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# Set Inner Amenability for Semigroups

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**Abstract**

In this paper, we present a new concept of inner amenability for a non-empty arbitrary subset $A$ of discrete semigroup $S$ called $A$-inner amenablility. This condition is considerably weaker than ordinary inner amenability. Further, we show some relationships between this version of inner amenability and Følner's condition.

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## 1. Introduction

Throughout this paper, $S$ will denote a discrete semigroup. We shall use $\ell^\infty(S)$ to denote the Banach space of bounded real-valued functions on $S$ with the supremum norm. For every subset $A$ of $S$, let $\chi_A$ denote its characteristic function, that is

$$\chi_A(s) = \begin{cases} 1 & s \in A \\ 0 & s \notin A \end{cases}$$

A mean is a linear functional $m \in \ell^\infty(S)^*$ such that $m(\chi_S) = \|m\| = 1$. For each $s \in S$ and $f \in \ell^\infty(S)$ we define ${}_sf$ and $f_s$ on $S$ by $({}_sf)(t) = f(st)$ and $(f_s)(t) = f(ts)$ for all $t \in S$. We say that $m \in \ell^\infty(S)^*$ is invariant if $m({}_sf) = m(f) = m(f_s)$ for all $s \in S$ and $f \in \ell^\infty(S)$. A semigroup $S$ is said to be amenable if it has an invariant mean $m$ on $\ell^\infty(S)$. Also, let $\ell^1(S)$ denote the Banach space of all real-valued functions $\varphi$ on $S$ such that $\|\varphi\|_1 := \sum_{x \in S} |\varphi(x)| < \infty$. With pointwise addition and scalar multiplication, and with convolution

$$(\varphi * \psi)(x) = \sum_{st=x} \varphi(s)\psi(t) \quad (x \in S),$$

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as product, $\ell^1(S)$ is a Banach algebra.

We say that $m \in \ell^\infty(S)^*$ is inner invariant mean if

$$m({}_sf) = m(f_s),$$

for all $s \in S$ and $f \in \ell^\infty(S)$. Following Ling[11], a semigroup $S$ is said to be inner amenable if it has an inner invariant mean $m$ on $\ell^\infty(S)$.

We will show that many results concerning inner amenability of semigoups have similar analogues for $A$-inner amenability. Finally, a number of equivalent conditions characterizing $A$-inner amenable semigroups is given.

## 2. Set inner amenability for semigroup

We start off with the following definition, which is the most important here.

**Definition 2.1.** *Let $S$ be a semigroup and $\phi \neq A \subseteq S$. We say that a mean $m$ on $\ell^\infty(S)$, is an inner $A$-invariant mean if for all $a \in A$ and $f \in \ell^\infty(S)$ we have*

$$m({}_af) = m(f_a).$$

*A semigroup $S$ which admits inner $A$-invariant means is called $A$-inner amenable.*

In other words, invarince of $m$ is only required in the subsets of $S$. It follows immediately that every inner amenable semigroup is $A$-inner amenable for all subsets $A$ of $S$. But the converse is not true in genaral. (see Examples 3.2 and 3.4)

For an arbitrary non-empty subset $A$ of semigroup $S$, we denote by $\mathcal{H}(A)$, the real linear span of functions of the form ${}_af - f_a$, where $a \in A$ and $f \in \ell^\infty(S)$. In the following theorem, a sequence of characterizations of $A$-inner amenable semigroup is given.

**Theorem 2.2.** *Let $S$ be a semigroup with non-empty subset $A$. Then the following proporties are equivalent:*

*(a) $S$ is an $A$-inner amenable semigroup.*

*(b) for every $h \in \mathcal{H}(A)$, $\sup\{h(x) : x \in S\} \geq 0$.*

*(c) $\inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\} = 1$.*

*Proof.* $(a) \Rightarrow (b)$. Let $m$ be an inner $A$-invariant mean on $\ell^\infty(S)$. If $h \in \mathcal{H}(A)$, then $\sup\{h(x) : x \in S\} \geq m(h) = 0$. Thus, the property (b) holds.

$(b) \Rightarrow (c)$. For every $h \in \mathcal{H}(A)$, we have

$$0 \leq \sup\{-h(x) : x \in S\} = -\inf\{h(x) : x \in S\}.$$

This shows that, $\inf\{h(x) : x \in S\} \leq 0$. Hence, for any $\epsilon > 0$, there exists $x_0 \in S$ such that $h(x_0) < \epsilon$, and so $1 - h(x_0) > 1 - \epsilon$. Therefore, $\|1 - h\|_\infty \geq 1$ for any $h \in \mathcal{H}(A)$. But $0 \in \mathcal{H}(A)$, $\inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\} \leq \|1 - 0\|_\infty = 1$.

$(c) \Rightarrow (a)$. Assume that the property (c) holds. Now by the Hahn-Banach theorem, there exists a linear functional $m$ on $\ell^\infty(S)$ with norm one such that $m(\mathcal{H}(A)) = \{0\}$ and $m(1) = \inf\{\|1 - h\|_\infty : h \in \mathcal{H}(A)\}$. So $m$ is an inner $A$-invariant mean on $\ell^\infty(S)$. □

A non-empty subset $A$ of $S$ is said to act injectively on the left (right) of semigroup $S$, if $ax = ay$ $(xa = ya)$ implies $x = y$ for every $a \in A, x, y \in S$. We say that $A$ acts injectively on the semigroup $S$, if it acts on both left and right of $S$. In particular, if $S$ is a cancellative semigroup, then every non-empty subset of $S$ acts injectively on $S$.

**Theorem 2.3.** *Let $A$ act injectively on the left of semigroup $S$. Then $S$ is $A$-inner amenable if and only if $\mathcal{H}(A)$ is not norm dense in $\ell^\infty(S)$.*

*Proof.* We suppose that $m$ be a nonzero self-adjoint functional $m \in \ell^\infty(S)^*$ such that $m(\mathcal{H}(A)) = 0$. Consider the decomposition $m = m^+ - m^-$, such that

$$m^+(f) = \sup\{m(g) : 0 \le g \le f\}$$

and

$$m^-(f) = -\inf\{m(g) : 0 \le g \le f\},$$

for all $f \in \ell^\infty(S)$ with $f \ge 0$. A similar proof of Teorem 2 of [11], shows that $m^+$ and $m^-$ are inner $A$-invariant mean on $\ell^\infty(S)$. □

In the following proposition, we see that increasing union of a family of $A$-inner amenable semigroups is $A$-inner amenable.

**Proposition 2.4.** *Let $\{S_\alpha\}_{\alpha\in I}$ be a family of subsemigroups of $S$ such that for each $\alpha \in I$, $S_\alpha$ is $A_\alpha$-inner amenable and $A = \bigcup_{\alpha\in I} A_\alpha$ with the following conditions:*

*$(a)$ for each $S_\alpha, S_\beta$ that are $A_\alpha$-inner amenable and $A_\beta$-inner amenable, respectively, there exists $S_\gamma \supseteq S_\alpha \cup S_\beta$ such that $S_\gamma$ is $A_\gamma$-inner amenable with $A_\gamma \supseteq A_\alpha \cup A_\beta$.*

*$(b)$ $S = \bigcup_{\alpha\in I} S_\alpha$.*

*Then $S$ is $A$-inner amenable.*

*Proof.* Assume that $h = \sum_{k=1}^{n} ({}_{a_k}(f_k) - (f_k)_{a_k})$ such that $f_k \in \ell^\infty(S)$, $a_k \in A$. By the assumption, there exists a $S_\lambda$ such that $a_k \in A_\lambda$. Since $S_\lambda$ is $A_\lambda$-inner amenable, it follows from Theorem 2.2, $\sup\{h(x) : x \in S_\lambda\} \ge 0$. In particular, $\sup\{h(x) : x \in S\} \ge 0$. Again by Theorem 2.2, $S$ is $A$-inner amenable.

□

**Remark 2.5.** *A subsemigroup of an $A$-inner amenable semigroup need not be $A$-inner amenable. As an example let $S$ be any non $A$-inner amenable semigroup, and let $S^o$ contain $S$ and one new element $o$ such that $os = so = oo = o$, and $S$ is a subsemigroup of $S^o$. Then $S^o$ has an inner $A$-invariant mean: $m(f) = f(o)$, whereas $S$ is not an $A$-inner amenable.*

**Theorem 2.6.** *Let $T$ be a subsemigroup of $S$ and $A \subseteq T$. Then $T$ is an $A$-inner amenable if and only if $S$ is an $A$-inner amenable with mean $m$ such that $m(\chi_T) = 1$.*

*Proof.* Let $\theta : T \longrightarrow S$ be the embedding map. Then it induces $\overline{\theta} : \ell^\infty(S) \longrightarrow \ell^\infty(T)$ by $\overline{\theta}(f) = f|_T$. It is easily that $\overline{\theta}$ is bounded and linear. Consider $\overline{\theta}^* : \ell^\infty(T)^* \longrightarrow \ell^\infty(S)^*$. Now suppose that $m \in \ell^\infty(T)^*$ is an inner $A$-invariant mean. Clearly $\overline{\theta}^*(m)$ is a mean on $\ell^\infty(S)$. Also, for any $f \in \ell^\infty(S)$, $a \in A$, it is easy to see that

$$\overline{\theta}({}_af) = ({}_af)|_T = {}_a(f|_T) = {}_a(\overline{\theta}(f)),$$

and

$$\overline{\theta}(f_a) = (f_a)|_T = (f|_T)_a = (\overline{\theta}(f))_a.$$

Therefore, for all $f \in \ell^\infty(S)$, $a \in A$, we get

$$(\overline{\theta}^*(m))({}_af) = m(\overline{\theta}({}_af)) = m({}_a(\overline{\theta}(f))) = m((\overline{\theta}(f))_a) = (\overline{\theta}^*(m))(f_a).$$

This means that, $\overline{\theta}^*(m)$ is an inner $A$-invariant mean on $\ell^\infty(S)$. Also,

$$(\overline{\theta}^*(m))(\chi_T) = m(\overline{\theta}(\chi_T)) = m(\chi_T|_T) = m(1) = 1.$$

Conversely, Suppose that $m$ is an inner $A$-invariant mean on $\ell^\infty(S)$ such that $m(\chi_T) = 1$. Define the mapping $\varphi : \ell^\infty(T) \longrightarrow \ell^\infty(S)$ by

$$\varphi(f)(t) = \begin{cases} f(t) & t \in T \\ 0 & t \in S\backslash T \end{cases}$$

It is obvious that $\varphi$ is a bounded and linear. Consider $\varphi^* : \ell^\infty(S)^* \longrightarrow \ell^\infty(T)^*$. For any $f \in \ell^\infty(T)$ with $f \geq 0$, we have $\varphi(f) \geq 0$. It is easy to see that $\varphi^*(m)$ is a mean on $\ell^\infty(T)$. Also, for any $f \in \ell^\infty(T)$, $a \in A$ and $t \in T$, we get

$$(\varphi({}_a f) - {}_a(\varphi(f)))(t) = ({}_a f)(t) - (\varphi(f))(at) = f(at) - f(at) = 0.$$

So, $(\varphi({}_a f) - {}_a(\varphi(f)))|_T = 0$, and

$$|\varphi({}_a f) - {}_a(\varphi(f))| \leq ||\varphi({}_a f) - {}_a(\varphi(f))||_u \chi_{S\backslash T}.$$

This implies that $m(\varphi({}_a f) - {}_a(\varphi(f))) = 0$, or, $m(\varphi({}_a f)) = m({}_a(\varphi(f)))$. Similarly, one can show that $m(\varphi(f_a)) = m((\varphi(f))_a)$.

Therefore,

$$\begin{aligned} (\varphi^*(m))({}_a f) &= m(\varphi({}_a f)) = m({}_a(\varphi(f))) \\ &= m((\varphi(f))_a) = m(\varphi(f_a)) \\ &= (\varphi^*(m))(f_a). \end{aligned}$$

This shows that, $\varphi^*(m)$ is an inner $A$-invariant mean on $\ell^\infty(T)$. □

Given semigroups $S$ and $T$, a map $\varphi : S \longrightarrow T$ is called a homomorphism if it satisfies

$$\varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2) \ (s_1, s_2 \in S).$$

**Theorem 2.7.** *Let $S$, $T$ be semigroups and $\varphi$ be a homomorphism of $S$ onto $T$. If $S$ is $A$-inner amenable, then $T$ is $\varphi(A)$-inner amenable.*

*Proof.* Assume that $m$ is an inner $A$-invariant mean on $\ell^\infty(S)$. Put $m_o(f) = m(fo\varphi)$ for each $f \in \ell^\infty(T)$. Now for every $s \in S$, $b \in B = \varphi(A)$ and $f \in \ell^\infty(T)$ we have

$${}_b fo\varphi(s) = f(b\varphi(s)) = f(\varphi(a)\varphi(s)) = f(\varphi(as)) = (fo\varphi)(as) = {}_a(fo\varphi)(s),$$

and

$$f_b o\varphi(s) = f(\varphi(s)b) = f(\varphi(s)\varphi(a)) = f(\varphi(sa)) = (fo\varphi)(sa) = (fo\varphi)_a(s),$$

where $a \in A$ is such that $\varphi(a) = b$. So, ${}_b fo\varphi = {}_a(fo\varphi)$ and $f_b o\varphi = (fo\varphi)_a$. It follows from this relations that

$$m_o({}_b f) = m({}_b fo\varphi) = m({}_a(fo\varphi)) = m((fo\varphi)_a) = m(f_b o\varphi) = m_o(f_b).$$

Thus $m_o$ is an inner $\varphi(A)$-invariant mean. □

Let $S$ and $T$ be semigroups. Then $S \times T$ is a semigroup with the operation $(s_1, t_1)(s_2, t_2) = (s_1 s_2, t_1 t_2)$ for all $s_1, s_2 \in S$ and $t_1, t_2 \in T$. Also we can consider $\ell^\infty(S \times T)$ as a Banach $S \times T$-bimodule via

$$({}_{(s,t)}f)(s', t') = f(ss', tt'), \quad \text{and} \quad (f_{(s,t)})(s', t') = f(s's, t't),$$

for all $s, s' \in S$, $t, t' \in T$ and $f \in \ell^\infty(S \times T)$. The homomorphisms $\pi_S : S \times T \longrightarrow S$ and $\pi_T : S \times T \longrightarrow T$ with $\pi_S(s, t) = s$, $\pi_T(s, t) = t$, respectively, are called projection homomorphisms.

**Theorem 2.8.** *Let $S$, $T$ be semigroups such that $\ell^\infty(S \times T) = \ell^\infty(S) \times \ell^\infty(T)$. $S$ and $T$ are $A$-inner amenable and $B$-inner amenable, respectively if and only if $S \times T$ is $(A \times B)$-inner amenable.*

*Proof.* Suppose that $m$ and $n$ are inner $A$-invariant and inner $B$-invariant means for $\ell^\infty(S)$ and $\ell^\infty(T)$, respectively. Define the mean $m_o$ on $\ell^\infty(S\times T)$ by $m_o(f,g)=m(f)n(g)$ for all $f\in\ell^\infty(S)$ and $g\in\ell^\infty(T)$. Then for each $(a,b)\in A\times B$

$$\begin{aligned} m_o({}_{(a,b)}(f,g)) &= m_o({}_af,{}_bg)=m({}_af)n({}_bg)\\ &= m(f_a)n(g_b)=m_o(f_a,g_b)\\ &= m_o((f,g)_{(a,b)}). \end{aligned}$$

This means that $m_o$ is inner $(A\times B)$-invariant mean.

Conversely, suppose that $S\times T$ is $(A\times B)$-inner amenable. Then by projection homomorphism $\pi_S(A\times B)=A$ and Theorem 2.7, we obtain that $S$ is $A$-inner amenable. Similarly, we conclude that $T$ is $B$-inner amenable. □

**Theorem 2.9.** *Let $S$, $T$ be two semigroups such that $S$ and $T$ are $A$-amenable and $B$-inner amenable, respectively. Then $S\times T$ is $(A\times B)$-inner amenable.*

*Proof.* Suppose that $m$ be an $A$-invariant mean on $\ell^\infty(S)$ and $n$ be an inner $B$-invariant mean on $\ell^\infty(T)$. For each $f\in\ell^\infty(S\times T)$ and $(s,t)\in S\times T$, we consider $f_T\in\ell^\infty(T)$ and $f_S^t\in\ell^\infty(S)$ by $f_S^t(s)=f(s,t)$ and $f_T(t)=m(f_S^t)$. Now, define the mean $m_o$ on $\ell^\infty(S\times T)$ by

$$m_o(f)=n(f_T)\ \text{ for all }\ f\in\ell^\infty(S\times T).$$

For every $(a,b)\in A\times B$ it follows that $({}_{(a,b)}f)_S^t={}_a(f_S^{bt})$ and $(f_{(a,b)})_S^t=(f_S^{tb})_a$.
Furthermore, for every $t\in T$

$$\begin{aligned} ({}_{(a,b)}f)_T(t) &= m(({}_{(a,b)}f)_T^t)=m({}_a(f_S^{bt}))\\ &= m(f_S^{bt})=(f_T(bt))\\ &= {}_b(f_T)(t). \end{aligned}$$

That is, $({}_{(a,b)}f)_T={}_b(f_T)$. Similarly, one find that $(f_{(a,b)})_T=(f_T)_b$.

For every $f\in\ell^\infty(S\times T)$ and $(a,b)\in A\times B$ we get

$$\begin{aligned} m_o({}_{(a,b)}f) &= n(({}_{(a,b)}f)_T)=n({}_b(f_T))\\ &= n((f_T)_b)=n((f_{(a,b)})_T)\\ &= m_o(f_{(a,b)}). \end{aligned}$$

It follows that $m$ is an inner $(A\times B)$-invariant mean. □

## 3. Examples of *A*-inner amenability

**Example 3.1.** *If there exists an element $x$ in semigroup $S$ that commutes with all $a\in A$, then the Dirac measure $\delta_x$ for all $f\in\ell^\infty(S)$ is an inner $A$-invariant mean on $\ell^\infty(S)$.*

$$\delta_x({}_af)=f(ax)=f(xa)=\delta_x(f_a).$$

In the following examples, we study $A$-inner amenability over a left (right) zero semigroup, that is a semigroup whose multiplication is defined by $st=s$ $(st=t)$ for all $s,t\in S$. We denote the cardinal number of a set $A$ by $|A|$.

**Example 3.2.** *Let $S$ be a left zero semigroup, then for any subset $A$ of $S$:*

*(i) if $|A|=1$, then $S$ is $A$-inner amenable;*

*(ii) if $|A|\geq 2$, then $S$ is not $A$-inner amenable.*

*Proof.* (i) Assume that $A = \{a\}$. Define $m \in \ell^\infty(S)^*$ by $m(f) = f(a)$ for every $f \in \ell^\infty(S)$. Then we obtain $m({}_af) = {}_af(a) = f(aa) = f_a(a) = m(f_a)$. This shows that $S$ is $A$-inner amenable.

(ii) Clearly for each $a \in A$ we have

$$ {}_af = f(a) \quad \text{and} \quad f_a = f. $$

Now, if we suppose that $S$ is $A$-inner amenable with an inner $A$-invariant mean $m$, then for every $a \in A$ and $f \in \ell^\infty(S)$, we have $m({}_af) = m(f_a)$. Therefore $f(a) = m(f)$. Now if we consider $a \neq b \in A$ and $f = \chi_{\{a\}}$ then we obtain

$$ 1 = f(a) = m(f) = f(b) = 0. $$

This is a contradiction. □

**Example 3.3.** *Let $\mathbb{F}_2$ be free group on two generators $a$ and $b$. If $A$ is the set of elements of $\mathbb{F}_2$ that begin with $a$ or $a^{-1}$ when written as reduced words. Then $\mathbb{F}_2$ is not $A$-inner amenable.*

*Proof.* We consider $f = \chi_A$ and

$$ h = (({}_{ba^{-1}}f)_{ab^{-1}} - {}_{ab^{-1}}({}_{ba^{-1}}f)) + (({}_{b^{-1}a^{-1}}f)_{aba} - {}_{aba}({}_{b^{-1}a^{-1}}f)). $$

Clearly $h \in \mathcal{H}(A)$. Now by Theorem 2.2, it is enough to prove that the function $h$ has the property that, $\sup\{h(x) : x \in \mathbb{F}_2\} < 0$. For each $x \in \mathbb{F}_2$ we have

$$ h(x) = f(ba^{-1}xab^{-1}) + f(b^{-1}a^{-1}xaba) - f(ax) - f(x). $$

Now the argument as in the proof of Theorem (17.16) of [7] shows $\sup\{h(x) : x \in \mathbb{F}_2\} \leq -1$. □

By use of Theorem 2.2, in the following example, we study $A$-inner amenability over a right zero semigroup.

**Example 3.4.** *Let $S$ be a right zero semigroup, then for any subset $A$ of $S$ we have*

*(i) if $|A| = 1$, then $S$ is $A$-inner amenable.*

*(ii) if $|A| \geq 2$, then $S$ is not $A$-inner amenable.*

*Proof.* (i) Assume that $A = \{a\}$. Since for every $h \in \mathcal{H}(A)$ and $x \in S$ we have

$$ \begin{aligned} h(x) &= \sum_{k=1}^{n}((f_k)_a - {}_a(f_k))(x) \\ &= \sum_{k=1}^{n}(f_k(xa) - f_k(ax)) \\ &= \sum_{k=1}^{n}(f_k(a) - f_k(x)). \end{aligned} $$

Then by set $x = a$ we have $\sup\{h(x) : x \in S\} \geq 0$. This shows that $S$ is $A$-inner amenable.

(ii) For $a \neq b \in A$, we take $h = ({}_a(\chi_{\{a\}}) - (\chi_{\{a\}})_a) + ({}_b(\chi_{\{b\}}) - (\chi_{\{b\}})_b)$. Hence for each $x \in S$ we obtain

$$ \begin{aligned} h(x) &= ({}_a(\chi_{\{a\}}) - (\chi_{\{a\}})_a)(x) + ({}_b(\chi_{\{b\}}) - (\chi_{\{b\}})_b)(x) \\ &= (\chi_{\{a\}}(ax) - \chi_{\{a\}}(xa)) + (\chi_{\{b\}}(bx) - \chi_{\{b\}}(xb)) \\ &= \chi_{\{a\}}(x) + \chi_{\{b\}}(x) - 2. \end{aligned} $$

and this implise that $\sup\{h(x) : x \in S\} \leq -1$. Hence by theorem 2.2, $S$ is not $A$-inner amenable. □

## 4. Følner's condition

Before stating the following theorem, recall that a mean in $\ell^1(S)$ is called a finite mean if it is a convex combination of the Dirac measures. We shall use $\Phi$ denote the set of all finite means and $\delta_x$ denotes the Dirac measure at $x \in S$. It is obvious that $\Phi$ is convex subset of $\ell^1(S)$ . In fact, $\Phi$ is convex hull of $S$.

**Theorem 4.1.** *Let $S$ be a semigroup and $A \subseteq S$. Then the following statements are equivalent:*

*(a) $S$ is $A$-inner amenable.*

*(b) there is a net $(\varphi_\alpha)$ of finite means such that $\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a \longrightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.*

*(c) there is a net $(\psi_\alpha)$ of finite means such that $\|\delta_a * \psi_\alpha - \psi_\alpha * \delta_a\|_1 \longrightarrow 0$ for every $a \in A$.*

*Proof.* $(a) \Rightarrow (b)$. Let $m$ be an inner $A$-invariant mean on $\ell^\infty(S)$. Since $m \in \ell^\infty(S)^*$, we can find a net $(\varphi_\alpha)$ of finite means such that $\lim_\alpha \varphi_\alpha = m$ in the weak* topology of $\ell^\infty(S)^*$. Then for all $f \in \ell^\infty(S)$ and $a \in A$,

$$f(\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a) = \varphi_\alpha({}_af) - \varphi_\alpha(f_a) \longrightarrow m({}_af) - m(f_a) = 0.$$

It follows that $\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a \longrightarrow 0$ in the weak topology of $\ell^1(S)$, for every $a \in A$.

$(b) \Rightarrow (c)$. Let $(\varphi_\beta)$ be a net as in (b). Using an idea of Ling [11], we define linear map $T : \ell^1(S) \longrightarrow \prod_{a\in A} \ell^1(S)$ by $(T(\varphi))(a) = \delta_a * \varphi - \varphi * \delta_a$ for every $\varphi \in \ell^1(S), a \in A$. Now by assumption, $(T(\varphi_\beta))(a) = \delta_a * \varphi_\beta - \varphi_\beta * \delta_a \longrightarrow 0$ weakly in $\ell^1(S)$, for every $a \in A$. This means that zero lies in the weak closure of $T(\Phi)$. Since $\prod_{a\in A} \ell^1(S)$ with product of the norm topology is a locally convex space and $\Phi$ is convex, the closure of $T(\Phi)$ in this topology contains 0. Thus, there exists a subnet $(\varphi_\alpha) \subseteq (\varphi_\beta)$ such that $\|\delta_a * \psi_\alpha - \psi_\alpha * \delta_a\|_1 \longrightarrow 0$ for every $a \in A$.

$(c) \Rightarrow (b)$. Since convergence in norm implies convergence in weak topology, this implication is trivial.

$(b) \Rightarrow (a)$. Let $(\varphi_\alpha)$ be a net satisfying the convergence in (b). By Alaoglu's theorem, it has a weak* convergent subnet. By passing to such a subnet if necessary, there is a $m \in \ell^\infty(S)^*$ such that $\lim_\alpha \varphi_\alpha = m$ in the weak* topology of $\ell^\infty(S)^*$ . Therefore $m$ is a mean on $\ell^\infty(S)$, and for all $a \in A$, $f \in \ell^\infty(S)$

$$m({}_af) - m(f_a) = \lim_\alpha(\varphi_\alpha({}_af) - \varphi_\alpha(f_a)) = \lim_\alpha(\delta_a * \varphi_a - \varphi_\alpha * \delta_a)(f) = 0.$$

□

For each $s \in S$ we put $s^{-1}A = \{t \in S : st \in A\}$ and $As^{-1} = \{t \in S : ts \in A\}$. We also note that $\frac{1}{|A|}\chi_A$ defines an element in $\ell^1(S)$.

**Lemma 4.2.** *Let $A$ acts injectively on the right of semigroup $S$, then for every $B \subseteq A$ and $a \in A$*

$$||\chi_B * \delta_a - \delta_a * \chi_B||_1 = 2|Ba \backslash aB|.$$

*Proof.* For $a \in A$ and $B \subseteq A$, we get

$$\begin{aligned}(\delta_a * \chi_B)(x) &= \sum_{as=x} \chi_B(s) \\ &= \sum_{s\in a^{-1}\{x\}} \chi_B(s) \\ &= |B \cap a^{-1}\{x\}|.\end{aligned}$$

Similarly, we obtain $(\chi_B * \delta_a)(x) = |B \cap \{x\}a^{-1}|$. It is easy to see that

$$(\chi_B * \delta_a - \delta_a * \chi_B)(x) = \begin{cases} |B \cap \{x\}a^{-1}| & \text{if } x \in Ba\backslash aB \\ -|B \cap a^{-1}\{x\}| & \text{if } x \in aB\backslash Ba \\ |B \cap \{x\}a^{-1}| - |B \cap a^{-1}\{x\}| & \text{if } x \in aB \cap Ba \\ 0 & \text{if } x \notin aB \cup Ba \end{cases}$$

Since $A$ acts injectively on the right of semigroup $S$, then for each $x \in Ba$ we obtain $|B \cap \{x\}a^{-1}| = 1$. This implies that

$$\begin{aligned}
\|\chi_B * \delta_a - \delta_a * \chi_B\|_1 &= \sum_{x \in Ba \setminus aB} 1 + \sum_{x \in aB \setminus Ba} |B \cap a^{-1}\{x\}| + \sum_{x \in aB \cap Ba} (|B \cap a^{-1}\{x\}| - 1) \\
&= |Ba \setminus aB| + \sum_{x \in Ba} |B \cap a^{-1}\{x\}| - |aB \cap Ba| \\
&= |Ba \setminus aB| + |B| - |aB \cap Ba| \\
&= |Ba \setminus aB| + |Ba| - |aB \cap Ba| \\
&= |Ba \setminus aB| + |Ba \setminus aB| \\
&= 2|Ba \setminus aB|
\end{aligned}$$

□

**Theorem 4.3.** *Let $A$ act injectively on the right of semigroup $S$. If for any finite set $F \subseteq A$ and any $\varepsilon > 0$, there exists a finite non-empty set $B \subseteq A$ such that $|Ba \setminus aB| < \epsilon |B|$ for all $a \in F$, then $S$ is $A$-inner amenable.*

*Proof.* By the assumption there exists a net of finite non-empty sets $B_\alpha \subseteq A$ such that

$$|B_\alpha a \setminus aB_\alpha| / |B_\alpha| \longrightarrow 0 \text{ for all } a \in A.$$

By Lemma 4.2, we have

$$||\chi_{B_\alpha} * \delta_a - \delta_a * \chi_{B_\alpha}||_1 = |B_\alpha a \setminus aB_\alpha|.$$

Set $\varphi_\alpha = |B_\alpha|^{-1} \chi_{B_\alpha}$. Then for $\alpha$, and $a \in A$

$$||\delta_a * \varphi_\alpha - \varphi_\alpha * \delta_a||_1 \longrightarrow 0.$$

Now the proof is complete by Theorem 4.1. □

**Remark 4.4.** *The assumption of Theorem 4.3 'that $A$ acts injectively on the right of semigroup $S$' is necessary. In fact, any right zero semigroup $S$ is not $A$-inner amenable if $A$ has at least two elements (see Example 3.2).*

## References


[1] C. A. Akemann, *Operator algebras associated with fuschian groups*, Houston J. Math. 7 (1981), 295-301.
[2] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer–Verlag, Berlin, 1973.
[3] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys, 7, American Math. Soc., Providence, 1961.
[4] M. M. Day, *Amenable semigroups*, Illinois. J. Math. 1 (1957), 509-544. New York, 1969, 5-53.
[5] E. G. Effros, Property and inner amenability, Proc. Amer. Math. Soc. 47 (1975), 483-486.
[6] E. Følner, *On groups with full Banach mean values*, Math. Scand., 3 (1955), 243-254.
[7] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis I.* Springer Verlag, 1963.
[8] J. M. Howie, *An introduction to semigroup theory*, Academic Press, London, 1976.
[9] J. L. Kelley, *General Topology.* Van Nostrand, 1955.
[10] A. T. Lau and A. L. T. Paterson, Inner amenable locally compact groups, Trans. Amer. Math. Soc. 325 (1991), 155-169.
[11] J. M. Ling, *Inner amenable semigroups*, J. Math. Soc. Japan. 49 (1997), 603-616.
[12] I. Namioka, *Fϕlner's conditions for amenable semi-groups*, Math. Scand. 15 (1964), 18-28.
[13] A. L. T. Paterson, *Amenability.* American Mathematical Society, 1988.
[14] J. P. Pier, *Amenable Locally Compact Groups.* Wiley-Interscience, 1984.
[15] C. K. Yuan, Conjugate convolutions and inner invariant means, J. Math. Anal. Appl. 157 (1991), 166-178.