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*Research Article*

# Certain Class of Bi-Bazilevič Functions with Bounded Boundary Rotation Involving Sălăgean Operator

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ABSTRACT. In the present paper, we consider certain classes of bi-univalent Bazilevič functions with bounded boundary rotation involving Sălăgean operator to obtain the estimates of their second and third coefficients. Further, certain special cases are also indicated. Some interesting remarks about the results presented here are also discussed.

**Keywords:** Analytic function, bi-univalent, Bazilevič functions, Salăgeăn operator.

**2020 Mathematics Subject Classification:** 30C45.

## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse $f^{-1}$, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^2 - 5a_2 a_3 + a_4\right) w^4 + \dots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f$ and $f^{-1}$ are univalent in $\mathbb{U}$. Let $\Sigma$ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples in the class $\Sigma$ (see [26]).

For functions $f \in \mathcal{A}$, Sălăgean [27] (see also [4] and [28]) defined the linear operator $\mathcal{D}^m : \mathcal{A} \to \mathcal{A}$ ($m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{N} = \{1, 2, 3, \dots\}$) as follows:

$$\mathcal{D}^0 f(z) = f(z),$$

$$\mathcal{D}^1 f(z) = \mathcal{D}f(z) = z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

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and (in general)

$$\mathcal{D}^m f(z) = \mathcal{D}\left(\mathcal{D}^{m-1} f(z)\right) = z + \sum_{n=2}^{\infty} n^m a_n z^n. \tag{1.3}$$

From (1.3), we can easily deduce that

$$\mathcal{D}^{m+1} f(z) = z\left(\mathcal{D}^m f(z)\right)'. \tag{1.4}$$

Let $\mathcal{P}_k^\lambda(\alpha)$ be the class of analytic functions $p(z)$ in $\mathbb{U}$ normalized by $p(0) = 1$ and satisfying

$$\int_0^{2\pi} \left| \frac{\Re\left\{e^{i\lambda} p(z)\right\} - \alpha \cos\lambda}{1-\alpha} \right| d\theta \leq k\pi \cos\lambda, \tag{1.5}$$

where $z = re^{i\theta}, 0 \leq r < 1, |\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1$ and $k \geq 2$. The class $\mathcal{P}_k^\lambda(\alpha)$ was introduced and studied by Moulis [16] (see also Aouf [3] and Noor et al. [21]). We note that
(i) $\mathcal{P}_k^0(0) = \mathcal{P}_k$, is the class of functions have their real parts bounded in the mean on $\mathbb{U}$, introduced by Robertson [25] and studied Pinchuk [24];
(ii) $\mathcal{P}_k^\lambda(0) = \mathcal{P}_k^\lambda$, is the class of functions introduced by Robertson [25] and he derived a variational formula for functions in this class;
(iii) $\mathcal{P}_k^0(\alpha) = \mathcal{P}_k(\alpha)$, is the class of functions introduced by Padmanabhan and Parvatham [23] (see also Umarani and Aouf [31]);
(iv) $\mathcal{P}_2^0(\alpha) = \mathcal{P}(\alpha)$, is the class of functions with positive real part of order $\alpha$, $0 \leq \alpha < 1$;
(v) $\mathcal{P}_2^0(0) = \mathcal{P}$, is the class of functions having positive real part for $z \in \mathbb{U}$.

Using Salăgeăn operator $\mathcal{D}^m$ and the class $\mathcal{P}_k$, we now introduce the following subclass of Bi-Bazilevič analytic functions of the class $\Sigma$ as follows:

**Definition 1.1.** *A function $f \in \Sigma$ is said to be in the class $\mathcal{B}_\Sigma^m(\gamma, \delta, b; k)$ if it satisfies the following subordination condition:*

$$1 + \frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^\delta + \gamma \frac{\mathcal{D}^{m+1} f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^\delta - 1\right] \in \mathcal{P}_k \tag{1.6}$$

*and*

$$1 + \frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m g(w)}{w}\right)^\delta + \gamma \frac{\mathcal{D}^{m+1} g(w)}{\mathcal{D}^m g(w)}\left(\frac{\mathcal{D}^m g(w)}{w}\right)^\delta - 1\right] \in \mathcal{P}_k, \tag{1.7}$$

*where $g = f^{-1}, b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, \gamma, \delta \in \mathbb{C}, m \in \mathbb{N}_0, k \geq 2$ and all powers are understood as principle values.*

Taking additional choices of $m, \gamma, \delta, k$ and $b$, the class $\mathcal{B}_\Sigma^m(\gamma, \delta, b; k)$ reduces to the following subclasses of $\Sigma$:

(i) $\mathcal{B}_\Sigma^0(\gamma, \delta, 1; k) = \mathcal{B}_\Sigma(\gamma, \delta; k)$

$$= \left\{ f \in \Sigma : (1-\gamma)\left(\frac{f(z)}{z}\right)^\delta + \gamma \frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^\delta \in \mathcal{P}_k \right.$$

$$\left. \text{and } \ (1-\gamma)\left(\frac{g(w)}{w}\right)^\delta + \gamma \frac{wg'(w)}{g(w)}\left(\frac{g(w)}{w}\right)^\delta \in \mathcal{P}_k \right\};$$

(ii) $\mathcal{B}_{\Sigma}^{0}\left(\gamma,\delta,1-\eta;2\right)=\mathcal{B}_{\Sigma}\left(\gamma,\delta,\eta\right)\left(0\leq\eta<1\right)$ (see [15] for $f\in\mathcal{A}$)(see also [29])

$$=\left\{f\in\Sigma:\Re\left\{\left(1-\gamma\right)\left(\frac{f\left(z\right)}{z}\right)^{\delta}+\gamma\frac{zf^{'}\left(z\right)}{f\left(z\right)}\left(\frac{f\left(z\right)}{z}\right)^{\delta}\right\}>\eta\right.$$
$$\left.\text{and }\Re\left\{\left(1-\gamma\right)\left(\frac{g\left(w\right)}{w}\right)^{\delta}+\gamma\frac{wg^{'}\left(w\right)}{g\left(w\right)}\left(\frac{g\left(w\right)}{w}\right)^{\delta}\right\}>\eta\right\};$$

(iii) $\mathcal{B}_{\Sigma}^{0}\left(\gamma,1,1;k\right)=\mathcal{B}_{\Sigma}\left(\gamma;k\right)$

$$=\left\{f\in\Sigma:\left(1-\gamma\right)\frac{f\left(z\right)}{z}+\gamma f^{'}\left(z\right)\in\mathcal{P}_{k}\text{ and }\left(1-\gamma\right)\frac{g\left(w\right)}{w}+\gamma g^{'}\left(w\right)\in\mathcal{P}_{k}\right\};$$

(iv) $\mathcal{B}_{\Sigma}^{0}\left(\gamma,1,1-\eta;2\right)=\mathcal{B}_{\Sigma}\left(\gamma,\eta\right)\left(0\leq\eta<1\right)$ (see [10] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{\left(1-\gamma\right)\frac{f\left(z\right)}{z}+\gamma f^{'}\left(z\right)\right\}>\eta\text{ and }\Re\left\{\left(1-\gamma\right)\frac{g\left(w\right)}{w}+\gamma g^{'}\left(w\right)\right\}>\eta\right.$$

(v) $\mathcal{B}_{\Sigma}^{0}\left(1,\delta,1-\eta;2\right)=\mathcal{B}_{\Sigma}\left(\delta,\eta\right)\left(0\leq\eta<1\right)$ (see [22] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{\frac{zf^{'}\left(z\right)}{f\left(z\right)}\left(\frac{f\left(z\right)}{z}\right)^{\delta}\right\}>\eta\text{ and }\Re\left\{\frac{wg^{'}\left(w\right)}{g\left(w\right)}\left(\frac{g\left(w\right)}{w}\right)^{\delta}\right\}>\eta\right\};$$

(vi) $\mathcal{B}_{\Sigma}^{0}\left(1,0,b;k\right)=\mathcal{S}_{\Sigma}\left(b;k\right)$ (see Nasr and Aouf [20] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{1}{b}\left(\frac{zf^{'}\left(z\right)}{f(z)}-1\right)\in\mathcal{P}_{k}\text{ and }1+\frac{1}{b}\left(\frac{wg^{'}\left(w\right)}{g(w)}-1\right)\in\mathcal{P}_{k}\right\};$$

(vii) $\mathcal{B}_{\Sigma}^{0}\left(1,0,b;2\right)=\mathcal{S}_{\Sigma}\left(b\right)$ (see Nasr and Aouf [19] for $f\in\mathcal{A}$) (see also [5])

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{1}{b}\left(\frac{zf^{'}\left(z\right)}{f(z)}-1\right)\right\}>0\text{ and }\Re\left\{1+\frac{1}{b}\left(\frac{wg^{'}\left(w\right)}{g(w)}-1\right)\right\}>0\right\};$$

(viii) $\mathcal{B}_{\Sigma}^{1}\left(1,0,b;k\right)=\mathcal{C}_{\Sigma}\left(b;k\right)$ (see Nasr and Aouf [20] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{1}{b}\frac{zf^{''}\left(z\right)}{f'(z)}\in\mathcal{P}_{k}\text{ and }1+\frac{1}{b}\frac{wg^{''}\left(w\right)}{g'(w)}\in\mathcal{P}_{k}\right\};$$

(ix) $\mathcal{B}_{\Sigma}^{1}\left(1,0,b;2\right)=\mathcal{C}_{\Sigma}\left(b\right)$ (see Nasr and Aouf [18] for $f\in\mathcal{A}$) (see also [5])

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{1}{b}\frac{zf^{''}\left(z\right)}{f'(z)}\right\}>0\text{ and }\Re\left\{1+\frac{1}{b}\frac{wg^{''}\left(w\right)}{g'(w)}\right\}>0\right\};$$

(x) $\mathcal{B}_{\Sigma}^{0}\left(1,0,1;k\right)=\mathcal{S}_{\Sigma}\left(k\right)$ (see Pinchuk [24] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\frac{zf^{'}\left(z\right)}{f(z)}\in\mathcal{P}_{k}\text{ and }\frac{wg^{'}\left(w\right)}{g(w)}\in\mathcal{P}_{k}\right\};$$

(xi) $\mathcal{B}_{\Sigma}^{1}(1,0,1;k)=\mathcal{C}_{\Sigma}(k)$ (see Pinchuk [24] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:1+\frac{zf^{''}(z)}{f^{'}(z)}\in\mathcal{P}_k \text{ and } 1+\frac{wg^{''}(w)}{g^{'}(w)}\in\mathcal{P}_k\right\};$$

(xii) $\mathcal{B}_{\Sigma}^{0}(1,0,1-\eta;2)=\mathcal{S}_{\Sigma}(\eta)\,(0\leq\eta<1)$ (see [9] and [30])

$$=\left\{f\in\Sigma:\Re\left(\frac{zf^{'}(z)}{f(z)}\right)>\eta \text{ and } \Re\left(\frac{wg^{'}(w)}{g(w)}\right)>\eta\right\};$$

(xiii) $\mathcal{B}_{\Sigma}^{1}(1,0,1-\eta;2)=\mathcal{C}_{\Sigma}(\eta)\,(0\leq\eta<1)$ (see [9] and [30])

$$=\left\{f\in\Sigma:\Re\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)>\eta \text{ and } \Re\left(1+\frac{wg^{''}(w)}{g^{'}(w)}\right)>\eta\right\};$$

(xiv) $\mathcal{B}_{\Sigma}^{0}\left(\gamma,\delta,(1-\alpha)e^{-i\lambda}\cos\lambda;k\right)=\mathcal{B}_{\Sigma}(\gamma,\delta,\alpha,\lambda;k)\left(|\lambda|<\frac{\pi}{2},0\leq\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\left[(1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta}+\gamma\frac{zf^{'}(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\right]-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$
$$\left.\text{and } \frac{e^{i\lambda}\left[(1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta}+\gamma\frac{wg^{'}(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\right]-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:(1-\gamma)\left(\frac{f(z)}{z}\right)^{\delta}+\gamma\frac{zf^{'}(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\delta}\in\mathcal{P}_k^{\lambda}(\alpha)\right.$$
$$\left.\text{and } (1-\gamma)\left(\frac{g(w)}{w}\right)^{\delta}+\gamma\frac{wg^{'}(w)}{g(w)}\left(\frac{g(w)}{w}\right)^{\delta}\in\mathcal{P}_k^{\lambda}(\alpha)\right\};$$

(xv) $\mathcal{B}_{\Sigma}^{0}\left(1,0,be^{-i\lambda}\cos\lambda;2\right)=\mathcal{S}_{\Sigma}^{\lambda}(b)\left(|\lambda|<\frac{\pi}{2},b\in\mathbb{C}^{*}\right)$ (see Al-Oboudi and Haidan [2] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf^{'}(z)}{f(z)}-1\right)\right\}>0\right.$$
$$\left.\text{and } \Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{wg^{'}(w)}{g(w)}-1\right)\right\}>0\right\};$$

(xvi) $\mathcal{B}_{\Sigma}^{1}\left(1,0,be^{-i\lambda}\cos\lambda;2\right)=\mathcal{C}_{\Sigma}^{\lambda}(b)\left(|\lambda|<\frac{\pi}{2},b\in\mathbb{C}^{*}\right)$ (see Al-Oboudi and Haidan [2] for $f\in\mathcal{A}$)

$$=\left\{f\in\Sigma:\Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{zf^{''}(z)}{f^{'}(z)}\right)\right\}>0\right.$$
$$\left.\text{and } \Re\left\{1+\frac{e^{i\lambda}}{b\cos\lambda}\left(\frac{wg^{''}(w)}{g^{'}(w)}\right)\right\}>0\right\};$$

(xvii) $\mathcal{B}_{\Sigma}^{0}\left(1,0,(1-\alpha)\,e^{-i\lambda}\cos\lambda;k\right)=\mathcal{S}_{\alpha}^{\lambda}\left(k\right)\left(|\lambda|<\frac{\pi}{2},0\leq\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\frac{zf^{'}(z)}{f(z)}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$

$$\left.\text{and }\frac{e^{i\lambda}\frac{wg^{'}(w)}{g(w)}-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:\frac{zf^{'}(z)}{f(z)}\in\mathcal{P}_k^{\lambda}(\alpha)\text{ and }\frac{wg^{'}(w)}{g(w)}\in\mathcal{P}_k^{\lambda}(\alpha)\right\};$$

(xviii) $\mathcal{B}_{\Sigma}^{1}\left(1,0,(1-\alpha)\,e^{-i\lambda}\cos\lambda;k\right)=\mathcal{C}_{\alpha}^{\lambda}\left(k\right)\left(|\lambda|<\frac{\pi}{2},0\leq\alpha<1\right)$

$$=\left\{f\in\Sigma:\frac{e^{i\lambda}\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right)-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right.$$

$$\left.\text{and }\frac{e^{i\lambda}\left(1+\frac{wg^{''}(w)}{g^{'}(w)}\right)-\alpha\cos\lambda-i\sin\lambda}{(1-\alpha)\cos\lambda}\in\mathcal{P}_k\right\}$$

or

$$=\left\{f\in\Sigma:1+\frac{zf^{''}(z)}{f^{'}(z)}\in\mathcal{P}_k^{\lambda}(\alpha)\text{ and }1+\frac{wg^{''}(w)}{g^{'}(w)}\in\mathcal{P}_k^{\lambda}(\alpha)\right\}.$$

In order to establish our main results, we need the following lemma:

**Lemma 1.1.** [3, Theorem 5 with $p=1$] *If* $p(z)=1+\sum_{n=1}^{\infty}c_nz^n\in\mathcal{P}_k^{\lambda}(\alpha)$ *in* $\mathbb{U}$, *then*

$$|c_n|\leq(1-\alpha)\,k\cos\lambda\qquad(n\in\mathbb{N})\,. \tag{1.8}$$

*The result is sharp. Equality is attained for the odd coefficients and even coefficients, respectively, for the functions*

$$p_1(z)=1+(1-\alpha)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z}{1+z}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z}{1-z}\right)-1\right],$$

$$p_2(z)=1+(1-\alpha)\cos\lambda\,e^{-i\lambda}\left[\left(\frac{k+2}{4}\right)\left(\frac{1-z^2}{1+z^2}\right)-\left(\frac{k-2}{4}\right)\left(\frac{1+z^2}{1-z^2}\right)-1\right].$$

**Remark 1.1.** *For* $\lambda=\alpha=0$ *in Lemma 1.1, we obtain the result obtained by Goswami et al.* [11] *for the class* $\mathcal{P}_k$.

Lewin [13] defined the class of bi-univalent functions and obtained the bound for the second coefficient. Brannan and Taha [9] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced the concept of bi-starlike functions and the bi-convex functions, and obtained estimates for the initial coefficients. Recently, Srivastava et al. [26], Ali et al. [1], Frasin and Aouf [10], Goyal and Goswami [12] and many others (see [6], [7], [8], [14], [17] and [32]) have introduced and investigated subclasses of bi-univalent functions and obtained non-sharp bounds for the initial coefficients.

In the present paper, we estimates on the coefficients for second and third coefficients for the functions in the subclass $\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$ and its special subclasses.

## 2. Main results

Unless otherwise mentioned, we assume throughout this paper that $g=f^{-1}, b\in\mathbb{C}^{*}, \gamma, \delta\in\mathbb{C}, k\geq 2, m\in\mathbb{N}_0$ and all powers are understood as principle values.

**Theorem 2.1.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$ with $\delta\neq 1-\frac{3^m}{2^{2m-1}}$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_2|\leq\min\left\{\sqrt{\frac{|b|\,k}{|(\delta-1)\,2^{2m-1}+3^m|\,|\delta+2\gamma|}},\frac{|b|\,k}{2^m\,|\delta+\gamma|}\right\} \tag{2.9}$$

*and*

$$|a_3|\leq\frac{|b|\,k}{3^m\,|\delta+2\gamma|}\min\left\{\begin{array}{c}1+\frac{3^m}{|(\delta-1)2^{2m-1}+3^m|};1+\frac{|\delta+2\gamma||1-\delta||b|k}{2|\delta+\gamma|^2};\\ 1+\frac{|\delta+2\gamma||\delta-1||b|k}{2|\delta+\gamma|^2}+\frac{3^m|\delta+2\gamma||b|k}{2^{2m-1}|\delta+\gamma|^2}\end{array}\right\}. \tag{2.10}$$

*Proof.* If $f\in\mathcal{B}_{\Sigma}^{m}(\gamma,\delta,b;k)$, according to the Definition 1.1, we have

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}-1\right]=p(z) \tag{2.11}$$

and

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}g(w)}{\mathcal{D}^m g(w)}\left(\frac{\mathcal{D}^m g(w)}{w}\right)^{\delta}-1\right]=q(w), \tag{2.12}$$

where $p(z), q(w)\in\mathcal{P}_k$ and $g=f^{-1}$. Using the fact that the functions $p(z)$ and $q(w)$ have the following Taylor expansions

$$p(z)=1+p_1z+p_2z^2+... \tag{2.13}$$

and

$$q(w)=1+q_1w+q_2w^2+.... \tag{2.14}$$

Since

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}f(z)}{\mathcal{D}^m f(z)}\left(\frac{\mathcal{D}^m f(z)}{z}\right)^{\delta}-1\right]$$

$$=1+\left(\frac{\delta+\gamma}{b}\right)2^m a_2 z+\left(\frac{\delta+2\gamma}{b}\right)\left[3^m a_3+\frac{\delta-1}{2}2^{2m}a_2^2\right]z^2+... \tag{2.15}$$

and according to (1.2), we have

$$1+\frac{1}{b}\left[(1-\gamma)\left(\frac{\mathcal{D}^{m}g\left(w\right)}{w}\right)^{\delta}+\gamma\frac{\mathcal{D}^{m+1}g\left(w\right)}{\mathcal{D}^{m}g(w)}\left(\frac{\mathcal{D}^{m}g\left(w\right)}{w}\right)^{\delta}-1\right]$$

$$(2.16)\qquad =1-\left(\frac{\delta+\gamma}{b}\right)2^{m}a_{2}w+\left(\frac{\delta+2\gamma}{b}\right)\left[\left(2a_{2}^{2}-a_{3}\right)3^{m}+\frac{\delta-1}{2}2^{2m}a_{2}^{2}\right]w^{2}+...$$

from (2.13) and (2.14), combined with (2.15) and (2.16), it follows that

$$(2.17)\qquad p_{1}=\left(\frac{\delta+\gamma}{b}\right)2^{m}a_{2},$$

$$(2.18)\qquad p_{2}=\left(\frac{\delta+2\gamma}{b}\right)\left[3^{m}a_{3}+\frac{\delta-1}{2}2^{2m}a_{2}^{2}\right],$$

$$(2.19)\qquad q_{1}=-\left(\frac{\delta+\gamma}{b}\right)2^{m}a_{2},$$

$$(2.20)\qquad q_{2}=\left(\frac{\delta+2\gamma}{b}\right)\left[\left(2a_{2}^{2}-a_{3}\right)3^{m}+\frac{\delta-1}{2}2^{2m}a_{2}^{2}\right].$$

Now, from (2.18) and (2.20), we deduce that

$$(2.21)\qquad a_{2}^{2}=\frac{b\left(p_{2}+q_{2}\right)}{\left[\left(\delta-1\right)2^{2m}+\left(2\right)3^{m}\right]\left(\delta+2\gamma\right)}$$

and

$$(2.22)\qquad a_{3}-a_{2}^{2}=\frac{b\left(p_{2}-q_{2}\right)}{2\left(\delta+2\gamma\right)3^{m}}.$$

Using (2.21) in (2.22), we obtain

$$(2.23)\qquad a_{3}=\frac{b}{\delta+2\gamma}\left[\frac{p_{2}-q_{2}}{\left(2\right)3^{m}}+\frac{p_{2}+q_{2}}{\left(\delta-1\right)2^{2m}+\left(2\right)3^{m}}\right].$$

From (2.17) and (2.18), we get

$$(2.24)\qquad a_{3}=\frac{b}{3^{m}\left(\delta+2\gamma\right)}\left[p_{2}+\frac{\left(\delta+2\gamma\right)\left(1-\delta\right)p_{1}^{2}b}{2\left(\delta+\gamma\right)^{2}}\right],$$

while from (2.19) and (2.20), we deduce that

$$(2.25)\qquad a_{3}=\frac{b}{3^{m}\left(\delta+2\gamma\right)}\left[-q_{2}+\frac{\left(\delta+2\gamma\right)\left(\delta-1\right)bq_{1}^{2}}{2\left(\delta+\gamma\right)^{2}}+\frac{2\left(\delta+2\gamma\right)3^{m}bq_{1}^{2}}{2^{2m}\left(\delta+\gamma\right)^{2}}\right].$$

Combining (2.17) and (2.21) for the computation of the upper-bound of $|a_2|$, and (2.23), (2.24) and (2.25) for the computation of $|a_3|$, by using Lemma 1.1 (with $\alpha=\lambda=0$), we easily find the estimates of Theorem 2.1. This completes the proof of Theorem 2.1. □

Taking $m=0$ and $b=1$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_{\Sigma}(\gamma,\delta;k)$.

**Corollary 2.1.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}(\gamma,\delta;k)$ with $\delta\neq-1$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_{2}|\leq\min\left\{\sqrt{\frac{2k}{|\delta+1|\,|\delta+2\gamma|}},\frac{k}{|\delta+\gamma|}\right\}$$

*and*

$$|a_3| \leq \frac{k}{|\delta+2\gamma|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma|\,|1-\delta|\,k}{2\,|\delta+\gamma|^2}; 1+\frac{|\delta+2\gamma|\,|\delta+3|\,k}{2\,|\delta+\gamma|^2}\right\}.$$

Taking $m=0, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_\Sigma\,(\gamma,\delta,\eta)$.

**Corollary 2.2.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma\,(\gamma,\delta,\eta)$ with $0\leq\eta<1$, $\delta\neq-1$, $\delta\neq-\gamma$ and $\delta\neq-2\gamma$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4\,(1-\eta)}{|\delta+1|\,|\delta+2\gamma|}}, \frac{2\,(1-\eta)}{|\delta+\gamma|}\right\}$$

*and*

$$|a_3| \leq \frac{2\,(1-\eta)}{|\delta+2\gamma|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma|\,|1-\delta|\,(1-\eta)}{|\delta+\gamma|^2}; 1+\frac{|\delta+2\gamma|\,|\delta+3|\,(1-\eta)}{|\delta+\gamma|^2}\right\}.$$

Taking $m=0, \delta=1, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following corollary which improves the result of Frasin and Aouf [10, Theorem 3.2].

**Corollary 2.3.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma\,(\gamma,\eta)$ with $0\leq\eta<1$, $\gamma\neq-1$ and $\gamma\neq-\frac{1}{2}$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2\,(1-\eta)}{|2\gamma+1|}}, \frac{2\,(1-\eta)}{|\gamma+1|}\right\}$$

*and*

$$|a_3| \leq \frac{2\,(1-\eta)}{|2\gamma+1|} \min\left\{2, 1+\frac{4\,|2\gamma+1|\,(1-\eta)}{|\gamma+1|^2}\right\}.$$

Taking $m=0, \gamma=1, b=1-\eta\,(0\leq\eta<1)$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_\Sigma\,(\delta,\eta)$.

**Corollary 2.4.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_\Sigma\,(\delta,\eta)$ with $\delta\neq-1$ and $\delta\neq-2$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{4\,(1-\eta)}{|\delta+1|\,|\delta+2|}}, \frac{2\,(1-\eta)}{|\delta+1|}\right\}$$

*and*

$$|a_3| \leq \frac{2\,(1-\eta)}{|\delta+2|} \min\left\{1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2|\,|1-\delta|\,(1-\eta)}{|\delta+1|^2}; 1+\frac{|\delta+2|\,|\delta+3|\,(1-\eta)}{|\delta+1|^2}\right\}.$$

Taking $\delta=m=0, \gamma=1$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}_\Sigma\,(b)$.

**Corollary 2.5.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}_\Sigma\,(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{2\,|b|}, 2\,|b|\right\}$$

*and*

$$|a_3| \leq |b| \min\left\{3, 1+2\,|b|\right\}.$$

Taking $\delta=0, m=1, \gamma=1$ and $k=2$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}_\Sigma\,(b)$.

**Corollary 2.6.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{C}_{\Sigma}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{|b|}, |b|\right\}$$

*and*

$$|a_3| \leq \frac{|b|}{3} \min\left\{4, 1+2|b|\right\}.$$

Taking $m = 0$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{B}_{\Sigma}(\gamma, \delta, \alpha, \lambda; k)$.

**Corollary 2.7.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{B}_{\Sigma}(\gamma, \delta, \alpha, \lambda; k)$ with $\delta \neq -1$, $\delta \neq -\gamma$ and $\delta \neq -2\gamma$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{2k(1-\alpha)\cos\lambda}{|\delta+1||\delta+2\gamma|}}, \frac{k(1-\alpha)\cos\lambda}{|\delta+\gamma|}\right\}$$

*and*

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{|\delta+2\gamma|} \min\left\{\begin{array}{c} 1+\frac{2}{|\delta+1|}; 1+\frac{|\delta+2\gamma||(1-\delta)|k(1-\alpha)\cos\lambda}{2|\delta+\gamma|^2}; \\ 1+\frac{|\delta+2\gamma||\delta+5|k(1-\alpha)\cos\lambda}{2|\delta+\gamma|^2} \end{array}\right\}.$$

Taking $m = \delta = 0, \gamma = 1, k = 2$ and $b \to be^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}^{\lambda}_{\Sigma}(b)$.

**Corollary 2.8.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}^{\lambda}_{\Sigma}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{2|b|\cos\lambda}, 2|b|\cos\lambda\right\}$$

*and*

$$|a_3| \leq |b|\cos\lambda \, \min\left\{3, 1+2|b|\cos\lambda\right\}.$$

Taking $m = \gamma = 1, \delta = 0, k = 2$ and $b \to be^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}^{\lambda}_{\Sigma}(b)$.

**Corollary 2.9.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{C}^{\lambda}_{\Sigma}(b)$, then*

$$|a_2| \leq \min\left\{\sqrt{|b|\cos\lambda}, |b|\cos\lambda\right\}$$

*and*

$$|a_3| \leq \frac{|b|\cos\lambda}{3} \min\left\{4, 1+2|b|\cos\lambda\right\}.$$

Taking $\delta = m = 0, \gamma = 1$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{S}^{\lambda}_{\alpha}(k)$.

**Corollary 2.10.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{S}^{\lambda}_{\alpha}(k) \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$, then*

$$|a_2| \leq \min\left\{\sqrt{k(1-\alpha)\cos\lambda}, k(1-\alpha)\cos\lambda\right\}$$

*and*

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{2} \min\left\{3, 1+k(1-\alpha)\cos\lambda\right\}.$$

Taking $\delta = 0, \gamma = m = 1$ and $b = (1-\alpha)e^{-i\lambda}\cos\lambda \left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$ in Theorem 2.1, we obtain the following result for the functions belonging to the class $\mathcal{C}^{\lambda}_{\alpha}(k)$.

**Corollary 2.11.** *Let $f(z)$ given by (1.1) belongs to the class $\mathcal{C}_{\alpha}^{\lambda}(k)$ $\left(|\lambda| < \frac{\pi}{2}, 0 \leq \alpha < 1\right)$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{k(1-\alpha)\cos\lambda}{2}}, \frac{k(1-\alpha)\cos\lambda}{2}\right\}$$

*and*

$$|a_3| \leq \frac{k(1-\alpha)\cos\lambda}{6}\min\{4, 1+k(1-\alpha)\cos\lambda\}.$$

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