Conference Proceeding Science and Technology, 3(1), 2020, 160–165

*Conference Proceeding of 3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)*

ISSN: 2651-544X
http://dergipark.gov.tr/cpost

# New Answers to the Rhoades' Open Problem and the Fixed-Circle Problem

*Nihal Taş*[1*]

[1] *Balıkesir University, Department of Mathematics, 10145 Balıkesir, Turkey, ORCID: 0000-0002-4535-4019*
** Corresponding Author E-mail: nihaltas@balikesir.edu.tr*

**Abstract:** Recently, the Rhoades' open problem which is related to the discontinuity at fixed point of a self-mapping and the fixed-circle problem which is related to the geometric meaning of the set of fixed points of a self-mapping have been studied using various approaches. Therefore, in this paper, we give some solutions to the Rhoades' open problem and the fixed-circle problem on metric spaces. To do this, we inspire from the Meir-Keeler type, Ciric type and Caristi type fixed-point theorems. Also, we use the simulation functions and Wardowski's technique to obtain new fixed-circle results.

**Keywords:** Fixed circle, Fixed disc, Fixed point, Metric space.

## 1 Introduction

Fixed-point theory, was started with the Banach's contraction principle [1] **,** is very important in the different areas of mathematics such as applied mathematics, topology, analysis etc. This principle has been generalized using the various techniques since there exist some examples of a self-mapping does not satisfy the Banach's contraction principle but has a fixed point. One of these techniques is to generalize the used contractive condition such as Ciric type contractive condition, Meir-Keeler type contractive condition etc. (for example, see [19] and the references therein). Another technique is to generalize the used metric spaces (for example, see [7] and the references therein). Recently, the geometric properties of fixed points have been investigated as a new generalization of the fixed-point theory (see [11]-[12]-[22]).

Also many contractive conditions require that a self-mapping is continuous at fixed point but there are some contractive conditions which do not require that the self-mapping to be continuous. In this context, the following open questions raised by Rhoades [20] and Özgür et al. [12] have been extensively studied, respectively:

Does there exist a contractive condition which is strong enough to generate a fixed point but which does not force the self-mapping to be continuous at the fixed point?

What are the geometric properties of fixed points in which case a self-mapping has more than one fixed point?

Many authors have investigated new solutions using various approaches to above first question. For example, in [14], Pant obtained a solution using the function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for each $t > 0$ and the number defined as

$$m(x, y) = \max\{d(x, Tx), d(y, Ty)\}.$$

**Theorem 1.** *[14] Let $(X, d)$ be a complete metric space and $T : X \to X$ a self-mapping such that*
*$(i)$ $d(Tx, Ty) \leq \phi(m(x, y))$,*
*$(ii)$ Given $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\varepsilon < m(x, y) < \varepsilon + \delta \Longrightarrow d(Tx, Ty) \leq \varepsilon.$$

*Then $T$ has a unique fixed point $z$ in $X$ and $T$ is continuous at $z$ if and only if $\lim\limits_{x\to z} m(x, z) = 0$.*

After then, various solutions to first question have been given (see [2]-[4]-[11]**-** [16]- [17]**-**[18]**-**[21] for more details). The obtained results are gained great importance since there exist some applications of the obtained solutions to some other areas such as discontinuous neural networks, simulation functions, biology etc. (see [6]-[11]**-**[16]- [17]**-**[18]**-**[21]**-**[24]).

More recently, the fixed-circle (or fixed-disc) problem has been studied using some classical fixed-point techniques related to second question. For example, Özgür and Taş obtained a solution to this question using Caristi's inequality (see [3]) as follows:

**Theorem 2.** *[12] Let $(X, d)$ be a metric space and $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ any circle on $X$. Let us define the mapping $\varphi : X \to [0, \infty)$ such that $\varphi(x) = d(x, x_0)$ for all $x \in X$. If there exists a self-mapping $T : X \to X$ satisfying*
*$(C1)$ $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$,*
*$(C2)$ $d(Tx, x_0) \geq r$ for each $x \in C_{x_0,r}$,*
*then the circle $C_{x_0,r}$ is a fixed circle of $f$.*

Motivated by this fact, the fixed-circle problem has brought a new light to the fixed-point theory and geometric thinking since this problem has been studied as a geometric approach to the generalization of fixed-point theory. For this purpose, some known techniques used in fixed-point theorems are adapted to the fixed-circle problem on metric spaces and some generalized metric spaces (see [11]-[12]-[13]-[16]-[17]-[22]-[23]).

In this paper, we give some solutions to the Rhoades' open problem using the Meir-Keeler type, Ciric type and Caristi type fixed-point theorems with a different approach similar to the given approaches in [3]-[5]-[8]- [10]. We investigate some answers to the fixed-circle problem using the Meir-Keeler type, Ciric type and Caristi type techniques with three approaches. Finally, we give some consequences of our obtained theorical results.

## 2 Main Results

In this section, we give new solutions to the Rhoades' open problem and the fixed-circle problem on metric spaces using the Meir-Keeler type, Ciric type and Caristi type techniques.

### 2.1 New discontinuity results on metric spaces

Throughout this paper, we assume that $(X,d)$ is a complete metric space, $T : X \to X$ a self-mapping and $\varphi : X \to [0,\infty)$ a lower semicontinuous and bounded below function. In this section, we use the following number

$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(y,Tx), d(x,Ty)\}.$$

**Theorem 3.** *If the following condition holds for all $x,y \in X$*
$(MKCC)$ *Given $\varepsilon > 0$ there exists a $\delta > 0$ such that $d(x,Tx) > 0$ implies*

$$\varepsilon \leq [\varphi(x) - \varphi(Tx)] N(x,y) < \varepsilon + \delta \Longrightarrow d(Tx,Ty) < \varepsilon,$$

*then given $x \in X$, the sequence of iterates $\{T^n x\}$ is a Cauchy sequence and $\lim_{n\to\infty} T^n x = z$ for some $z \in X$.*

*Proof:* Using the condition $(MKCC)$, we obtain that if $d(x,Tx) > 0$ then

$$d(Tx,Ty) < [\varphi(x) - \varphi(Tx)] N(x,y). \tag{1}$$

Let $x_0 \in X$ and let us define a sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1}$,that is, $x_n = T^n x_0$. If $x_n = x_{n+1}$ for some $n$ then $x_n = x_{n+1} = x_{n+2} = ...$, that is, $\{x_n\} = \{T^n x\}$ is a Cauchy sequence and $x_n$ is a fixed point of $T$. Thus, without loss of generality, suppose that $x_n \neq x_{n+1}$ for each $n$ and $c_n = d(x_{n-1}, x_n)$. Using the inequality (1), we get

$$\begin{aligned} c_{n+1} &= d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) < [\varphi(x_{n-1}) - \varphi(x_n)] N(x_{n-1}, x_n) \\ &= [\varphi(x_{n-1}) - \varphi(x_n)] \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1})\} \\ &= [\varphi(x_{n-1}) - \varphi(x_n)] \alpha_N. \end{aligned} \tag{2}$$

**Case 1**: If $\alpha_N = d(x_{n-1}, x_n)$, then using the inequality (2), we have

$$c_{n+1} = d(x_n, x_{n+1}) < [\varphi(x_{n-1}) - \varphi(x_n)] c_n$$

and so

$$0 < \frac{c_{n+1}}{c_n} < \varphi(x_{n-1}) - \varphi(x_n),$$

for each $n \in \mathbb{N}$. Therefore the sequence $\{\varphi(x_n)\}$ is nonincreasing and positive whence it converges to some $t \geq 0$. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{m=1}^{n} \frac{c_{m+1}}{c_m} &< \sum_{m=1}^{n} [\varphi(x_{m-1}) - \varphi(x_m)] = \varphi(x_0) - \varphi(x_n) \\ &\to \varphi(x_0) - t < \infty \text{ as } n \to \infty \end{aligned}$$

and

$$\sum_{m=1}^{n} \frac{c_{m+1}}{c_m} < \infty \Longrightarrow \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 0.$$

Hence for $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{c_{n+1}}{c_n} \leq \alpha \text{ for all } n \geq n_0$$

and we have

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \text{ for all } n \geq n_0.$$

**Case 2**: If $\alpha_N = d(x_n, x_{n+1})$, then using the inequality (2), we have

$$d(x_n, x_{n+1}) < [\varphi(x_{n-1}) - \varphi(x_n)] d(x_n, x_{n+1}).$$

By the similar approach used in Case 1, we say that $\{\varphi(x_n)\}$ is a positive and nonincreasing sequence and so it converges to some $t \geq 0$. Since $d(x_n, x_{n+1}) > 0$, we get

$$1 < \varphi(x_{n-1}) - \varphi(x_n) \to 0 \text{ as } n \to \infty,$$

a contradiction.

**Case 3**: If $\alpha_N = d(x_{n-1}, x_{n+1})$, then using the inequality (2), we have

$$\begin{aligned} c_{n+1} &= d(x_n, x_{n+1}) < [\varphi(x_{n-1}) - \varphi(x_n)]\, d(x_{n-1}, x_{n+1}) \\ &\leq [\varphi(x_{n-1}) - \varphi(x_n)]\, (d(x_{n-1}, x_n) + d(x_n, x_{n+1})) \\ &= [\varphi(x_{n-1}) - \varphi(x_n)]\, (c_n + c_{n+1}) \end{aligned}$$

and so

$$0 < \frac{c_{n+1}}{c_n + c_{n+1}} < \varphi(x_{n-1}) - \varphi(x_n).$$

The sequence $\{\varphi(x_n)\}$ converges to some $t \geq 0$ since it is a positive and nonincreasing sequence. For each $n \in \mathbb{N}$, we obtain

$$\begin{aligned} \sum_{m=1}^{n} \frac{c_{m+1}}{c_m + c_{m+1}} &< \sum_{m=1}^{n} [\varphi(x_{m-1}) - \varphi(x_m)] = \varphi(x_0) - \varphi(x_n) \\ &\to \varphi(x_0) - t < \infty \text{ as } n \to \infty \end{aligned}$$

and

$$\sum_{m=1}^{n} \frac{c_{m+1}}{c_m + c_{m+1}} < \infty \Longrightarrow \lim_{n\to\infty} \frac{c_{n+1}}{c_n + c_{n+1}} = 0.$$

Hence for $\alpha' \in \left(0, \frac{1}{2}\right)$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{c_{n+1}}{c_n + c_{n+1}} \leq \alpha' \text{ for all } n \geq n_0$$

and we have

$$d(x_n, x_{n+1}) \leq \alpha d(x_{n-1}, x_n) \text{ for all } n \geq n_0,$$

where $\alpha = \frac{\alpha'}{1-\alpha'}$.

Now we show that $\{x_n\}$ is a Cauchy sequence and $\{x_n\}$ converges to some $a \in X$. Under the above cases, we say that the sequence $\{d(x_n, x_{n+1})\}$ is bounded below and nonincreasing. Therefore, it converges to some $x \geq 0$. Since $\alpha < 1$, we easily prove $x = 0$. For each $n_1, n_2 \in \mathbb{N}$ $(n_1 > n_2)$, we get

$$d(x_{n_1}, x_{n_2}) \leq \sum_{m=n_2}^{n_1-1} d(x_m, x_{m+1}) \leq \frac{\alpha^{n_2}}{1-\alpha} d(x_0, x_1),$$

that is,

$$\lim_{n\to\infty} \sup \{d(x_{n_1}, x_{n_2}) : n_1 > n_2\} = 0.$$

Consequently, the sequence $\{x_n\}$ is Cauchy and there exists $a \in X$ such that $\{x_n\} \to a$ since $(X, d)$ is a complete metric space. □

A self-mapping $T$ of a metric space $X$ is called $k$-continuous, $k = 1, 2, 3, ...$, if $T^k x_n \to Tt$ whenever $\{x_n\}$ is a sequence in $X$ such that $T^{k-1} x_n \to t$ (see [15] for more details).

**Theorem 4.** *Let $T$ satisfies the condition $(MKCC)$. If $T$ is k-continuous then $T$ has a fixed point $z$. Also, $T$ is continuous at $z$ if and only if*

$$\lim_{x\to z} [\varphi(x) - \varphi(Tx)]\, N(x, z) = 0.$$

*Proof:* Let $x_0 \in X$ and let us define a sequence $\{x_n\}$ in $X$ by $x_n = Tx_{n-1}$, that is, $x_n = T^n x_0$. Using Theorem 3, we say that $\{x_n\}$ is a Cauchy sequence. Hence there exists a point $a \in X$ such that $\{x_n\} \to a$ since $(X, d)$ is a complete metric space. Also we have $T^p x_n \to a$ for each $p \geq 1$.

Let $T$ be a $k$-continuous self-mapping. $k$-continuity of $T$ implies that $T^k x_n \to Ta$ since $T^{k-1} x_n \to a$ and so we get $Ta = a$ as $T^k x_n \to a$. Therefore, $a$ is a fixed point of $T$. It is also easy to prove that $T$ is continuous at $a$ if and only if

$$\lim_{x\to a} [\varphi(x) - \varphi(Tx)]\, N(x, a) = 0.$$

□

### 2.2 *Some fixed-circle results on metric spaces*

In this section, let the number $r$ be defined as

$$r = \inf \{d(x, Tx) : x \neq Tx, x \in X\} \tag{3}$$

and the function $\varphi : X \to [0, \infty)$ defined as

$$\varphi(x) = d(x, Tx), \tag{4}$$

for all $x \in X$.

In the following theorem, we inspire from the Meir-Keeler type, Ciric type and Caristi type fixed-point theorems to obtain a new fixed-circle theorem.

At first, we recall the notions of a fixed circle and a fixed disc.

Let $(X, d)$ be a metric space, $C_{x_0,r} = \{x \in X : d(x, x_0) = r\}$ a circle and $T : X \to X$ a self-mapping. If $Tx = x$ for every $x \in C_{x_0,r}$ then $C_{x_0,r}$ is called as the fixed circle of $T$ [12].

If $Tx = x$ for every $x \in D_{x_0,r} = \{x \in X : d(x, x_0) \leq r\}$ then $D_{x_0,r}$ is called as the fixed disc of $T$ (see [13] and the references therein).

**Theorem 5.** *Let $(X, d)$ be a metric space, $T : X \to X$ a self-mapping, $r$ defined as in (3) and $\varphi$ defined as in (4). If there exists $x_0 \in X$ such that*

1. $d(x_0, Tx) \leq r$ *and* $0 \leq \varphi(x) \leq 1$ *for all* $x \in C_{x_0,r}$,
2. *For all* $x \in X$,
$$\varphi(x) > 0 \Longrightarrow \varphi(x) < [\varphi(x) - \varphi(x_0)]\, N(x, x_0),$$

*then $Tx_0 = x_0$ and the circle $C_{x_0,r}$ is a fixed circle of $T$.*

*Proof:* Let $r = 0$. Then we have $C_{x_0,r} = \{x_0\}$. On the contrary, we assume that $\varphi(x_0) > 0$. Using the condition (2), we get
$$\varphi(x_0) = d(x_0, Tx_0) < [\varphi(x_0) - \varphi(x_0)]\, N(x_0, x_0) = 0,$$
a contradiction. Thus, it should be $\varphi(x_0) = 0$, that is,
$$Tx_0 = x_0. \tag{5}$$
Let $r > 0$ and $x \in C_{x_0,r}$. Now we show that $T$ fixes the circle $C_{x_0,r}$. To do this, we suppose that $\varphi(x) > 0$. Again, using the conditions (1), (2) and the equality (5), we obtain
$$\begin{aligned} \varphi(x) &= d(x, Tx) < [\varphi(x) - \varphi(x_0)]\, N(x, x_0) \\ &= d(x, Tx) \max\{r, d(x, Tx), 0, d(x_0, Tx)\} \\ &= d(x, Tx) \max\{d(x, Tx), d(x_0, Tx)\} = d(x, Tx)d(x, Tx) \end{aligned}$$
and so
$$d(x, Tx) < [d(x, Tx)]^2,$$
a contradiction. Hence it should be $x = Tx$. Consequently, $T$ fixes the circle $C_{x_0,r}$. □

To obtain another fixed-circle theorem, we use the family of simulation functions. Hence, we recall the definition of a simulation function. The function $\zeta : [0, \infty)^2 \to \mathbb{R}$ is said to be a simulation function, if the followings hold:
$(\zeta_1)$ $\zeta(0, 0) = 0$,
$(\zeta_2)$ $\zeta(t, s) < s - t$ for all $s, t > 0$,
$(\zeta_3)$ If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that
$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$
then
$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$
The set of all simulation functions is denoted by $\mathcal{Z}$ [9].

**Theorem 6.** *Let $(X, d)$ be a metric space, $T : X \to X$ a self-mapping, $r$ defined as in (3) and $\varphi$ defined as in (4). If there exists $x_0 \in X$ and $\zeta \in \mathcal{Z}$ such that*

1. $\varphi(x_0) = 0$,
2. $d(x_0, Tx) \leq r$ *and* $0 \leq \varphi(x) \leq 1$ *for all* $x \in C_{x_0,r}$,
3. *For all* $x \in X$,
$$\varphi(x) > 0 \Longrightarrow \zeta\left(\varphi(x), \varphi(x)N(x, x_0)\right) \geq 0,$$

*then the circle $C_{x_0,r}$ is a fixed circle of $T$.*

*Proof:* Let $r = 0$. Then we have $C_{x_0,r} = \{x_0\}$. By the condition (1), we know $\varphi(x_0) = 0$, that is, $d(x_0, Tx_0) = 0$. Hence we get $x_0 = Tx_0$. Now we assume that $r > 0$ and $x \in C_{x_0,r}$ be any point such that $x \neq Tx$, that is, $\varphi(x) = d(x, Tx) > 0$. Using the conditions (1), (2), (3) and $(\zeta_2)$, we obtain
$$\begin{aligned} 0 &\leq \zeta\left(\varphi(x), \varphi(x)N(x, x_0)\right) < \varphi(x)N(x, x_0) - \varphi(x) \\ &= d(x, Tx)\, [d(x, Tx) - 1] \end{aligned}$$
and so
$$1 \leq d(x, Tx),$$
a contradiction. Therefore, it should be $x = Tx$, that is, $T$ fixes the circle $C_{x_0,r}$. □

We obtain a new fixed-circle result using a different approach. At first, we recall the definition of the following family of functions which was introduced by Wardowski in [25].

**Definition 7.** *[25] Let $\mathbb{F}$ be the family of all functions $F:(0,\infty)\to\mathbb{R}$ such that*
$(F_1)$ *$F$ is strictly increasing,*
$(F_2)$ *For each sequence $\{\alpha_n\}$ in $(0,\infty)$ the following holds*

$$\lim_{n\to\infty}\alpha_n=0 \text{ if and only if } \lim_{n\to\infty}F(\alpha_n)=-\infty,$$

$(F_3)$ *There exists $k\in(0,1)$ such that $\lim\limits_{\alpha\to 0^+}\alpha^k F(\alpha)=0$.*

Some examples of functions that satisfies the conditions $(F_1)$, $(F_2)$ and $(F_3)$ of before definition are $F(x)=\ln(x)$, $F(x)=\ln(x)+x$, $F(x)=-\frac{1}{\sqrt{x}}$ and $F(x)=\ln(x^2+x)$ (see [25] for more details).

**Theorem 8.** *Let $(X,d)$ be a metric space, $T:X\to X$ a self-mapping, $r$ defined as in $(3)$ and $\varphi$ defined as in $(4)$. If there exists $x_0\in X$, $t>0$ and $F\in\mathbb{F}$ such that*

1. *$d(x_0,Tx)\le r$ and $0\le\varphi(x)\le 1$ for all $x\in C_{x_0,r}$,*
2. *For all $x\in X$,*

$$\varphi(x)>0\Longrightarrow t+F(\varphi(x))\le F\left(\left[\varphi(x)-\varphi(x_0)\right]N(x,x_0)\right),$$

*then $Tx_0=x_0$ and the circle $C_{x_0,r}$ is a fixed circle of $T$.*

*Proof:* Let $r=0$. Then we have $C_{x_0,r}=\{x_0\}$. As an immediate consequence of the condition (2), we get $x_0=Tx_0$. Now suppose that $r>0$ and $x\in C_{x_0,r}$ be any point $x\ne Tx$. Then using the conditions (1), (2) and the strictly increasing property of $F$, we find

$$\begin{aligned} t+F(\varphi(x)) &= t+F(d(x,Tx))\le F\left(\left[\varphi(x)-\varphi(x_0)\right]N(x,x_0)\right)\\ &= F\left(d(x,Tx)\max\{r,d(x,Tx),d(x_0,Tx)\}\right)\\ &= F\left(\left[d(x,Tx)\right]^2\right)\le F\left(d(x,Tx)\right),\end{aligned}$$

a contradiction. Hence it should be $Tx=x$. Consequently, the circle $C_{x_0,r}$ is a fixed circle of $T$. □

### 2.3 Consequences

If we consider Theorem 3, we obtain the following corollary.

**Corollary 1.** *If the following condition holds for all $x,y\in X$*
$(i)$ *Given $\varepsilon>0$ there exists a $\delta>0$ such that $d(x,Tx)>0$ implies*

$$\varepsilon\le\left[\varphi(x)-\varphi(Tx)\right]d(x,y)<\varepsilon+\delta\Longrightarrow d(Tx,Ty)<\varepsilon,$$

*then given $x\in X$, the sequence of iterates $\{T^n x\}$ is a Cauchy sequence and $\lim\limits_{n\to\infty}T^n x=z$ for some $z\in X$. If $T$ is k-continuous then $T$ has a fixed point $z$.*

On the other hand, Theorem 5, Theorem 6 and Theorem 8 are considered as new fixed-disc results as seen in the following corollaries:

**Corollary 2.** *Let $(X,d)$ be a metric space, $T:X\to X$ a self-mapping, $r$ defined as in $(3)$ and $\varphi$ defined as in $(4)$. If there exists $x_0\in X$ such that*

1. *$d(x_0,Tx)\le r$ and $0\le\varphi(x)\le 1$ for all $x\in D_{x_0,r}$,*
2. *For all $x\in X$,*

$$\varphi(x)>0\Longrightarrow\varphi(x)<\left[\varphi(x)-\varphi(x_0)\right]N(x,x_0),$$

*then $Tx_0=x_0$ and the circle $D_{x_0,r}$ is a fixed disc of $T$.*

**Corollary 3.** *Let $(X,d)$ be a metric space, $T:X\to X$ a self-mapping, $r$ defined as in $(3)$ and $\varphi$ defined as in $(4)$. If there exists $x_0\in X$ and $\zeta\in\mathcal{Z}$ such that*

1. *$\varphi(x_0)=0$,*
2. *$d(x_0,Tx)\le r$ and $0\le\varphi(x)\le 1$ for all $x\in D_{x_0,r}$,*
3. *For all $x\in X$,*

$$\varphi(x)>0\Longrightarrow\zeta\left(\varphi(x),\varphi(x)N(x,x_0)\right)\ge 0,$$

*then the circle $D_{x_0,r}$ is a fixed disc of $T$.*

**Corollary 4.** *Let $(X,d)$ be a metric space, $T:X\to X$ a self-mapping, $r$ defined as in $(3)$ and $\varphi$ defined as in $(4)$. If there exists $x_0\in X$, $t>0$ and $F\in\mathbb{F}$ such that*

1. $d(x_0, Tx) \leq r$ *and* $0 \leq \varphi(x) \leq 1$ *for all* $x \in D_{x_0,r}$,
2. *For all* $x \in X$,

$$\varphi(x) > 0 \Longrightarrow t + F(\varphi(x)) \leq F\left([\varphi(x) - \varphi(x_0)]\, N(x, x_0)\right),$$

*then* $Tx_0 = x_0$ *and the circle* $D_{x_0,r}$ *is a fixed disc of* $T$.

In Theorem 6 and Corollary 4, we can use the following simulation functions given in [9] to obtain new fixed-circle and fixed-disc results:

1. $\zeta_1(t,s) = \lambda s - t$,
2. $\zeta_2(t,s) = s - \varphi(s) - t$,
3. $\zeta_3(t,s) = s\varphi(s) - t$,
4. $\zeta_4(t,s) = \eta(s) - t$,
5. $\zeta_5(t,s) = s - \int\limits_0^t \phi(u)du$.

## *3* References


1 S. Banach, *Sur les operations dans les ensembles abstrait et leur application aux equations integrals*, Fundam. Math. **3** (1922), 133-181.
2 R.K. Bisht, R. P. Pant, *A remark on discontinuity at fixed point*, J. Math. Anal. Appl. **445**(2) (2017) 1239-1242.
3 J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Am. Math. Soc. **215** (1976), 241-251.
4 U. Çelik, N. Özgür, *A new solution to the discontinuity problem on metric spaces*, Turk. J. Math. **44** (2020), 1115-1126.
5 Lj. B. Ciric, *A generalization of Banach's contraction principle*, Proc. Am. Math. Soc. **45**(2) (1974), 267-273.
6 L. J. Cromme, I. Diener, *Fixed point theorems for discontinuous mapping*, Math. Program. **51**(1-3) (1991), 257-267.
7 T. Dosenovic, S. Radenovic, S. Sedghi, *Generalized metric spaces: survey*, TWMS J. Pure Appl. Math. **9**(1) (2018), 3-17.
8 E. Karapınar, F. Khojasteh, W. Shatanawia, *Revisiting Ciric-Type Contraction with Caristi's Approach*, Symmetry **11** (2019), 726.
9 F. Khojasteh, S. Shukla, S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat **29**(6) (2015), 1189-1194.
10 A. Meir, E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl. **28** (1969), 326-329.
11 N. Y. Özgür, N. Taş, *Some fixed-circle theorems and discontinuity at fixed circle*, AIP Conference Proceedings **1926** (2018), 020048.
12 N. Y. Özgür, N. Taş, *Some fixed-circle theorems on metric spaces*, Bull. Malays. Math. Sci. Soc. **42**(4) (2019), 1433-1449.
13 N. Y. Özgür, *Fixed-disc results via simulation functions*, Turk. J. Math. **43**(6) (2019), 2794-2805.
14 R. P. Pant, *Discontinuity and fixed points*, J. Math. Anal. Appl. **240**(1) (1999), 284-289.
15 A. Pant, R. P. Pant, *Fixed points and continuity of contractive maps*, Filomat **31**(11) (2017), 3501-3506.
16 R. P. Pant, N. Y. Özgür, N. Taş, *Discontinuity at fixed points with applications*, Bull. Belg. Math. Soc. - Simon Stevin **26** (2019), 571-589.
17 R. P. Pant, N. Y. Özgür, N. Taş, *On discontinuity problem at fixed point*, Bull. Malays. Math. Sci. Soc. **43**(1) (2020), 499-517.
18 M. Rashid, I. Batool, N. Mehmood, *Discontinuous mappings at their fixed points and common fixed points with applications*, J. Math. Anal. **9**(1) (2018), 90-104.
19 B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Am. Math. Soc. **226** (1977), 257-290.
20 B. E. Rhoades, *Contractive definitions and continuity*, Contemp. Math. **72** (1988), 233-245.
21 N. Taş, N. Y. Özgür, *A new contribution to discontinuity at fixed point*, Fixed Point Theory **20**(2) (2019), 715-728.
22 N. Taş, N. Y. Özgür, N. Mlaiki, *New types of $F_C$-contractions and the fixed-circle problem*, Mathematics **6**(10) (2018), 188.
23 N. Taş, *Bilateral-type solutions to the fixed-circle problem with rectified linear units application*, Turk. J. Math. **44** (2020), 1330-1344.
24 M. J. Todd, *The computation of fixed points and applications*, Berlin, Heidelberg, New York: Springer-Verlag, 1976.
25 D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. **2012** (2012), 94.