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Research Article

On waiting time distribution of runs in a Fibonacci and Lucas sequences

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Abstract

There is an increasingly needed to development a new mathematical apparatus concerned with the description, prediction, and understanding of natural phenomena in a precise manner. The purpose of this study is to extend the mathematical framework of Fibonacci and Lucas sequences for underpinned and establishing a modern mathematical formulas. For this purpose, we derive analytical formulas of order k-Fibonacci and order k-Lucas sequences simultaneously based on Bernoulli sequence. Furthermore, exploiting a relationship with the k th order Fibonacci and Lucas sequence, we study the probability distribution function (pdf) of the waiting time (W (k)).

Keywords: Bernoulli trials, Order k - Fibonacci sequence, Order k -Lucas sequence.

Bir Fibonacci ve Lucas dizisinde tekrarların bekleme süresi üzerine

Öz

Doğal fenomenlerin kesin bir manada tanımlanması, öngörülmesi ve anlaşılmasıyla ilgili yeni matematiksel aygıtlar geliştirme ihtiyacı giderek artmaktadır. Bu çalışmanın amacı, Fibonacci ve Lucas dizilerinin matematiksel çerçevesini modern matematiksel formüller ile desteklemek amacıyla genişletmektir. Bu amaçla, order-k Fibonacci ve order-k Lucas dizileri ile eş zamanlı olarak Bernoulli dizisine dayanarak analitik formüllerin türetilmesi amaçlanmaktadır. Ayrıca, k-yıncı mertebeden Fibonacci ve Lucas dizileri arasındaki bir ilişkiden yararlanarak (W (k)) bekleme süresinin olasılık dağılım fonksiyonu (pdf) çalışılacaktır.

Anahtar Kelimeler: Bernoulli denemeleri, k-mertebeli Fibonacci dizisi, k-mertebeli Lucas dizisi.

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1. Introduction

In Bernoulli trials, there is a series of results about distribution theories on runs (Greenberg, 1970; Klots & Park, 1972; Saperstein, 1973; Koutras, 1996, 1997; Chaves & de Souza, 2007; Aki & Hirano, 2007; Kim,S et al,2013). KOUTRAS (1996) investigated the exact distribution of the waiting time in a sequence of independent and identically distributed (iid) Bernoulli trials. Formulae are provided for terms of certain generalized Fibonacci numbers and polynomials are also included. Chaves et al., (2007) used j-step Fibonacci numbers to derive the expected value and the variance for the distribution of the waiting time for n consecutive successes in a Bernoulli sequences. Singh et al., (2014) presented some generalized identities on the products of k-Fibonacci and k-Lucas numbers to establish connection formulas between them with the help of Binet's formula. Kim,S et al, (2013) derived probabilty distribution of W(k) for both independent and homogeneous two-state Markovian Bernoulli trials, using a generalized Fibonacci sequence of order k. Öcal, A el al, (2005) give some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers and obtained the Binet's formula for these sequences.

There can be many applications: Bernoulli trials to get a cluster of positive or negative response on certain treatment to a DNA sequence. A sequence of n Bernoulli trials contains as many runs of length k as there are non-overlapping uninterrupted succession of exactly k 1's or 0's.

We will follow the same previous methodology, but our ideas are completely different. That mean we are trying to derive a new formula for order k-Fibonacci and order k- Lucas sequences through sequences of Bernoulli trials.

In this work, we study the sequence of Bernoulli trials under some conditions and exploiting a relationship with the Fibonacci and Lucas sequences.

2. Sequence of Bernoulli trials

We will start a sequence of Bernoulli trials of size n by toss a coin n times, and obtaining a sequence i ones consecutive on the n^{th} trial at the end of sequence with less than i zeros or ones consecutive previously where $i \geq 2$.

Let $s_n^{(i)}$ denote the number of cases where a sequence of i ones consecutive on the n^{th} trial at the end of sequence with less than i zeros or ones consecutive previously.

To derive the value of $s_n^{(i)}$ we note that:

Case 2.1 When $\boldsymbol{i = 2}$

The sequence as shown in table 1.

Table 1: Sequence of Bernoulli when i=2

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{s_n^{(2)}}$	0	1	1	1	1	1	1	1	1	1	1	1

We can see comparing the previous values of $s_{n-2}^{(2)}$ with those values of $F_n^{(1)}$ shown in table 2. This shows that $s_{n-2}^{(2)} = F_n^{(1)}$.

Lemma 2.1. The number of $s_n^{(2)}$ where $n \geq i$, "i.e. sequences", follows the Fibonacci sequence of order 1.

A proof is easily established by table 1 that all the number of $s_n^{(2)}$ is equal to 1.

2.1.1.Probability distribution function of $W_n^{(i)}$

Let define independent Bernoulli variables x_i , with $p(x_i = 1) = p\ and,\ p(x_i = 0) = q$.

2.1 Definition(1) : For any two probabilities Z and D, let's define them by two equations.

$$Z_n = qD_{n-1}\ \ and\ \ \ D_n = pZ_{n-1}\ for\ n = 2,3,4,5, \ldots,$$

$$with\ \ Z_1 = 1\ , D_1 = 1.$$

The new formula for either consecutive k ones or k zeros for the first time at end sequence with less than k ones or k zeros before.

We are derivation of pdf of F_n^1 is as follows.

Theorem (1). For n = 2, the pdf of F_n^1 is

$$P(F_{n-1}^1 = n) = \begin{cases} 0 & n < 2 \\ (p^2+q^2)(pq)^{\frac{n-2}{2}} & if\ (n > 1) is\ an\ even\ number \\ (pq)^{\frac{n-1}{2}} & if\ (n > 2) is\ an\ odd\ number \end{cases}$$

Proof.

$$p(F_1^1 = 2) = pp + qq = p^2 + q^2$$

$$p(F_2^1 = 3) = qp^2 + pq^2 = pq$$

$$p(F_3^1 = 4) = pq(p^2 + q^2)$$

$$p(F_4^1 = 5) = qpq(p^2 + q^2) = (pq)^2$$

$$p(F_5^1 = 6) = p^2q^2(p^2 + q^2)$$

$$p(F_6^1 = 7) = qpqpq(p^2 + q^2) = (pq)^3$$

.

$$(p^2 + q^2)(pq)^{\frac{n-2}{2}}\ \ \ if\ (n > 1) is\ an\ even\ number$$

$$(pq)^{\frac{n-1}{2}}\ \ \ \ \ \ if\ (n > 2) is\ an\ odd\ number$$

Case 2.2. When i = 3

The sequence as shown in the following Table 2.

Table 2: Sequence of Bernoulli when i=3

n	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{s_n^{(3)}}$	0	0	1	1	2	3	5	8	13	21	34	55

From table 2, We define the second order recurrence relations

$$s_n^{(3)} = s_{n-1}^{(3)} + s_{n-2}^{(3)} \quad n \geq 5 \qquad (2.1)$$

With the initial conditions

$s_1^{(3)} = 0, s_2^{(3)} = 0, s_3^{(3)} = 1, s_4^{(3)} = 1$

2.2.Definition 2 For any four real numbers a_0, a_1, b_0 and b_1 , the sequences $\{w_n^2\}_{n=0}^{\infty}$ is defined recursively by

$$w_n^{(2)} = \left\{ b_0 s_{n+1}^{(3)} + b_1\, s_{n+2}^{(3)} \qquad n \geq 2 \right\}$$

where $\quad w_0^{(2)} = a_0, w_1^{(2)} = a_1$ (2.2)

where $s_n^{(3)}$ is above definition.

Special cases When

1) $a_0 = 1\,, a_1 = 1, b_0 = 1, b_1 = \quad 1$

than $\{W_n^2\}_{n=0}^{\infty}$ *is Fibonacci sequence*

2) $a_0 = 2, a_1 = 1, b_0 = 2, b_1 = 1$

than $\{W_n^2\}_{n=0}^{\infty}$ is Lucas sequence

Notes. We can see comparing the previous values of $s_{n-3}^{(3)}$ with those values of F_n^2 shown in table 2. This shows that $s_{n+3}^{(3)} = F_n^2$.

Theorem 2 The sequences $\{W_n^{(2)}\}_{n=0}^{\infty}$ satisfy following

$1 -\; L_n = 2F_{n-2} +\; F_{n-1}$ (2.3)

$2 -\; L_n = F_{n-2} +\; F_n$ (2.4)

$3 -\; L_n = 2F_n -\; F_{n-1}$ (2.5)

$4 -\; L_n = 2F_{n+1} - 3F_{n-1}$ (2.6)

Proof.

1) $L_n^{(2)} = 2S_{n+1}^3 + S_{n+2}^3$

We know $S_{n+3}^3 = \; F_n^2$ then $L_n^2 = \; 2F_{n-2}^2 + \; F_{n-1}^2$.

2) $$\begin{aligned} L_n^{(2)} &= 2S_{n+1}^3 + S_{n+2}^3 \; = S_{n+1}^3 + (S_{n+1}^3 + S_{n+2}^3) \\ &= S_{n+1}^3 + S_{n+3}^3 \rightarrow \; L_n^2 = \; F_{n-2}^2 + \; F_n^3. \end{aligned}$$

3) $$\begin{aligned} L_n^{(2)} &= 2S_{n+1}^3 + S_{n+2}^3 = 2\,(S_{n+3}^3 - S_{n+2}^3) + S_{n+2}^3 \\ &= 2S_{n+3}^3 - S_{n+2}^3 \rightarrow \; L_n^2 = 2F_n^2 -\; F_{n-1}^2. \end{aligned}$$

4) $$\begin{aligned} L_n^{(2)} &= 2S_{n+1}^3 + S_{n+2}^3 \; = 2S_{n+3}^3 - 2S_{n+2}^3 + S_{n+2}^3 \\ &= 2S_{n+4}^3 - 2S_{n+2}^3 - S_{n+2}^3 = 2S_{n+4}^3 - 3S_{n+2}^3 \\ &= 2F_{n+1}^{(2)} - 3F_{n-1}^{(2)}. \end{aligned}$$

Theorem 3 The generating function of sequence $\{W_n^2\}_{n=0}^{\infty}$ is given by

$$D_{(x)} = \sum_{n=0}^{\infty} w_n\, x^n$$
$$= \frac{a_0 + (a_1 - a_0)x + (b_0 + b_1 - a_0 - a_1)x^2 + (b_1 - a_1)x^3}{1 - x - x^2}$$

Special cases

$1 -$ When $a_0 = 1\,, a_1 \; = \; 1\,, b_0 \; = \; 1\,, b_1 = 1$

$$D_{(x)} = \frac{1}{1 - x - x^2}$$

is generating function of Fibonacci sequence.

$2 -$ When $a_0 \; = \; 2\,, \qquad a_1 = 1\,, \qquad b_0 = 2\,, \qquad b_1 = 1$

$$D_{(x)} = \frac{2 - x}{1 - x - x^2}$$ is generating function of Lucas sequence.

2.1.2 The distribution of order 2 -Fibonacci and Lucas

We use only elementary facts about Fibonacci and Lucas numbers to derive the waiting time for n consecutive successes in Bernoulli sequences. In the case of the Bernoulli parameter $p = \frac{1}{2}$ the exact distribution is obtained.

Let $W_n^{(2)}$ be the random variable determined by the number for a run of j consecutive successes in a Bernoulli experiment, as defined in the preliminaries. The exact distribution of $W_n^{(2)}$ seems to be intractable, except for $p = \frac{1}{2}$. In this case, every equal length sequence has the same probability and the distribution of $W_n^{(2)}$ is nicely given:

2.1.3 The distribution of $W_n^{(2)}$

The distribution of $W_n^{(2)}$ is nicely given by:

$$P\big(W_{n-2}^{(2)} = n\big) = \begin{cases} 0 & n < 2 \\ W_{n-2}^{(2)} \left(\dfrac{1}{2}\right)^{n+i} & n = 2,3,4,5,\ldots \\ o & O.W \end{cases}$$

There are two cases depends on $W_n^{(2)}$ and i:

The first case: Where $w_0^{(2)} = 1, w_1^{(2)} = 1,$

$w_n^{(2)} = \left\{ s_{n+1}^{(3)} + \; s_{n+2}^{(3)} \qquad n \geq 2 \right\}$ and i=0 Then

$$P\big(W_{n-2}^{(2)} = n\big) = \begin{cases} 0 & n < 2 \\ W_{n-2}^{(2)} \left(\dfrac{1}{2}\right)^{n} & n = 2,3,4,5,\ldots \\ o & O.W \end{cases}$$

This distribution is Fibonacci distribution.

And we can rewrite the distribution of $W_n^{(2)}$ as

$$P\big(W_{n-2}^{(2)} = n\big) = \begin{cases} 0 & n < 2 \\ F_{n-2}^{(2)} \left(\frac{1}{2}\right)^{n} & n = 2,3,4,5,\ldots \\ o & O.W \end{cases}$$

This distribution as Shane(1973) defined.

The second case: Where $w_0^{(2)} = 1, w_1^{(2)} = 3,$

$w_n^{(2)} = \left\{ 2s_{n+1}^{(3)} + \; s_{n+2}^{(3)} \qquad n \geq 2 \right\}$, and i=1

This distribution as Lucas distribution.

And we can rewrite the distribution of $W_n^{(2)}$ as

$$P\big(W_{n-2}^{(2)} = n\big) = \begin{cases} 0 & n < 2 \\ L_{n-1}^{(2)} \left(\dfrac{1}{2}\right)^{n+1} & n = 2,3,4,5,\ldots \\ o & O.W \end{cases}$$

The distribution of $W_n^{(2)}$ can called distribution of waiting time. And The Fibonacci and Lucas numbers allow the calculation of the probability distribution function for waiting time for a run of n successes in a sequence of Bernoulli trials. Obtained the closed form for the Fibonacci and Lucas numbers.

2.1.4 Probability distribution function of $W_n^{(i)}$

2.3 Definition(3) : For any two probabilities Z and D, let's define them by two equations

$$Z_n = qD_{n-1} + q^2D_{n-2} \quad and \quad D_n = pZ_{n-1} + p^2Z_{n-2}\ for\ n = 3,4,5,\ldots,$$

$$with \quad Z_1 = 1, Z_2 = q, D_1 = 1, D_2 = p$$

The pdf of F_n^2 is given using the above definition.

Theorem (4). For n = 3, the pdf of F_{n-2}^2 is

$$P(F_{n-2}^2 = n) = \{Z_{n-2}p^3 + D_{n-2}q^3\}$$

$$where\ Z_n = qD_{n-1} + q^2D_{n-2}$$

$$D_n = pZ_{n-1} + p^2Z_{n-2}\ for\ n = 3,4,5 \ldots..with\ Z_1 = 1, Z_2 = q$$

$D_1 = 1$ and $D_2 = p$.

Proof. First, note that

$$P(F_1^2 = 3) = ppp + qqq = p^3 + q^3 = Z_1p^3 + D_1q^3$$

$$P(F_2^2 = 4) = qppp + pqqq = qp^3 + pq^3 = Z_2p^3 + D_2q^3$$

$$P(F_3^2 = 5) = pqp^3 + qqp^3 + qpq^3 + ppq^3 = Z_3p^3 + D_3q^3\ where$$

$$Z_3 = qD_2 + q^2D_1 = pq + qq\ and\ D_3 = pZ_2 + p^2Z_1 = pq + pp$$

$$P(F_4^2 = 6) = pqqp^3 + qpqp^3 + ppqp^3 + qppq^3 + pqpq^3 + qqpq^3$$

$$= 2q^2p^4 + qp^5 + 2p^2q^4 + pq^5$$

where $Z_4 = qD_3 + q^2D_2 = qpq + ppq + pqq = 2pq^2 + qp^2\ and$

$$D_4 = pZ_3 + p^2Z_2 = ppq + pqq + ppq = 2qp^2 + pq^2.$$

$P\left(F_{n-2}^{(2)} = n\right) = \{Z_{n-2}p^3 + D_{n-2}q^3\}$ This result as Kim, S., Park, C., & Oh, J. (2013) defined.

2.4 Definition(4) : For any two probabilities z and D, let's define them by two equations

$$Z_n = qD_{n-1} + q^2D_{n-2} \quad \text{and} \quad D_n = pZ_{n-1} + p^2Z_{n-2}\ for\ n = 3,4,5,\ldots,$$

with $Z_1 = 1, Z_2 = (q+1), D_1 = 1, D_2 = (p+1)$

The pdf of $L_n^{(2)}$ is given using the above definition.

Theorem (5). For n = 3, the pdf of $L_{n-2}^{(2)}$ is

$$P\left(L_{n-2}^{(2)} = n\right) = Z_{n-2}p^3 \qquad where$$

$$Z_n = qD_{n-1} + q^2D_{n-2} \quad \text{and} \quad D_n = pZ_{n-1} + p^2Z_{n-2}$$

$$for\ n = 3,4,5,\ldots$$

with $Z_1 = 1, Z_2 = (q+1), D_1 = 1, D_2 = (p+1)$.

Proof. First, note that

$$P\left(L_1^{(2)} = 3\right) = Z_1p^3 = ppp \quad ,$$

$$P\left(L_2^{(2)} = 4\right) = Z_2p^3 = p^3(q+1) = qppp + ppp$$

$$Z_3 = qD_2 + q^2D_1 = qp + q + q^2, D_3 = pZ_2 + p^2Z_1 = pq + p + p^2$$

$$P\left(L_3^{(2)} = 5\right) = Z_3p^3 = pqppp + qppp + qqppp$$

$$Z_4 = qD_3 + q^2D_2 = qpq + qp + qp^2 + pq^2 + q^2,$$

$$D_4 = pZ_3 + p^2Z_2 = pqp + pq + pq^2 + qp^2 + p^2$$

$$P(L_4^2 = 6) = Z_4p^3 = qpqp^3 + qpp^3 + qp^2p^3 + pq^2p^3 + q^2p^3$$

.

.

$$P(L_{n-2}^2 = n) = Z_{n-2}p^3$$

Some numerical examples:

The pdf of W_n^i for a special case of $p = \frac{1}{2}$ is obtained numerically, using $F_{n-2}^1, F_{n-2}^2, L_{n-2}^2$.

1- **For n=2**

$$P(F_0^1 = 2) = pp + qq = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} = F_0^1\left(\frac{1}{2}\right)^1$$

$$P(F_1^1 = 3) = qpp + pqq = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = F_1^1(\frac{1}{2})^2$$

$$P(F_2^1 = 4) = pqpp + qpqq = \frac{1}{16} + \frac{1}{16} = \frac{1}{8} = F_2^1(\frac{1}{2})^3$$

.

$$P(F_{n-2}^1 = n) = \cdots pqpqpqpp + \cdots qpqpqpqq = (\frac{1}{2})^{n-1} = F_{n-2}^1(\frac{1}{2})^{n-1}$$

where $F_{n-2}^1 = 1, n = 2,3.4, \ldots$

2- $\boldsymbol{For\ n = 3\ and\ W_n^2 = F_{n-2}^2}$

$P(F_1^2 = 3) = ppp + qqq = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} = F_1^2(\frac{1}{2})^2$

$P(F_2^2 = 4) = qp^3 + pq^3 = \frac{1}{16} + \frac{1}{16} = \frac{1}{8} = F_2^2(\frac{1}{2})^3$

$P(F_3^2 = 5) = pqp^3 + qqp^3 + qpq^3 + ppq^3$
$= \frac{1}{32} + \frac{1}{32} + \frac{1}{32} + \frac{1}{32} = \frac{1}{8} = \frac{2}{16} = F_3^2(\frac{1}{2})^4$

$P(F_4^2 = 6) = 2q^2p^4 + qp^5 + 2p^2q^4 + pq^5$
$= \frac{2}{64} + \frac{1}{64} + \frac{2}{64} + \frac{1}{64} = \frac{3}{32} = F_4^2(\frac{1}{2})^5$

$P(F_{n-2}^2 = n) = F_{n-2}^2(\frac{1}{2})^{n-1}$ *where* $F_{n-2}^2 = (1,1,2,3,5,8,13,..)$

And the sum over all F_{n-2}^1 and F_{n-2}^2 are equal to 1 that are

$$\sum_{n=2}^{\infty} F_{n-2}^1 (\frac{1}{2})^{n-1} = 1 \text{ and } \sum_{n=3}^{\infty} F_{n-2}^2 (\frac{1}{2})^{n-1} = 1.$$

1- For $n = 3$ and $W_n^2 = L_{n-2}^2$

$P(L_1^2 = 3) = ppp = \frac{1}{8} = L_1^2(\frac{1}{2})^3$

$P(L_2^2 = 4) = ppp + qppp = \frac{1}{8} + \frac{1}{16} = \frac{3}{16} = L_2^2(\frac{1}{2})^4$

$P(L_3^2 = 5) = pqppp + qppp + qqppp = \frac{1}{32} + \frac{1}{16} + \frac{1}{32} = \frac{4}{32}$
$= L_3^2(\frac{1}{2})^5$

$P(L_4^2 = 6) = 2q^2p^4 + qp^4 + qp^5 + q^2p^3$
$= \frac{2}{64} + \frac{1}{32} + \frac{1}{64} + \frac{1}{32} = \frac{7}{64} = L_4^2(\frac{1}{2})^6$

.

$P(L_{n-2}^2 = n) = L_{n-2}^2(\frac{1}{2})^n$ where $L_i^2 = (1,3,4,7,11,18,29, \ldots..)$

And the sum over all L_{n-2}^2 is equal to 1 that

$$\sum_{n=3}^{\infty} L_{n-2}^2 (\frac{1}{2})^n = 1$$

2.2.Distribution function of W_n^i for either consecutive k ones or k zeros for the first time in a two state Markovian trial

Let's define a sequence of Markov dependent Bernoulli variables x_i, with $P(x_i = 1) = p, P(x_i = 0) = q$ and with Markov dependence as following: $P(x_{i+1} = 1 \backslash x_i = 1) = a$ and $P(x_{i+1} = 0 \backslash x_i = 0) = b$

2.2.1 Probability distribution function of F_n^1

Theorem (6). the probability distribution function of F_n^1 is

$$P(F_{n-2}^1 = n) = \begin{cases} 0 & n < 2 \\ [(1-a)(1-b)]^{\frac{n-2}{2}}(pa + qb) & n \text{ is even} \\ [(1-a)(1-b)]^{\frac{n-3}{2}}(q(1-b)a + p(1-a)b) & n \text{ is odd} \end{cases}$$

Proof. Starting with

$P(F_0^1 = 2) = pp + qq = pa + qb$

$P(F_1^1 = 3) = qpp + pqq = q(1-b)a + p(1-a)q$

$P(F_2^1 = 4) = pqpp + qpqq$
$= p(1-a)(1-b)a + q((1-b)(1-a)b$
$= [(1-a)(1-b)](pa + qb)$

$P(F_3^1 = 5) = qpqpp + pqpqq$
$= q(1-b)(1-a)(1-b)a$
$+ p(1-a)(1-b)(1-a)b$
$= [(1-a)(1-b)][q(1-b)a + p(1-a)b]$

$P(F_4^1 = 6) = pqpqpp + qpqpqq$
$= p(1-a)(1-b)(1-a)(1-b)a +$

$q(1-b)(1-a)(1-b)(1-a)b$
$= [(1-a)(1-b)]^2(pa + qb)$

$P(F_5^1 = 7) = qpqpqpp + pqpqpqq$
$= q(1-b)(1-a)(1-b)(1-a)(1-b)a$
$+ p(1-a)(1-b)(1-a)(1-b)(1-a)b$
$= [(1-a)(1-b)]^2[q(1-b)a$
$+ p(1-a)b]$

2.2.2 Probability distribution function of L_n^2

Theorem (7). the probability distribution function of L_n^2 is

$P(L_{n-2}^2 = n) = Z_{n-2}aa \quad n \geq 3 \quad$ *where* Z_n
$= (1-b)D_{n-1} + (1-b)bD_{n-2}$

and $D_n = (1-a)Z_{n-1} + (1-a)aZ_{n-2}$

$n = 3,4,5,6, \ldots$ and

$Z_1 = p$, $Z_2 = q(1-b) + p$, $D_1 = q$ and $D_2 = p(1-a) + q$

Proof. Starting with

$(L_1^2 = 3) = \{111\} \rightarrow P(L_{n-2}^2 = 3) = ppp = paa = Z_1aa$

$(L_2^2 = 4) = \{111{,}0111\} \rightarrow P(L_2^2 = 4) = ppp + qppp$
$= paa + q(1-b)aa = Z_2aa$

$(L_3^2 = 5) = \{0111{,}00111{,}10111\} = qppp + qqppp + pqppp$
$= q(1-b)aa + qb(1-b)aa$
$+ p(1-a)(1-b)aa = \quad Z_3aa \quad$ *Where* Z_3
$= (1-b)D_2 + (1-b)bD_1$
$= \ p(1-b)(1-a) + q(1-b)$
$+ (1-b)bq$

$(L_4^2 = 6) = \{00111, 10111, 100111 + 110111 + 010111\}$
$= \quad qqppp + pqppp + pqqppp + ppqppp$
$+ qpqppp$
$= qb(1-b)aa + p(1-a)(1-b)aa$
$+ p(1-a)b(1-b)aa$
$+ pa(1-a)(1-b)aa$
$+ q(1-b)(1-a)(1-b)aa$
$= Z_4 aa \; where \quad Z_4$
$= (1-b)D_3 + (1-b)bD_2$
$= q(1-b)(1-a)(1-b) +$

$p(1-a)(1-b) + ap(1-b)(1-a) + bp(1-b)(1-a)$
$+ qb(1-b)$

.

$P(L_{n-2}^2 = n) = Z_{n-2} aa.$

Case 2.3. When i = 4

The sequence as shown in Table 3.

Table 3: Sequence of Bernoulli when i=4

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$s_n^{(4)}$	0	0	0	1	1	2	4	7	13	24	44	81	149

From table 3, We define the third order recurrence relations

$$s_n^{(4)} = s_{n-1}^{(4)} + s_{n-2}^{(4)} + s_{n-3}^{(4)} \quad n \geq 6 \qquad (2.7)$$

with the initial conditions

$$s_1^{(4)} = 0, s_2^{(4)} = 0, s_3^{(4)} = 0, s_4^{(4)} = 1, s_5^{(4)} = 1.$$

2.5 Definition 5. For any six real numbers a_0, a_1, a_2 b_0, b_1 and b_2 , the sequences $\{w_n^{(3)}\}_{n=0}^{\infty}$ is defined recursively by

$$W_0^{(3)} = a_0, w_1^{(3)} = a_1 \;, w_2^3 = a_2$$

$$W_n^{(3)} = \sum_{k=0}^{2} b_k \; S_{n+k-1}^{(4)} \qquad n \geq 3 \qquad (2.8)$$

where $s_n^{(4)}$ is above definition.

Special cases.

$1 - \; a_0 = 0, a_1 = 0, a_2 = 1, b_l = 1$ where $l = 0,1,2$

Then $W_0^{(3)} = 0 \,, W_1^{(3)} = 0$ and $W_2^{(3)} = 1$

$$W_n^{(3)} = s_{n-1}^{(4)} + s_n^{(4)} + s_{n+1}^{(4)} \qquad n \geq 3$$

$\{W_n^3\}_{n=0}^{\infty}$ is Tribonacci sequence

$2 - \; a_0 = 3, a_1 = 1, a_2 = 3 \,, b_0 = 3 \,, b_1 = 4, b_2 = 7$ then

$W_0^{(3)} = 3 \,, W_1^{(3)} = 1$ and $W_2^{(3)} = 3$

$$W_n^{(3)} = 3s_{n-1}^{(4)} + 4s_n^{(4)} + 7s_{n+1}^{(4)} \qquad n \geq 3$$

$\{W_n^{(3)}\}_{n=0}^{\infty}$ is Tribonacci – Lucas sequence

Notes: we can see comparing the previous values of $s_{n+3}^{(4)}$ with those values of $F_n^{(3)}$ shown in table 3. This shows that $s_{n+3}^{(4)} = F_n^{(3)}$

Theorem 8 The sequences $\{W_n^{(3)}\}_{n=0}^{\infty}$ satisfy following

$$1 - \; L_n^{(3)} = F_{n+1}^{(3)} + 2F_n^{(3)} + 3F_{n-1}^{(3)} \qquad (2.9)$$

$$2 - \; L_n^{(3)} = 3F_{n+2}^{(3)} - 2F_{n+1}^{(3)} - F_n^{(3)} \qquad (2.10)$$

$$3 - \; L_n^{(3)} = 2F_{n+2}^{(3)} - F_{n+1}^{(3)} + F_{n-1}^{(3)} \qquad (2.11)$$

Proof.

$1 - \; L_n^{(3)} = 3S_{n-1}^{(4)} + 4S_n^{(4)} + 7S_{n+1}^{(4)}$

$= 3\left(S_{n-1}^{(4)} + S_n^{(4)} + S_{n+1}^{(4)}\right) + S_n^{(4)} + 4S_{n+1}^{(4)}$

$= 3S_{n+2}^{(4)} + S_n^{(4)} + 4S_{n+1}^{(4)}$

$= 3S_{n+2}^{(4)} + S_{n+3}^{(4)} - S_{n+2}^{(4)} - S_{n+1}^{(4)} + 4S_{n+1}^{(4)}$

$= S_{n+3}^{(4)} + 2S_{n+2}^{(4)} + 3S_{n+1}^{(4)}$ where $s_{n+2}^{(4)} = F_n^{(3)}$ Then

$L_n^{(3)} = F_{n+1}^{(3)} + 2F_n^{(3)} + 3F_{n-1}^{(3)}$

$2 - \; L_n^{(3)} = 3S_{n-1}^{(4)} + 4S_n^{(4)} + 7S_{n+1}^{()4} = 3S_{n+2}^{(4)} + S_n^{(4)} + 4S_{n+1}^{(4)}$

$= 3S_{n+2}^{(4)} + S_{n+3}^{(4)} - S_{n+2}^{(4)} + 3S_{n+4}^{(4)} - 3S_{n+3}^{(4)} - S_{n+2}^{(4)}$

$= 3S_{n+4}^{(4)} - 2S_{n+3}^{(4)} - S_{n+2}^{(4)} = 3F_{n+2}^{(3)} - 2F_{n+1}^{(3)} - F_n^{(3)}$

$3 - \; L_n^{(3)} = 3S_{n-1}^{(4)} + 4S_n^{(4)} + 7S_{n+1}^{(4)} = 3S_{n+2}^{(4)} + S_n^{(4)} + 4S_{n+1}^{(4)}$

$= 2S_{n+2}^{(4)} + S_{n+3}^{(4)} + 3S_{n+1}^{(4)} = 2S_{n+4}^{(4)} - S_{n+3}^{(4)} + s_{n+1}^{(4)}$

$L_n^{(3)} = 2F_{n+2}^{(3)} - F_{n+1}^{(3)} + F_{n-1}^{(3)}$

Theorem 9 The generating function of sequence $\{W_n^{(3)}\}_{n=0}^{\infty}$ is given by.

$$D_{(x)} = \sum_{n=0}^{\infty} w_n \, x^n$$
$$= \frac{a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (b_2 - a_2 - a_1 - a_0)x^3 + (b_1 - a_2 - a_1)x^4 + (b_0 - a_2)x^5}{1 - x - x^2 - x^3}$$

Special cases.

$1 - \;$ When $a_0 = 0, a_1 = 0, a_2 = 1, b_l = 1$ where $l = 0,1,2$

then $D_x = \sum_{n=0}^{\infty} W_n^{(3)} \, x^n = \frac{x^2}{1 - x - x^2 - x^3}.$

D_x is generating function of Tribonacci

$2 - \;$ when $a_0 = 3, a_1 = 1, a_2 = 3 \,, b_0 = 3 \,, b_1 = 4, b_2 = 7$

then $D_{(x)} = \sum_{n=0}^{\infty} w_n \, x^n = \frac{3 - 2x - x^2}{1 - x - x^2 - x^3}.$

$D_{(x)}$ is generating function of Tribonacci – Lucas

Case 2.4. When $i \geq 5$

The sequence as shown in Table 4.

Table 4: Sequence of Bernoulli when $i \geq 5$

n	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$s_n^{(5)}$	1	1	2	4	8	15	29	56	108	208	401	773	1490	2872	5536
$s_n^{(6)}$	0	1	1	2	4	8	16	31	61	120	236	464	912	1793	3525
$s_n^{(7)}$	0	0	1	1	2	4	8	16	32	63	125	248	492	976	1936
$s_n^{(8)}$	0	0	0	1	1	2	4	8	16	32	64	127	253	504	1004
$s_n^{(9)}$	0	0	0	0	1	1	2	4	8	16	32	64	128	255	509
$s_n^{(10)}$	0	0	0	0	0	1	1	2	4	8	16	32	64	128	256

From table 4, $S_n^{(i)} = \sum_{k=1}^{i-1} S_{n-k}^{(i)} \quad n \geq i+2$ (2.12)

with the initial conditions

$S_n^{(i)} = 0\,, n = 1,2, \ldots i-1\ , S_i^{(i)} = S_{i+1}^{(i)} = 1$

We can see comparing the previous values of $s_{n+3}^{(i+1)}$ with those values of $F_n^{(i)}$ shown in table 4. This shows that $s_{n+3}^{(i+1)} = F_n^{(i)}$

Then $F_n^{(i)} = \sum_{k=1}^{i-1} F_{n-k}^{(i)}\ , n \geq i$ and $F_n^{(i)} = 0\,,$

$$n = 1,2,.,i-3\,, F_{i-1}^{(i)} = F_{i-2}^{(i)} = 1.$$

2.6 Definition 6 For any real numbers a_i and b_i the sequences $\{W_n^{(i)}\}_{n=0}^{\infty}$ is defined recursively by

$$W_n^{(i)} = \sum_{k=2}^{i+1} b_{i-k+1}\, S_{n+(-1)^m(k-i)}^{(i+1)}\,, \qquad n \geq i,$$

$$W_j^i = a_j\,, j = 0,1,.,i-1 \qquad (2.13)$$

$b_j\,, j = 0,1 \ldots . i-1$

where $s_n^{(i)}$ is above definition.

Special cases.

$1-$ When $a_j = 0$ where $j = 0,1, \ldots \ldots \ldots, i-2, a_{i-1} = 1,$

$b_j = 1, j = 0,1, \ldots \ldots \ldots .., i-1, m = 2$ Then

$W_j^{(i)} = 0\,, j = 0,1, \ldots ., i-2, W_{i-1}^{(i)} = 1$

$W_n^{(i)} = \sum_{k=2}^{i+1} b_{i-k+1}\ S_{n+k-i}^{(i+1)} \qquad n \geq i$

$\{W_n^{(i)}\}_{n=0}^{\infty}$ is order $-$ i Fibonacci sequence where $i \geq 4$

$2-$ when $a_0 = i, a_1 = 1, a_j = 2a_{j-1}+1\ , j = 2,3, \ldots, i-1$

$b_0 = 3, b_1 = 4, b_2 = i + b_1\,, b_j = b_{j-1} - 1\,, j = 3,4, \ldots ., i-1, m = 1$

Then $W_0^{(i)} = i, W_1^{(i)} = 1, W_j^{(i)} = 2W_{j-1}^{(i)} + 1, j = 2,3, \ldots, i-2$

$$w_n^{(i)} = \sum_{k=2}^{i+1} b_{i-k+1}\, S_{n+i-k}^{(i+1)} \qquad n \geq i$$

$\{W_n^{(i)}\}_{n=0}^{\infty}$ is order $-$ i Lucas sequence where $i \geq 4$

Theorem 10 the sequences $\{W_n^{(3)}\}_{n=0}^{\infty}$ satisfy following

$L_n^{(n)} = F_{2n}^{(n)}$

Theorem 11 (Generating function of $\{W_n^{(i)}\}_{n=0}^{\infty}, i \geq 4$) . The generating function of sequence $\{W_n^{(i)}\}_{n=0}^{\infty}$ is given by

$$D_{(x)}^i = \sum_{n=0}^{\infty} W_n^i x^n = \frac{\sum_{k=0}^{i-1}(a_k - a_{k-1} - \cdots \ldots .. - a_0)x^k}{1 - x - x^2 - \cdots \ldots \ldots \ldots - x^i}\ i \geq 4 \qquad (2.14)$$

Special cases.

$1-$ When $a_j = 0$ where $j = 0,1, \ldots \ldots \ldots, i-2, a_{i-1} = 1,$

Then $D_{(x)}^{(i)} = \sum_{n=0}^{\infty} W_n^{(i)} x^n = \frac{x^{i-1}}{1 - x - x^2 - \cdots - x^i}$

D_x^i is generating function of order $-$ i Fibonacci

$2-$ when $a_0 = i, a_1 = 1, a_j = 2a_{j-1}+1\ , j = 2,3, \ldots, i-2$

Then $D_{(x)}^i = \sum_{n=0}^{\infty} W_n^i x^n = \frac{\sum_{k=0}^{i-1}(a_k - a_{k-1} - \cdots \ldots .. - a_0)x^k}{1 - x - x^2 - \cdots \ldots \ldots \ldots - x^i}$

D_x^i is generating function of order $-$ i Lucas

3. Conclusions and Recommendations

The sequences of Bernoulli numbers allow to obtained Fibonacci and Lucas numbers.

New formula for sequences of Fibonacci and Lucas for a run of n successes in a sequence of Bernoulli trials can be obtained to a closed form for special values of the real numbers a and b.

Obtained the closed form for the order -i Fibonacci and Lucas and their generator function together.

Connection between Fibonacci and Lucas with the help of Bernoulli sequence were derived.

We studied the probability distribution function (pdf) of the waiting time (W (k)) problem both in independent and in homogeneous two state Markovian trials and express the pdf of W (2) for symmetric trials, we used a relationship with the 2th order Fibonacci and Lucas sequence.

4. Suggestions

The researcher recommends using the k-order relationship of the Fibonacci and Lucas sequences to study the probability distribution function of the waiting time

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