On the Dual Space of $L^{p(\cdot)}(\Omega)$ with $0<p(x)<1$

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ABSTRACT

In this research paper, we give a brief overview of the variable exponent Lebesgue spaces for $1\leq p(x)<\infty$. We also mention some applications of variable exponent Lebesgue spaces. We then mainly deal with continuous dual space of variable exponent Lebesgue spaces for $0<p(x)<1$. It is known that there exists no nonzero continuous linear functional on classical Lebesgue space L^p when $0<p<1$. We generalize this result to the variable exponent setting. We prove that if $p^+<1$, then the only continuous linear functional on $L^{p(\cdot)}(\Omega)$ $(0<p(x)<1)$ is the zero functional. However, it remains an open question whether there exists non zero continuous linear functional when $p_+=1$.

Keywords: Variable Exponent, Lebesgue Space, Linear Functional, Dual Space, Sequence of Function

$0<p(x)<1$ Durumunda $L^{p(.)}(\Omega)$ nin Dual Uzayı

ÖZ

Bu araştırma makalesinde, $1\leq p(x)<\infty$ durumu için, değişken üslü Lebesgue uzaylarının kısa genel bir tanıtımını veriyoruz. Değişken üslü Lebesgue uzaylarının bazı uygulamalarından da söz ediyoruz. Sonra, esas olarak, $0<p(x)<1$ durumu için, değişken üslü Lebesgue uzaylarının sürekli dual uzayı ile ilgileniyoruz. $0<p<1$ olduğunda, klasik Lebesgue L^p uzayında sıfır dışında sürekli lineer fonksiyonelin olmadığı bilinmektedir. Biz bu durumu değişken üslüye genelleştiriyoruz. $p^+<1$ olduğunda $L^{p(.)}(\Omega)$ $(0<p(x)<1$ üzerindeki tek sürekli lineer fonksiyonelin sıfır fonksiyoneli olduğunu ispatlıyoruz. Bununla birlikte, $p_+=1$ olduğunda, sıfırdan farklı sürekli lineer fonksiyonelin olup olmadığını sorusu açık kalmıştır.

Anahtar Kelimeler: Değişken Üs, Lebesgue Uzayı, Lineer Fonksiyonel, Dual Uzay, Fonksiyon Dizisi

1. INTRODUCTION

Before stating our results, let us first give an outline of $L^{p(.)}(\Omega)$ spaces with $1\leq p(x)<\infty$. In this paper we will always use the Lebesgue measure. The variable exponent Lebesgue spaces $L^{p(.)}$ are a generalization of the classical Lebesgue spaces L^p. $L^{p(.)}$ spaces can be traced back to Orlicz (1931). But the modern development started in a paper by Kováčik and Rákosník (1991). Also, it is a fact that these two spaces have many properties in common.These spaces have important applications in many fields e.g. in the study of PDEs, the calculus of variations and to problems in physics (See, for example (Amaziane et al., 2009: Acerbi and Mingione 2002 : Diening and Růžička 2003: Růžička 2000 : Levine et al., 2004)). We refer the reader to the books (Cruz-Uribe and Fiorenza 2013: Diening et al., 2011) for various properties and characterizations of $L^{p(.)}(\Omega)$ spaces. First, we recall the basic definitions and some classical properties of $L^{p(.)}(\Omega)$ spaces. Let $\Omega\subset\mathbb{R}^n$ be a measurable set. Let $p:\Omega\to[1,\infty)$ be a measurable bounded function, called a variable exponent on Ω. We let $P(\Omega)$ denote the set of all such functions. We also denote

$$p_-=\text{ess inf }\{p(x):x\in\Omega\},\quad p_+=\text{ess sup }\{p(x):x\in\Omega\}.$$

We define the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ to consist of all measurable functions $f:\Omega\to\mathbb{R}$ for which the modular functional

$$\rho_{p(.)}(f)=\int_{\Omega}|f(x)|^{p(x)}dx$$

finite. $L^{p(.)}(\Omega)$ is a Banach function space when endowed with the norm

$$\|f\|_{L^{p(.)}(\Omega)}=\|f\|_{p(.)}=\inf\left\{\lambda>0:\int_{\Omega}\left|\frac{f(x)}{\lambda}\right|^{p(x)}dx\leq 1\right\}.$$

If $p(.)=p$, a constant, then $L^{p(.)}(\Omega)$ and $L^{p}(\Omega)$ norms equal each other, and also these two spaces coincide.

For $p_{+}<\infty$, one of the most useful property is that in $L^{p(.)}(\Omega)$ spaces

$$\lim_{n\to\infty}\rho_{p(.)}(f_n-f)\to 0\Leftrightarrow \lim_{n\to\infty}\|f_n-f\|_{p(.)}\to 0$$

(Fan and Zhao 2001). Furthermore, the modular and the norm are related through the following inequalities

$$\min\left\{\|f\|_{p(.)}^{p_-},\|f\|_{p(.)}^{p+}\right\}\leq \rho_{p(.)}(f)\leq \max\left\{\|f\|_{p(.)}^{p_-},\|f\|_{p(.)}^{p+}\right\}.$$

Let $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ and $p(.)\in P(\Omega)$, then for any $f\in L^{p(.)}(\Omega)$ and $g\in L^{p'(.)}(\Omega)$ functions the variable exponent Lebesgue spaces have the following Hölder type inequality

$$\|fg\|_1\leq 2\|f\|_{p(.)}\|g\|_{p'(.)}.$$

If $g\in L^{p'(.)}(\Omega)$ and G is defined on $L^{p(.)}(\Omega)$ by

$$G(f)=\int_{\Omega}f(x)g(x)dx,\quad f\in L^{p(.)}(\Omega) \tag{1}$$

then G is a continuous linear functional on $L^{p(.)}(\Omega)$.

Theorem 1.1. Let $p(.)\in P(\Omega)$ with $p_{+}<\infty$. Then every linear continuous functional G on $L^{p(.)}(\Omega)$ can be represented in the form (1) with a unique function $g\in L^{p'(.)}(\Omega)$ (Kováčik and Rákosník 1991).

As an immediate consequence of this theorem, we see that if $p_+ < \infty$, then the dual of $L^{p(.)}(\Omega)$ is isometrically isomorphic $L^{p'(.)}(\Omega)$. The result is false if $p_+ = \infty$. For the proofs of these facts see (Kováčik and Rákosník 1991). In spite of these basic similarities between $L^{p(.)}(\Omega)$ and $L^{p}(\Omega)$, however, there is one fundamental difference between these two spaces. $L^{p(.)}(\Omega)$ spaces don't have the mean continuity property in such a way that if $p(.)$ is a non-constant continuous function in an open ball B, then there exists a function $f \in L^{p(.)}(\Omega)$ such that $f(x+h) \notin L^{p(.)}(B)$ for every $h \in \mathbb{R}^n$ with arbitrarily small norm (Kováčik and Rákosník 1991).

2. METHODS

2.1. The Spaces $L^{p(.)}(\Omega)$ with $0<p(x)<1$

We will now assume that $0<p(x)<1$. A measurable function $f:\Omega \to \mathbb{R}$ is in $L^{p(.)}(\Omega)$ for $0<p(x)<1$ if $\rho_{p(.)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty$. If $p(.) = p$, constant, it is known that $\|f\|_p = \left(\int_{\Omega} |f(x)|^p dx \right)^{\frac{1}{p}}$ does not define a norm on $L^p(\Omega)$ for $0<p<1$.

Lemma 2.1.1. Let $0<p<1$, then for α, β in $[0,\infty)$ the inequality

$$(\alpha+\beta)^p \le \alpha^p + \beta^p \tag{2}$$

holds (Bruckner et al., 1997).

By this lemma,

$$d(f,g) = \int_{\Omega} |f(x) - g(x)|^{p(x)} dx$$

defines a metric on $L^{p(.)}(\Omega)$ for $0<p(x)<1$. The triangle inequality follows from

$$|f(x)-g(x)|^{p(x)} \le |f(x)-h(x)|^{p(x)} + |h(x)-g(x)|^{p(x)}$$

a.e. on Ω. It follows from the inequality (2) that the spaces $L^{p(.)}(\Omega)$ for $0< p(x)< 1$ are linear spaces.

3. FINDINGS

The following theorem is our main result. Our proof is based on the classical (constant) L^p with $0<p<1$ approach.

Theorem 3.1. Let Ω be a measurable subset of $\mathbb{R}^n$. Then the spaces $L^{p(.)}(\Omega)$ for $0< p(x)< 1$ with $p_+ < 1$ admit no continuous linear functionals apart from the zero functional.

Proof. Let F be a nonzero linear functional on $L^{p(.)}(\Omega)$. We will show that F is discontinuous. There must be at least one function $f \in L^{p(.)}(\Omega)$ for which $F(f)=1$. There is a measurable set $A\subset\Omega$ such that

$$\int_{\Omega} |f\chi_A(x)|^{p(x)}dx = \frac{1}{2}\int_{\Omega} |f(x)|^{p(x)}dx$$

Let $g_1 = f\chi_A$ and $g_2 = f\chi_{\Omega-A}$. Therefore, $f = g_1 + g_2$ and

$$\int_{\Omega} |f(x)|^{p(x)}dx = \int_{\Omega} |g_1(x)|^{p(x)}dx + \int_{\Omega} |g_2(x)|^{p(x)}dx$$

Also, we have

$$\int_{\Omega} |g_1(x)|^{p(x)}dx = \int_{\Omega} |g_2(x)|^{p(x)}dx = \frac{1}{2}\int_{\Omega} |f(x)|^{p(x)}dx$$

Since $F(g_1+g_2) = F(g_1)+F(g_2)$ and $|F(g_1+g_2)| \le |F(g_1)|+|F(g_2)|$ then either $|F(g_1)| \ge \frac{1}{2}|F(f)|$ or $|F(g_2)| \ge \frac{1}{2}|F(f)|$. If the first holds true, let $f_1 = 2g_1$, otherwise, let, $f_1 = 2g_2$ (let us assume the first holds true). Then we have $|F(f_1)| \ge |F(f)| = 1$ and

$$\int_{\Omega} |f_1(x)|^{p(x)}dx = \int_{\Omega} |2g_1(x)|^{p(x)}dx$$

$$\leq 2^{p_+} \int_{\Omega} \left| g_1(x) \right|^{p(x)} dx$$

$$= 2^{p_+ - 1} \int_{\Omega} \left| f(x) \right|^{p(x)} dx$$

Now, we apply the same argument to f_1 rather than f. Then we find a function f_2 such that $\left| F(f_2) \right| \geq 1$ and

$$\int_{\Omega} \left| f_2(x) \right|^{p(x)} dx \leq 2^{p_+ - 1} \int_{\Omega} \left| f_1(x) \right|^{p(x)} dx \leq 2^{2(p_+ - 1)} \int_{\Omega} \left| f(x) \right|^{p(x)} dx$$

Continuing this process, we find a sequence of functions $\{ f_n \}$ in $L^{p(.)}(\Omega)$ such that $\left| F(f_n) \right| \geq 1$ and

$$\int_{\Omega} \left| f_n(x) \right|^{p(x)} dx \leq 2^{n(p_+ - 1)} \int_{\Omega} \left| f(x) \right|^{p(x)} dx$$

Since $p_+ < 1$ we have $f_n \to 0$ in $L^{p(.)}(\Omega)$, but $F(f_n)$ does not converge to 0. Hence F is necessarily discontinuous. This completes the proof.

4.DISCUSSION AND CONCLUSION

Finally, we note that for $p_+ = 1$, it remains an open question whether there exists non zero continuous linear functional on $L^{p(.)}(\Omega)$ space for $0 < p(x) < 1$. In particular, for $L^{p(x)}(0,1)$ with $p(x) = x$, do there exist any non zero continuous linear functional?

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