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Some Properties of Rough Statistical Convergence in 2-Normed Spaces

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Abstract: In this study, we introduce the concepts of rough statistical cluster point and rough statistical limit point of a sequence in 2-normed space and investigate some properties of these concepts.

Keywords: Rough convergence, 2-normed space, Rough cluster point, Rough limit point, Statistical convergence.

1 Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [14] and Schoenberg [35].

The concept of 2-normed spaces was initially introduced by Gähler [18, 19] in the 1960's. Since then, this concept has been studied by many authors. Gürdal and Pehlivan [23] studied statistical convergence, statistical Cauchy sequence and investigated some properties of statistical convergence in 2-normed spaces. Gürdal and Açık [24] investigated $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences in 2-normed spaces. Sarabadan and Talebi [33] studied statistical convergence and ideal convergence of sequences of functions in 2-normed spaces. Arslan and Dündar [1, 2] investigated the concepts of $\mathcal{I}$-convergence, $\mathcal{I}^*$-convergence, $\mathcal{I}$-Cauchy and $\mathcal{I}^*$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [9, 25, 28, 32, 34, 36, 38–40]).

The idea of rough convergence was first introduced by Phu [29] in finite-dimensional normed spaces. In [29], he showed that the set $\text{LIM}^r x_i$ is bounded, closed, and convex; and he introduced the notion of rough Cauchy sequence. He also investigated the relations between rough convergence and other convergence types and the dependence of $\text{LIM}^r x_i$ on the roughness degree r. In another paper [30] related to this subject, he defined the rough continuity of linear operators and showed that every linear operator $f : X \to Y$ is r -continuous at every point $x \in X$ under the assumption $dimY < \infty$ and $r > 0$ where X and Y are normed spaces. In [31], he extended the results given in [29] to infinite-dimensional normed spaces. Aytar [6] studied of rough statistical convergence and defined the set of rough statistical limit points of a sequence and obtained two statistical convergence criteria associated with this set and prove that this set is closed and convex. Also, Aytar [7] studied that the r-limit set of the sequence is equal to the intersection of these sets and that r-core of the sequence is equal to the union of these sets. Recently, Dündar and Çakan [11, 12] introduced the notion of rough $\mathcal{I}$-convergence and the set of rough $\mathcal{I}$-limit points of a sequence and Dündar [13] studied the notions of rough convergence, $\mathcal{I}_2$-convergence and the sets of rough limit points and rough $\mathcal{I}_2$-limit points of a double sequence. Arslan and Dündar [3, 4] introduced some concepts of rough convergence in 2-normed spaces. Also, Arslan and Dündar [5] studied rough statistical convergence in 2-normed spaces.

In this paper, we studied the concepts of rough statistical cluster point and rough statistical limit point of a sequence in 2-normed space and investigate some properties of these concepts. Also, we obtain an ordinary statistical convergence criteria associated with rough statistical cluster point of a sequence in 2-normed space. We note that our results and proof techniques presented in this paper are analogues of those in Aytar's [8] paper. Namely, the actual origin of most of these results and proof techniques is them papers. The following our theorems and results are the extension of theorems and results in [8].

Now, we recall the concept of 2-normed space, rough convergence and some fundamental definitions and notations (See [1–8, 10, 15–17, 20–27, 29–33, 37–40]).

Let r be a nonnegative real number and $\mathbb{R}^n$ denotes the real n-dimensional space with the norm $\|.\|$. Consider a sequence $x = (x_n) \subset \mathbb{R}^n$. The sequence $x = (x_n)$ is said to be r-convergent to L, denoted by $x_n \xrightarrow{r} L$ provided that

$$\forall \varepsilon > 0,\ \exists n_\varepsilon \in \mathbb{N} :\ n \geq n_\varepsilon \Rightarrow \|x_n - L\| < r + \varepsilon.$$

The set

$$\text{LIM}^r x := \{L \in \mathbb{R}^n : x_n \xrightarrow{r} L\}$$

is called the r-limit set of the sequence $x = (x_n)$. A sequence $x = (x_n)$ is said to be r-convergent if $\text{LIM}^r x \neq \emptyset$. In this case, r is called the convergence degree of the sequence $x = (x_n)$. For $r = 0$, we get the ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}$, and let us denote the set $\{k \in K : k \leq n\}$ by K_n. Then the natural density of K is given by

$$\delta(K) := \lim_{n\to\infty} \frac{|K_n|}{n},$$

where $|K_n|$ denotes the number of elements in K_n. Clearly, a finite subset has natural density zero and we have $\delta(K^c) = 1 - \delta(K)$, where $K^c := \mathbb{N} \setminus K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

A sequence $x = (x_n)$ is said to be r-statistically convergent to L, denoted by $x_n \stackrel{r-st}{\longrightarrow} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L\| \geq r + \varepsilon\}$ has natural density zero for $\varepsilon > 0$; or equivalently, if the condition $st - \limsup \|x_n - L\| \leq r$ is satisfied. In addition, we can write $x_n \stackrel{r-st}{\longrightarrow} L$ if and only if, the inequality $\|x_n - L\| < r + \varepsilon$ holds for every $\varepsilon > 0$ and almost all n.

Here r is called the statistical convergence degree. If we take $r = 0$, then we obtain the ordinary statistical convergence.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for roughness degree $r > 0$. So we have to consider the so-called r-statistical limit set of the sequence x, which is defined by

$$st - \text{LIM}^r x := \{L \in X : x_n \stackrel{r-st}{\longrightarrow} L\}.$$

The sequence x is said to be r-statistically convergent provided that $st - \text{LIM}^r x \neq \emptyset$.

Let $r \geq 0$. The vector $\lambda \in X$ is called the r-statistical cluster point of the sequence $x = (x_n)$ provided that

$$\delta(\{n \in \mathbb{N} : \|x_n - \lambda\| < r + \varepsilon\}) \neq 0,$$

for every $\varepsilon > 0$. We denote the set of all r-statistically cluster points the sequence x by Γ_x^r.

Let $r \geq 0$. The vector $\nu \in X$ is called the r-statistical limit point of the sequence $x = (x_n)$, provided that there is a nonthin subsequence (x_{n_k}) of (x_n) such that for every $\varepsilon > 0$ there exists a number $k_0 = k_{0(\varepsilon)} \in \mathbb{N}$ with $\|x_{n_k} - \nu\| < r + \varepsilon$ for all $k \geq k_0$. We denote the set of all r-statistical limit points the sequence x by Λ_x^r.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x, y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x, y\| = |x_1 y_2 - x_2 y_1|; \quad x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$. The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space.

A sequence $x = (x_n)$ in 2-normed space $(X, \|\cdot,\cdot\|)$ is said to be convergent to L in X if $\lim_{n\to\infty} \|x_n - L, z\| = 0$, for every $z \in X$. In such a case, we write $\lim_{n\to\infty} x_n = L$ and call L the limit of (x_n).

Let (x_n) be a sequence in $(X, \|.,.\|)$ 2-normed linear space and r be a non-negative real number. $x = (x_n)$ is said to be rough convergent (r-convergent) to L denoted by $x_n \stackrel{\|.,.\|}{\longrightarrow}_r L$ if

$$\forall \varepsilon > 0, \exists n_\varepsilon \in \mathbb{N} : n \geq n_\varepsilon \Rightarrow \|x_n - L, z\| < r + \varepsilon \tag{1}$$

or equivalently, if

$$\limsup \|x_n - L, z\| \leq r, \tag{2}$$

for every $z \in X$.

If (1) holds, L is an r-limit point of (x_n), which is usually no more unique (for $r > 0$). So, we have to consider the so-called r-limit set (or shortly r-limit) of (x_n) defined by

$$\text{LIM}_2^r x := \{L \in X : x_n \stackrel{\|.,.\|}{\longrightarrow}_r L\}. \tag{3}$$

The sequence (x_n) is said to be rough convergent if $\text{LIM}_2^r x \neq \emptyset$. In this case, r is called a convergence degree of (x_n). For $r = 0$ we have the classical convergence in 2-normed space again.

Let $(X, \|.,.\|)$ be a 2-normed space. A sequence $x = (x_n)$ in X said to be rough statistically convergent ($r_2 st$-convergent) to L, denoted by $x_n \stackrel{\|.,.\|}{\longrightarrow}_{r_2 st} L$, provided that the set $\{n \in \mathbb{N} : \|x_n - L, z\| \geq r + \varepsilon\}$ has natural density zero, for every $\varepsilon > 0$ and each nonzero $z \in X$; or equivalently, if the condition $st - \limsup \|x_n - L, z\| \leq r$ is satisfied. In addition, we can write $x_n \stackrel{\|.,.\|}{\longrightarrow}_{r_2 st} L$, if and only if, the inequality $\|x_n - L, z\| < r + \varepsilon$ holds almost all n.

In this convergence, r is called the statistical convergence degree. For $r = 0$, rough statistically convergence coincides with ordinary statistical convergence.

In general, the rough statistical limit of a sequence $x = (x_n)$ may not be unique for the roughness degree $r > 0$. So, we have to consider the so-called r-statistically limit set of the sequence x in X, which is defined by

$$st - \text{LIM}_2^r x := \{L \in X : x_n \stackrel{\|.,.\|}{\longrightarrow}_{r_2 st} L\}. \tag{4}$$

The sequence x is said to be r-statistically convergent provided that $st - \text{LIM}_2^r x \neq \emptyset$.

1.1 Main Results

In this section, we introduce the concept of rough statistical cluster points and rough statistical limit points of a sequence in 2-normed spaces. Also, we show that $r-\Gamma_x^2$ is closed and also,

$$r-\Lambda_x^2 \subseteq r-\Gamma_x^2$$

for a sequence$x=(x_n)$.

Definition 1. *Let $r \geq 0$. The vector $\lambda \in X$ is called the rough statistical cluster point of the sequence $x=(x_n)$ if for every $\varepsilon > 0$ and $z \in X$*

$$\delta(\{n \in \mathbb{N} : \|x_n - \lambda, z\| < r + \varepsilon\}) \neq 0.$$

We denote the set of all rough statistical cluster points of the sequence x in 2-normed space X by

$$r-\Gamma_x^2.$$

Here, if we take $r=0$, then we obtain the notion of ordinary statistical cluster point. It is clear that

$$r_1-\Gamma_x^2 \subseteq r_2-\Gamma_x^2,$$

for $r_1 \leq r_2$.

Aytar proved that the set Γ_x^r is closed. We will show that the set r-Γ_x^2 is closed, for each $r > 0$.

Theorem 1. *Let $x=(x_n)$ be a sequence in 2-normed space X. Then, for every $r \geq 0$, the set r-Γ_x^2 is closed.*

Proof: Let

$$r-\Gamma_x^2 \neq \emptyset$$

and consider a sequence

$$y=(y_n) \subseteq r-\Gamma_x^2$$

such that

$$\lim_{n\to\infty} y_n = L.$$

Let us show that

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}) \neq 0$$

for every $\varepsilon > 0$ and $z \in X$.

Fix $\varepsilon > 0$. Since

$$\lim_{n\to\infty} y_n = L,$$

there exists an $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\|y_n - L, z\| < \frac{\varepsilon}{2},$$

for all $n > n_0$ and every $z \in X$. Fix m_0 such that $m_0 > n_0$. Then, we have

$$\|y_{m_0} - L, z\| < \frac{\varepsilon}{2},$$

for every $z \in X$. Let m be any point of the set

$$\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\}.$$

Since

$$\|x_m - y_{m_0}, z\| < r + \frac{\varepsilon}{2},$$

we have

$$\begin{aligned} \|x_m - L, z\| &\leq \|x_m - y_{m_0}, z\| + \|y_{m_0} - L, z\| \\ &< r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= r + \varepsilon \end{aligned}$$

and so,

$$m \in \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\},$$

for every $z \in X$. Hence, we have

$$\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\} \subseteq \{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}. \tag{5}$$

Since

$$\delta\left(\left\{n \in \mathbb{N} : \|x_n - y_{m_0}, z\| < r + \frac{\varepsilon}{2}\right\}\right) \neq 0$$

by (5), we get

$$\delta(\{n \in \mathbb{N} : \|x_n - L, z\| < r + \varepsilon\}) \neq 0,$$

for every $z \in X$. Therefore, we have

$$L \in r - \Gamma_x^2.$$

□

We note that if we let

$$\lambda \in r - \Gamma_x^2,$$

then for every $z \in X$,

$$\delta(\{n : \|x_n - \lambda, z\| < r + \varepsilon\}) \neq 0.$$

By the statistical analogue of Bolzano-Weierstrass Theorem (see [37], Theorem 2), the subsequence $(x_n)_{n \in A}$ has a statistical cluster point, where

$$A = \{n : \|x_n - \lambda, z\| \leq r\},$$

for every $z \in X$. If we denote this statistical cluster point by γ then we have

$$\|\lambda - \gamma, z\| \leq r.$$

Therefore, we have that if

$$\lambda \in r - \Gamma_x^2,$$

then there exists a vector

$$\gamma \in \Gamma_x^2$$

such that

$$\|\lambda - \gamma, z\| \leq r.$$

We know that the sequence $x = (x_n)$ need not be statistically convergent in order that the inclusion

$$r - \Gamma_x^2 \subseteq st - LIM_2^r x$$

holds, but this sequence must be statistically convergent in order that the converse inclusion holds.

Definition 2. *Let $r \geq 0$. The vector γ in 2-normed space X is called the rough statistical limit point of the sequence $x = (x_n)$ in X, provided that there is a nonthin subsequence (x_{n_k}) of (x_n) such that for every $\varepsilon > 0$ there exists a number $k_0 = k_0(\varepsilon) \in \mathbb{N}$ with*

$$\|x_{n_k} - \gamma, z\| < r + \varepsilon,$$

for every $z \in X$ and all $k \geq k_0$. We denote the set of all rough statistical limit points of the sequence $x = (x_n)$ by

$$r - \Lambda_x^2.$$

Now we present a result which characterizes the set r-Λ_x^2. The proof is immediate by definitions.

Proposition 1. *We have*

$$\gamma \in r - \Lambda_x^2$$

if and only if there exists a nonthin subsequence (x_{n_k}) of (x_n) such that

$$\limsup_{k \to \infty} \|x_{n_k} - \gamma, z\| \leq r,$$

for every $z \in X$.

Theorem 2. *Let $x = (x_n)$ be a sequence in 2-normed space X. Then we have*

$$r - \Lambda_x^2 \subseteq r - \Gamma_x^2.$$

Conclusion

We gave definitions of rough statistical cluster point and rough statistical limit point of a sequence in 2-normed space. Our results include that for a sequence $x=(x_n)$, $r-\Gamma_x^2$ is closed and also,

$$r-\Lambda_x^2 \subseteq r-\Gamma_x^2.$$

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