only if $A_k \overset{[W_{N_{\sigma\theta}}^L]_p}{\sim} B_k$.

Proof. This can be obtained from Theorem 2.2 and Theorem 2.3.

Now, we shall state a theorem that gives a relation between $W_{\mathcal{J}_{\sigma\theta}}^L$ and $WS_{\sigma\theta}^L$.

Theorem 2.5 Two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$ -equivalent of multiple L if and only if these sequences are Wijsman asymptotically lacunary invariant statistical equivalent of multiple L.

Definition 2.3 Two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically lacunary $\mathcal{J}_\sigma^*$ -equivalent or Wijsman asymptotically $\mathcal{J}_{\sigma\theta}^*$ -equivalent of multiple L if and only if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{J}_{\sigma\theta})$ such that for each $x \in X$,

$$\lim_{k \to \infty} d_x(A_{m_k}, B_{m_k}) = L.$$

In this case, we write $A_k \overset{W_{\mathcal{J}_{\sigma\theta}^*}^L}{\sim} B_k$ and simply Wijsman asymptotically $\mathcal{J}_{\sigma\theta}^*$ -equivalent if $L = 1$.

Theorem 2.6 If two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{J}_{\sigma\theta}^*$ -equivalent of multiple L, then these sequences are Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$ -equivalent of multiple L.

Proof. Let $A_k \overset{W_{\mathcal{J}_{\sigma\theta}^*}^L}{\sim} B_k$. Then, there exists a set $H \in \mathcal{J}_{\sigma\theta}$ such that for $M = \mathbb{N} \backslash H = \{m_1 < m_2 < ... < m_k < ...\}$ and each $x \in X$,

$$\lim_{k\to\infty} d_x(A_{m_k}, B_{m_k}) = L. \quad (1)$$

Given $\varepsilon > 0$. By (1), there exists $k_0 \in \mathbb{N}$ such that

$$|d_x(A_{m_k}, B_{m_k}) - L| < \varepsilon,$$

for each $k > k_0$. Hence, for every $\varepsilon > 0$ and each $x \in X$, it is obvious that

$$\{k \in \mathbb{N}: |d_x(A_k, B_k) - L| \geq \varepsilon\} \subset H \cup \{m_1 < m_2 < \ldots < m_{k_0}\}. \quad (2)$$

Since $\mathcal{J}_{\sigma\theta}$ is admissible, the set on the right-hand side of (2) belongs to $\mathcal{J}_{\sigma\theta}$. Therefore, $A_k \overset{W^L_{\mathcal{J}_{\sigma\theta}}}{\sim} B_k$.

The converse of Theorem 2.6 holds if $\mathcal{J}_{\sigma\theta}$ has property (AP).

Theorem 2.7 Let $\mathcal{J}_{\sigma\theta}$ has property (AP). If two sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically $\mathcal{J}_{\sigma\theta}$-equivalent of multiple L, then these sequences are Wijsman asymptotically $\mathcal{J}^*_{\sigma\theta}$-equivalent of multiple L.

Proof. Suppose that $\mathcal{J}_{\sigma\theta}$ satisfies condition (AP) and $A_k \overset{W^L_{\mathcal{J}_{\sigma\theta}}}{\sim} B_k$. Then, for every $\varepsilon > 0$ and each $x \in X$, we have

$$\{k: |d_x(A_k, B_k) - L| \geq \varepsilon\} \in \mathcal{J}_{\sigma\theta}.$$

Put

$$E_1 = \{k: |d_x(A_k, B_k) - L| \geq 1\}$$

and

$$E_n = \left\{k: \frac{1}{n} \leq |d_x(A_k, B_k) - L| < \frac{1}{n-1}\right\},$$

for $n \geq 2 (n \in \mathbb{N})$. Obviously, $E_i \cap E_j = \varnothing$ for each $x \in X$ and $i \neq j$. By condition (AP), there exists a sequence of $\{F_n\}_{n\in\mathbb{N}}$ such that $E_j \Delta F_j$ are finite sets for $j \in \mathbb{N}$ and $F = (\cup_{j=1}^{\infty} F_j) \in \mathcal{J}_{\sigma\theta}$. It is enough to prove that for $V = \mathbb{N} \setminus F$ and each $x \in X$, we have

$$\lim_{\substack{k\to\infty \\ k\in V}} d_x(A_k, B_k) = L. \quad (3)$$

Let $\delta > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \delta$. Hence, for each $x \in X$

$$\{k: |d_x(A_k, B_k) - L| \geq \delta\} \subset \bigcup_{j=1}^{n+1} E_j.$$

Since $E_j \Delta F_j (j = 1, 2, \ldots, n+1)$ are finite sets, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{n+1} F_j\right) \cap \{k: k > k_0\} = \left(\bigcup_{j=1}^{n+1} E_j\right) \cap \{k: k > k_0\}. \quad (4)$$

If $k > k_0$ and $k \notin F$, then $k \notin U_{j=1}^{n+1} F_j$ and by (4), $k \notin U_{j=1}^{n+1} E_j$. But then, for each $x \in X$ we get

$$|d_x(A_k, B_k) - L| < \frac{1}{n+1} < \delta$$

and so (3) holds. Consequently, we have $A_k \overset{W^L_{\mathcal{J}^*_{\sigma\theta}}}{\sim} B_k$.

3. Conclusion

In this study, we introduced some asymptotically equivalence notions in Wijsman sense for sequences of sets. Also, relationships among the new asymptotically equivalence notions were given. Furthermore, the relationships among some of the new asymptotically equivalence notions introduced by us and other notions previously introduced on the asymptotically equivalence of sequences of sets were investigated.

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