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On the Trigonometric and p-Trigonometric Functions of Elliptical Complex Variables

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Abstract
In the early 2000s, the geometry of a one-parameter family of generalized complex number systems was studied (Math. Mag. **77**(2)(2004)). This family is denoted by $\mathbb{C}_p$. It is well known that $\mathbb{C}_p$ matches up with the elliptical complex number system when p is any negative real number. By using this system, Özen and Tosun expressed the elliptical complex valued trigonometric functions cosine, sine and p-trigonometric functions p-cosine, p-sine (Adv. Appl. Clifford Algebras **28**(3)(2018)). In this study, we introduce the remained elliptical complex valued trigonometric and p-trigonometric functions. Also we define the corresponding single-valued principal values of the inverse trigonometric and p-trigonometric functions by following the similar steps given in the literature.

Keywords: Generalized complex numbers, p-trigonometric functions, Elliptical complex numbers.
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1. Introduction

The generalized complex numbers were introduced by Yaglom [1] as in the following:

$$z = x + Iy \ \ (x, y \in \mathbb{R}), \ \ I^2 = Iq + p \ \ (q, p \in \mathbb{R})$$

where I denotes a formal quantity which is subject to the relation indicated above.

In [2], Harkins studied the geometry of a one parameter family of generalized complex number systems. In this one parameter family, $q = 0$ and $I^2 = p \in \mathbb{R}$. It is denoted by

$$\mathbb{C}_p = \left\{ x + Iy \ : \ x, y \in \mathbb{R}, \, I^2 = p, \ p \in \mathbb{R} \right\}.$$

In the special case $p < 0$, $\mathbb{C}_p$ corresponds to the set of elliptical complex numbers. Let this set be denoted by ${\mathbb{C}_p}^*$. That is,

$${\mathbb{C}_p}^* = \left\{ x + Iy \ : \ x, y \in \mathbb{R}, \, I^2 = p, \ p \in \mathbb{R}^- \right\}.$$

For $z_1 = (x_1 + Iy_1)$, $z_2 = (x_2 + Iy_2) \in {\mathbb{C}_p}^*$, addition and multiplication are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + Iy_1) + (x_2 + Iy_2) = (x_1 + x_2) + I(y_1 + y_2) \\ z_1 z_2 &= (x_1 x_2 + p y_1 y_2) + I(x_1 y_2 + x_2 y_1). \end{aligned}$$

As it is well known, ${\mathbb{C}_p}^*$ is a field under these two operations [2].

On the other hand, the p-magnitude of $z = x + Iy \in \mathbb{C}_p{}^*$ is $\|z\|_p = \sqrt{x^2 - py^2}$. As a result of this case, the unit circle in $\mathbb{C}_p{}^*$ is an Euclidean ellipse which is given by the equation $x^2 - py^2 = 1$. Specially, if $p = -1$ this ellipse matches the Euclidean unit circle [2].

Let $z = x + Iy$ be a number in $\mathbb{C}_p{}^*$. This number can be expressed with a position vector (see Figure 1.1). The arc of ellipse between this vector and the real axis determines an elliptic angle θ_p. This angle is called p-argument of z.

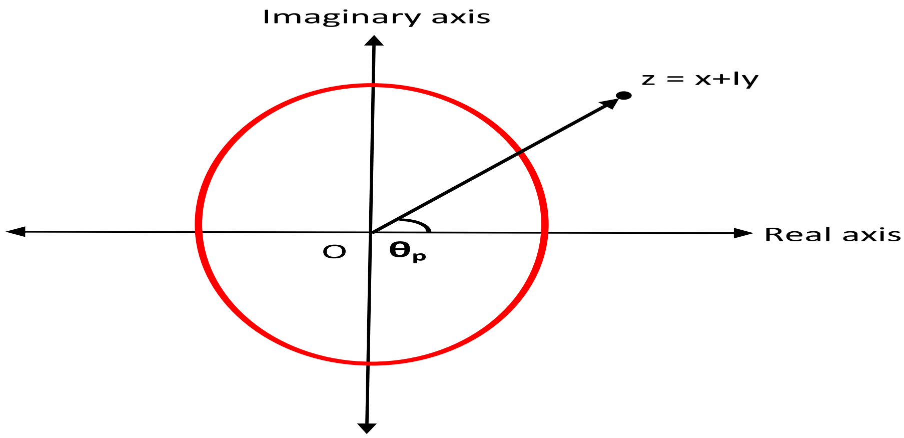

Figure 1.1. Elliptic angle in $\mathbb{C}_p{}^*$

On the other hand, the p-trigonometric functions p-cosine, p-sine and p-tangent are defined in $\mathbb{C}_p{}^*$ as follows [2]:

$$\cos_p(\theta_p) = \cos\left(\theta_p\sqrt{|p|}\right) \tag{1.1}$$

$$\sin_p(\theta_p) = \frac{1}{\sqrt{|p|}}\sin\left(\theta_p\sqrt{|p|}\right) \tag{1.2}$$

$$\tan_p(\theta_p) = \frac{\sin_p(\theta_p)}{\cos_p(\theta_p)}. \tag{1.3}$$

There can be found some interesting studies [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] on the generalized complex numbers and elliptical complex numbers in the literature.

Recently, Özen and Tosun have extended the trigonometric functions cosine, sine and p-trigonometric functions p-cosine, p-sine to the elliptical complex variables [3]. The functions cos, sin, $\cos_p$ and $\sin_p$ of an elliptical complex variable $\varphi_p = x + Iy \in \mathbb{C}_p$ are given as in the following

$$\cos(\varphi_p) = \cos(x)\cosh\left(y\sqrt{|p|}\right) - I\frac{1}{\sqrt{|p|}}\sin(x)\sinh\left(y\sqrt{|p|}\right) \tag{1.4}$$

$$\sin(\varphi_p) = \sin(x)\cosh\left(y\sqrt{|p|}\right) + I\frac{1}{\sqrt{|p|}}\cos(x)\sinh\left(y\sqrt{|p|}\right) \tag{1.5}$$

$$\cos_p(\varphi_p) = \cos_p(x)\cosh(py) + I\sin_p(x)\sinh(py) \tag{1.6}$$

$$\sin_p(\varphi_p) = \sin_p(x)\cosh(py) + I\frac{1}{p}\cos_p(x)\sinh(py) \tag{1.7}$$

in which case, φ_p is called elliptical complex angle. Also, these functions hold the following relations [3]:

$$\cos_p(\varphi_p) = \cos\left(\varphi_p\sqrt{|p|}\right)$$

$$\sin_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right).$$

Let the set of generalized complex numbers be showed with $\mathbb{C}_G$ in the case $I^2 = -q - rI\left(r^2 - 4q < 0\right)$. Thanks to Yaglom [1], it is known that there is an isomorphism between the set $\mathbb{C}_G$ and the set $\mathbb{C}$ as in the following:

$$\pi : \mathbb{C}_G \to \mathbb{C}$$

$$a_1 + b_1 I \to \pi(a_1 + b_1 I) = \left(a_1 - \frac{r}{2}b_1\right) + \left(\frac{b_1}{2}\sqrt{4q - r^2}\right)i.$$

If this isomorphism is restricted to the set of elliptical complex numbers, the following isomorphism

$$\begin{aligned} \pi^* \ : \ \mathbb{C}_p{}^* &\to \mathbb{C} \\ a_1 + b_1 I &\to \pi^*(a_1 + b_1 I) = a_1 + i b_1 \sqrt{|p|} \end{aligned}$$

is immediately written by considering $r = 0$ and $q = -p$. Here the statement $\sqrt{|p|}$ represents the positive square root of the positive number $|p|$. Throughout the paper the statement $\sqrt{|p|}$ will be used in this sense.

Theorem 1.1. *[13] For the elliptical complex valued sine and cosine functions, the equalities*

1. $\sin(\pi^*(\varphi_p)) = \pi^*(\sin(\varphi_p))$
2. $\cos(\pi^*(\varphi_p)) = \pi^*(\cos(\varphi_p))$

are satisfied where $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$.

The next two theorems, which reveal that the elliptical complex valued p-trigonometric functions $\cos_p(\varphi_p)$ and $\sin_p(\varphi_p)$ are surjective, can be given as consequences of the last theorem.

Theorem 1.2. *[3] For any elliptical complex number* $\psi_p = a + Ib \in \mathbb{C}_p{}^*$*, the equality* $\cos_p\left(\lambda_p^k\right) = \psi_p$ *is satisfied by the elliptical complex angles*

$$\lambda_p^k = \frac{Arg(u_k + iv_k)}{\sqrt{|p|}} + I\frac{\ln|u_k + iv_k|}{p}, \ \ k = 1,2$$

where $u_1 + iv_1, u_2 + iv_2 \in \mathbb{C}$ *are the complex numbers derived from the expression* $\left(a + ib\sqrt{|p|} + \sqrt{\left(a + ib\sqrt{|p|}\right)^2 - 1}\right)$.

Theorem 1.3. *[13] For any elliptical complex number* $\psi_p = a + Ib \in \mathbb{C}_p{}^*$*, the equality* $\sin_p\left(\chi_p^k\right) = \psi_p$ *is satisfied by the elliptical complex angles*

$$\chi_p^k = \frac{Arg(\varsigma_k + i\tau_k)}{\sqrt{|p|}} + I\frac{\ln|\varsigma_k + i\tau_k|}{p}, \ \ k = 1,2$$

where $\varsigma_1 + i\tau_1, \varsigma_2 + i\tau_2 \in \mathbb{C}$ *are complex numbers derived from the expression* $\left(i\left(a\sqrt{|p|} + ib|p|\right) + \sqrt{1 - \left(a\sqrt{|p|} + ib|p|\right)^2}\right)$.

Note that the last three theorems will be used to obtain single-valued principal values of the inverse cosine, sine, p-cosine and p-sine functions in Section 2.

Finally, we need to emphasize the principal square root of a complex number. Let $z = re^{i\varphi}$ be a complex number given by principal argument $-\pi < \varphi \le \pi$ in the polar form. As it is well-known in the literature, the principal square root of z is defined as $\sqrt{z} = \sqrt{r}e^{i\frac{\varphi}{2}}, \ -\frac{\pi}{2} < \frac{\varphi}{2} \le \frac{\pi}{2}$. We will use the statement "principle square root" in this sense throughout the rest of the paper.

2. Main Results

In this section, we obtain the elliptical complex valued tangent, cotangent, secant and cosecant functions. Then we define the corresponding single-valued principal values of the all inverse trigonometric functions by following the similar steps in [14]. Finally, we will repeat the same for $p-$trigonometric functions.

2.1 Results Related to Elliptical Complex-Valued Trigonometric Functions

In this subsection, firstly, we can give the following theorem by using the equations (1.4) and (1.5).

Theorem 2.1. *Tangent, cotangent, secant and cosecant functions of an elliptical complex variable* $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$ *are given as in the following:*

1. $\tan(\varphi_p) = \frac{\sin(\varphi_p)}{\cos(\varphi_p)} = \frac{\sin(2x)}{\cos(2x) + \cosh\left(2y\sqrt{|p|}\right)} + I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x) + \cosh\left(2y\sqrt{|p|}\right)}$,

2. $\cot(\varphi_p) = \frac{\cos(\varphi_p)}{\sin(\varphi_p)} = \frac{\sin(2x)\left(\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)\right)}{\sin^2(2x)+\sinh^2\left(2y\sqrt{|p|}\right)} - I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)\left(\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)\right)}{\sin^2(2x)+\sinh^2\left(2y\sqrt{|p|}\right)}$,

3. $\sec(\varphi_p) = \frac{1}{\cos(\varphi_p)} = \frac{2\cos(x)\cosh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)} + I\frac{2}{\sqrt{|p|}}\frac{\sin(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}$,

4. $\csc(\varphi_p) = \frac{1}{\sin(\varphi_p)} = \frac{2\sin(x)\cosh\left(y\sqrt{|p|}\right)}{\cosh\left(2y\sqrt{|p|}\right)-\cos(2x)} - I\frac{2}{\sqrt{|p|}}\frac{\cos(x)\sinh\left(y\sqrt{|p|}\right)}{\cosh\left(2y\sqrt{|p|}\right)-\cos(2x)}$.

Proof. We will prove the first item. The proofs of other items can be similarly completed.

1. By considering $|p| = -p$ and using some well-known trigonometric and hyperbolic identities, we get

$$\begin{aligned}
\tan(\varphi_p) &= \frac{\sin(\varphi_p)}{\cos(\varphi_p)} \\
&= \frac{\sin(x)\cosh\left(y\sqrt{|p|}\right) + I\frac{1}{\sqrt{|p|}}\cos(x)\sinh\left(y\sqrt{|p|}\right)}{\cos(x)\cosh\left(y\sqrt{|p|}\right) - I\frac{1}{\sqrt{|p|}}\sin(x)\sinh\left(y\sqrt{|p|}\right)} \\
&= \frac{\sin(x)\cos(x)\left(\cosh^2\left(y\sqrt{|p|}\right) - \sinh^2\left(y\sqrt{|p|}\right)\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} + \frac{I}{\sqrt{|p|}}\frac{\sinh\left(y\sqrt{|p|}\right)\cosh\left(y\sqrt{|p|}\right)\left(\cos^2(x)+\sin^2(x)\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} \\
&= \frac{2}{2}\frac{\sin(x)\cos(x)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} + \frac{I}{\sqrt{|p|}}\frac{2}{2}\frac{\sinh\left(y\sqrt{|p|}\right)\cosh\left(y\sqrt{|p|}\right)}{\cos^2(x)\cosh^2\left(y\sqrt{|p|}\right) + \sin^2(x)\sinh^2\left(y\sqrt{|p|}\right)} \\
&= \frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)} + I\frac{1}{\sqrt{|p|}}\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}.
\end{aligned}$$

□

Lemma 2.2. *For the elliptical complex valued tangent, cotangent, secant and cosecant functions, the equalities*

1. $\tan(\pi^*(\varphi_p)) = \pi^*(\tan(\varphi_p))$,
2. $\cot(\pi^*(\varphi_p)) = \pi^*(\cot(\varphi_p))$,
3. $\sec(\pi^*(\varphi_p)) = \pi^*(\sec(\varphi_p))$,
4. $\csc(\pi^*(\varphi_p)) = \pi^*(\csc(\varphi_p))$.

are satisfied where π^* is the aforesaid isomorphism and $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$.

Proof. We will prove the first and third item. Other items can be similarly proved.

1. It is very easy to see

$$\begin{aligned}
\pi^*(\tan(\varphi_p)) &= \pi^*\left(\left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + I\frac{1}{\sqrt{|p|}}\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right]\right) \\
&= \left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + i\frac{1}{\sqrt{|p|}}\sqrt{|p|}\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] \\
&= \left[\frac{\sin(2x)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right] + i\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos(2x)+\cosh\left(2y\sqrt{|p|}\right)}\right].
\end{aligned}$$

On the other hand, according to the theory of complex trigonometric functions (see [14, 15] for more details on the theory of complex trigonometric functions), it is clear that

$$\begin{aligned}\tan\left(\pi^*\left(\varphi_p\right)\right) &= \tan\left(x+iy\sqrt{|p|}\right)\\ &= \left[\frac{\sin\left(2x\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{\sinh\left(2y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

So, the proof is completed.

3. Similarly above, we have the equalities

$$\begin{aligned}\pi^*\left(\sec\left(\varphi_p\right)\right) &= \pi^*\left(\left[\frac{2\cos\left(x\right)\cosh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]+I\frac{2}{\sqrt{|p|}}\left[\frac{\sin\left(x\right)\sinh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]\right)\\ &= \left[\frac{2\cos\left(x\right)\cosh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\frac{2}{\sqrt{|p|}}\sqrt{|p|}\left[\frac{\sin\left(x\right)\sinh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]\\ &= \left[\frac{2\cos\left(x\right)\cosh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{2\sin\left(x\right)\sinh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

and

$$\begin{aligned}\sec\left(\pi^*\left(\varphi_p\right)\right) &= \sec\left(x+iy\sqrt{|p|}\right)\\ &= \left[\frac{2\cos\left(x\right)\cosh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right]+i\left[\frac{2\sin\left(x\right)\sinh\left(y\sqrt{|p|}\right)}{\cos\left(2x\right)+\cosh\left(2y\sqrt{|p|}\right)}\right].\end{aligned}$$

Thus the desired equality holds.

□

Theorem 2.3. *For any elliptical complex number $\psi_p = a+Ib \in \mathbb{C}_p{}^*$, the equalities $\sin\varphi_p = \psi_p$, $\cos\alpha_p = \psi_p$, $\tan\beta_p = \psi_p$, $\cot\gamma_p = \psi_p$, $\sec\theta_p = \psi_p$ and $\csc\delta_p = \psi_p$ are satisfied by the principal elliptical complex angles*

1. $\varphi_p = Arg\left(\sigma+i\omega\right) - I\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}$,

2. $\alpha_p = Arg\left(\varepsilon+i\kappa\right) - I\frac{\ln|\varepsilon+i\kappa|}{\sqrt{|p|}}$,

3. $\beta_p = \dfrac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2} + I\dfrac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}$,

4. $\gamma_p = \dfrac{Arg\left(\frac{-1-pb^2+a^2}{1+2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1+2b\sqrt{|p|}+a^2-pb^2}\right)}{-2} + I\dfrac{\ln\left|\frac{-1-pb^2+a^2}{1+2b\sqrt{|p|}+a^2-pb^2} - i\frac{2a}{1+2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}$,

5. $\theta_p = Arg\left(\eta+i\zeta\right) - I\frac{\ln|\eta+i\zeta|}{\sqrt{|p|}}$,

6. $\delta_p = Arg\left(\Omega+i\mho\right) - I\frac{\ln|\Omega+i\mho|}{\sqrt{|p|}}$,

where $\sigma+i\omega\in\mathbb{C}$, $\varepsilon+i\kappa\in\mathbb{C}$, $\eta+i\zeta\in\mathbb{C}$ *and* $\Omega+i\mho\in\mathbb{C}$ *are the principal complex values derived from the expressions* $\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)$, $\left(a+ib\sqrt{|p|}+\sqrt{\left(a+ib\sqrt{|p|}\right)^2-1}\right)$, $\left(\frac{1}{a+ib\sqrt{|p|}}+\sqrt{\frac{1}{\left(a+ib\sqrt{|p|}\right)^2}-1}\right)$ *and* $\left(\sqrt{1-\frac{1}{\left(a+ib\sqrt{|p|}\right)^2}}+\frac{i}{a+ib\sqrt{|p|}}\right)$*, respectively.*

Proof. Now, we will show that the first and third equalities are satisfied. Similar steps can be followed for the other equalities.

1. By considering Theorem 1.1 and the theory of complex trigonometric functions (see [14, 15] for more details), we can write

$$\begin{aligned}\sin(x+Iy)=a+Ib \quad &\Leftrightarrow \quad \pi^*(\sin(x+Iy))=\pi^*(a+Ib)\\ &\Leftrightarrow \quad \sin(\pi^*(x+Iy))=\pi^*(a+Ib)\\ &\Leftrightarrow \quad \sin\left(x+iy\sqrt{|p|}\right)=a+ib\sqrt{|p|}\\ &\Leftrightarrow \quad \arcsin\left(a+ib\sqrt{|p|}\right)=x+iy\sqrt{|p|}\\ &\Leftrightarrow \quad -i\log\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)=x+iy\sqrt{|p|}.\end{aligned}$$

The purpose of us is to get unique solutions for x and y. To do so, we use the principal value of arcsine function. It is determined by employing the principal value of the logarithm function and the principal value of the square-root function. By keeping these situations in mind, let us denote by $\sigma+i\omega$ the principal complex value derived from the expression $\left(i\left(a+ib\sqrt{|p|}\right)+\sqrt{1-\left(a+ib\sqrt{|p|}\right)^2}\right)$. Then we have

$$-i\,\mathrm{Log}(\sigma+i\omega)=x+iy\sqrt{|p|}.$$

This equation yields the followings

$$\begin{aligned}-i(\ln|\sigma+i\omega|+iArg(\sigma+i\omega)) &= x+iy\sqrt{|p|},\\ Arg(\sigma+i\omega)-i\ln|\sigma+i\omega| &= x+iy\sqrt{|p|}.\end{aligned}$$

Then we get the unique solutions for x and y as

$$x=Arg(\sigma+i\omega),\ y=-\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}.$$

Thus, we can conclude

$$\varphi_p=Arg(\sigma+i\omega)-I\frac{\ln|\sigma+i\omega|}{\sqrt{|p|}}.$$

3. Similarly above, we can write

$$\begin{aligned}\tan(x+Iy)=a+Ib \quad &\Leftrightarrow \quad \pi^*(\tan(x+Iy))=\pi^*(a+Ib)\\ &\Leftrightarrow \quad \tan(\pi^*(x+Iy))=\pi^*(a+Ib)\\ &\Leftrightarrow \quad \tan\left(x+iy\sqrt{|p|}\right)=a+ib\sqrt{|p|}\\ &\Leftrightarrow \quad \arctan\left(a+ib\sqrt{|p|}\right)=x+iy\sqrt{|p|}\\ &\Leftrightarrow \quad \frac{i}{2}\log\left(\frac{i+\left(a+ib\sqrt{|p|}\right)}{i-\left(a+ib\sqrt{|p|}\right)}\right)=x+iy\sqrt{|p|}.\end{aligned}$$

We aim to obtain the unique solutions for x and y. To do so, if we use the principal value of arctangent function which is determined by employing the principal value of the logarithm function, we have

$$\frac{i}{2}\operatorname{Log}\left(\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right)=x+iy\sqrt{|p|}.$$

This equation yields the followings

$$\frac{i}{2}\left(\ln\left|\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right|+iArg\left(\frac{a+i\left(1+b\sqrt{|p|}\right)}{-a+i\left(1-b\sqrt{|p|}\right)}\right)\right) = x+iy\sqrt{|p|},$$

$$\frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2}+i\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2} = x+iy\sqrt{|p|}.$$

In this case, we obtain the unique solutions for x and y as follows

$$x=\frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2},\quad y=\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}.$$

Therefore, we can conclude

$$\beta_p=\frac{Arg\left(\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right)}{-2}+I\frac{\ln\left|\frac{1+pb^2-a^2}{1-2b\sqrt{|p|}+a^2-pb^2}-i\frac{2a}{1-2b\sqrt{|p|}+a^2-pb^2}\right|}{2\sqrt{|p|}}. \quad \square$$

By taking into consideration Theorem 2.3, we can give the following corollary.

Corollary 2.4. *For any elliptical complex number $\psi_p=a+Ib\in\mathbb{C}_p{}^*$, the principal values of the inverse trigonometric functions:*

$$\begin{aligned}
\operatorname{Arcsin}(\psi_p) &= \varphi_p\\
\operatorname{Arccos}(\psi_p) &= \alpha_p\\
\operatorname{Arctan}(\psi_p) &= \beta_p\\
\operatorname{Arccot}(\psi_p) &= \gamma_p\\
\operatorname{Arcsec}(\psi_p) &= \theta_p\\
\operatorname{Arccsc}(\psi_p) &= \delta_p
\end{aligned}$$

can be expressed.

2.2 Results Related to Elliptical Complex-Valued p-Trigonometric Functions

In this subsection, firstly, let us define the elliptical complex valued p-trigonometric functions:

$$\frac{\sin_p(\varphi_p)}{\cos_p(\varphi_p)}=\tan_p(\varphi_p),\quad \frac{\cos_p(\varphi_p)}{\sin_p(\varphi_p)}=\cot_p(\varphi_p),\quad \frac{1}{\cos_p(\varphi_p)}=\sec_p(\varphi_p),\quad \frac{1}{\sin_p(\varphi_p)}=\csc_p(\varphi_p)$$

by means of the elliptical complex valued p-trigonometric functions

$$\cos_p(\varphi_p) = \cos\left(\varphi_p\sqrt{|p|}\right) = \cos_p(x)\cosh(py)+I\sin_p(x)\sinh(py)$$

and

$$\sin_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right) = \sin_p(x)\cosh(py)+I\frac{1}{p}\cos_p(x)\sinh(py)$$

given in (1.6) and (1.7).

As mentioned earlier in Section 1, real-valued p-trigonometric functions p-cosine, p-sine and p-tangent are defined in [2]. There is no such definition for neither cotangent function, secant function nor cosecant function. While the elliptical complex-valued functions $\cos_p(\varphi_p)$, $\sin_p(\varphi_p)$ and $\tan_p(\varphi_p)$ are extensions of real-valued functions p-cosine, p-sine and p-tangent, we can not say the same for the elliptical complex-valued functions $\cot_p(\varphi_p)$, $\sec_p(\varphi_p)$ and $\csc_p(\varphi_p)$. So, to use the notations $\cos_p(\varphi_p)$, $\sin_p(\varphi_p)$, $\tan_p(\varphi_p)$ and to use the statement "p-trigonometric function" are very natural for these functions. But, the reason of maintaining this situation for other functions $\cot_p(\varphi_p)$, $\sec_p(\varphi_p)$ and $\csc_p(\varphi_p)$ is not obvious. This reason is based on the relationships of these functions with the elliptical complex-valued trigonometric functions cotangent, secant and cosecant. Now, we give the next theorem including these relationships.

Theorem 2.5. *For any elliptical complex angle $\varphi_p = x + Iy \in \mathbb{C}_p{}^*$, the following equalities hold:*

1. $\tan_p(\varphi_p) = \frac{1}{\sqrt{|p|}} \tan\left(\varphi_p\sqrt{|p|}\right) = \frac{\sin_p(2x)}{\cos_p(2x)+\cosh(2yp)} + I\frac{\sinh(2yp)}{p\cos_p(2x)+p\cosh(2yp)}$,
2. $\cot_p(\varphi_p) = \sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right) = \frac{\sin_p(2x)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sin_p{}^2(2x)-\frac{1}{p}\sinh^2(2yp)\right)} + I\frac{\sinh(2yp)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sinh^2(2yp)-p\sin_p{}^2(2x)\right)}$,
3. $\sec_p(\varphi_p) = \frac{1}{\sqrt{|p|}}\sec\left(\varphi_p\sqrt{|p|}\right) = \frac{2\cos_p(x)\cosh(yp)}{\cos_p(2x)+\cosh(2yp)} - I\frac{2\sin_p(x)\sinh(yp)}{\cos_p(2x)+\cosh(2yp)}$,
4. $\csc_p(\varphi_p) = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) = \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp)-\cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp)-\cos_p(2x)}$.

Proof. We will prove the second and last items. The other items can be proved similarly.

2. It is easy to see the equality

$$\cot_p(\varphi_p) = \frac{\cos_p(\varphi_p)}{\sin_p(\varphi_p)} = \frac{\cos\left(\varphi_p\sqrt{|p|}\right)}{\frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right).$$

On the other hand, since $\varphi_p\sqrt{|p|} = x\sqrt{|p|} + Iy\sqrt{|p|}$,

$$\begin{aligned}
\sqrt{|p|}\cot\left(\varphi_p\sqrt{|p|}\right) &= \sqrt{|p|}\left[\frac{\sin\left(2x\sqrt{|p|}\right)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)} - \frac{I}{\sqrt{|p|}}\frac{\sinh(2y|p|)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)}\right] \\
&= \frac{\frac{1}{\sqrt{|p|}}\sin\left(2x\sqrt{|p|}\right)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\frac{1}{\left(\sqrt{|p|}\right)^2}\left(\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)\right)} - \frac{I}{|p|}\frac{\sinh(2y|p|)\left(\cos\left(2x\sqrt{|p|}\right)+\cosh(2y|p|)\right)}{\frac{1}{\left(\sqrt{|p|}\right)^2}\left(\sin^2\left(2x\sqrt{|p|}\right)+\sinh^2(2y|p|)\right)} \\
&= \frac{\sin_p(2x)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sin_p{}^2(2x)-\frac{1}{p}\sinh^2(2yp)\right)} + I\frac{\sinh(2yp)\left(\cos_p(2x)+\cosh(2yp)\right)}{\left(\sinh^2(2yp)-p\sin_p{}^2(2x)\right)}
\end{aligned}$$

can be written from the second item of Theorem 2.1. Then, we can immediately obtain the desired equality.

4. It is not difficult to find the equality

$$\csc_p(\varphi_p) = \frac{1}{\sin_p(\varphi_p)} = \frac{1}{\frac{1}{\sqrt{|p|}}\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\frac{1}{\sin\left(\varphi_p\sqrt{|p|}\right)} = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right).$$

Also, from the fourth item of Theorem 2.1

$$\begin{aligned}
\sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) &= \sqrt{|p|}\left[\frac{2\sin\left(x\sqrt{|p|}\right)\cosh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} - \frac{2I}{\sqrt{|p|}}\frac{\cos\left(x\sqrt{|p|}\right)\sinh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)}\right] \\
&= \frac{2|p|\frac{1}{\sqrt{|p|}}\sin\left(x\sqrt{|p|}\right)\cosh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} - 2I\frac{\cos\left(x\sqrt{|p|}\right)\sinh(y|p|)}{\cosh(2y|p|)-\cos\left(2x\sqrt{|p|}\right)} \\
&= \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp)-\cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp)-\cos_p(2x)}
\end{aligned}$$

can be written by keeping $\varphi_p\sqrt{|p|} = x\sqrt{|p|} + Iy\sqrt{|p|}$ in mind. From above, we immediately get

$$\csc_p(\varphi_p) = \sqrt{|p|}\csc\left(\varphi_p\sqrt{|p|}\right) = \frac{-2p\sin_p(x)\cosh(yp)}{\cosh(2yp) - \cos_p(2x)} + I\frac{2\cos_p(x)\sinh(yp)}{\cosh(2yp) - \cos_p(2x)}. \quad \square$$

Theorem 2.6. *For any elliptical complex number $\psi_p = a + Ib \in \mathbb{C}_p{}^*$, the equalities $\cos_p(\lambda_p) = \psi_p$, $\sin_p(\chi_p) = \psi_p$, $\tan_p(\Gamma_p) = \psi_p$, $\cot_p(\Lambda_p) = \psi_p$, $\sec_p(\Delta_p) = \psi_p$ and $\csc_p(\Upsilon_p) = \psi_p$ are satisfied by the principal elliptical complex angles*

1. $\lambda_p = \frac{Arg(u+iv)}{\sqrt{|p|}} + I\frac{\ln|u+iv|}{p}$,

2. $\chi_p = \frac{Arg(\varsigma+i\tau)}{\sqrt{|p|}} + I\frac{\ln|\varsigma+i\tau|}{p}$,

3. $\Gamma_p = \dfrac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|} + i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}} + I\dfrac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|} + i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}$,

4. $\Lambda_p = \dfrac{Arg\left(\frac{-a^2+pb^2-p}{-a^2+pb^2+2bp+p} + i\frac{2a\sqrt{|p|}}{-a^2+pb^2+2bp+p}\right)}{-2\sqrt{|p|}} + I\dfrac{\ln\left|\frac{-a^2+pb^2-p}{-a^2+pb^2+2bp+p} + i\frac{2a\sqrt{|p|}}{-a^2+pb^2+2bp+p}\right|}{2|p|}$

5. $\Delta_p = \frac{Arg(c+id)}{\sqrt{|p|}} + I\,\frac{\ln|c+id|}{p}$,

6. $\Upsilon_p = \frac{Arg(e+if)}{\sqrt{|p|}} + I\,\frac{\ln|e+if|}{p}$,

where $u+iv \in \mathbb{C}$, $\varsigma + i\tau \in \mathbb{C}$, $c+id \in \mathbb{C}$ and $e+if \in \mathbb{C}$ are the principal complex values which are derived from the expressions $\left(a+ib\sqrt{|p|} + \sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}\right)$, $\left(i\left(a\sqrt{|p|}+ib|p|\right) + \sqrt{1-\left(a\sqrt{|p|}+ib|p|\right)^2}\right)$, $\left(\frac{1}{a+ib\sqrt{|p|}} + \sqrt{\frac{1}{\left(a+ib\sqrt{|p|}\right)^2} - 1}\right)$, $\left(\sqrt{1-\frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}} + \frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)$, *respectively.*

Proof. We will show that the first, third and last equalities are satisfied. Similar steps can be followed for the other equalities.

1. Let us take into consideration the principle value of $\sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}$ and calculate the principle value of the statement $\left(a+ib\sqrt{|p|} + \sqrt{\left(a+ib\sqrt{|p|}\right)^2 - 1}\right)$. If we show this principle value with $u+iv$, Theorem 1.2 gives the proof of this item.

3. By considering Lemma 2.2 and the theory of complex trigonometric functions, we can write

$$\begin{aligned}
\tan_p(x+Iy) = a+Ib \quad &\Leftrightarrow \quad \frac{1}{\sqrt{|p|}}\tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right) = a+Ib \\
&\Leftrightarrow \quad \tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right) = a\sqrt{|p|} + Ib\sqrt{|p|} \\
&\Leftrightarrow \quad \pi^*\left(\tan\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right)\right) = \pi^*\left(a\sqrt{|p|} + Ib\sqrt{|p|}\right) \\
&\Leftrightarrow \quad \tan\left(\pi^*\left(x\sqrt{|p|} + Iy\sqrt{|p|}\right)\right) = \pi^*\left(a\sqrt{|p|} + Ib\sqrt{|p|}\right) \\
&\Leftrightarrow \quad \tan\left(x\sqrt{|p|} + iy|p|\right) = a\sqrt{|p|} + ib|p| \\
&\Leftrightarrow \quad \arctan\left(a\sqrt{|p|} + ib|p|\right) = x\sqrt{|p|} + iy|p| \\
&\Leftrightarrow \quad \frac{i}{2}\log\left(\frac{i+\left(a\sqrt{|p|}+ib|p|\right)}{i-\left(a\sqrt{|p|}+ib|p|\right)}\right) = x\sqrt{|p|} + iy|p|.
\end{aligned}$$

To get unique solutions for x and y is our aim. To do so, if we use the principal value of arctangent function which is determined by employing the principal value of the logarithm function, we obtain

$$\frac{i}{2}\operatorname{Log}\left(\frac{a\sqrt{|p|}+i(1+b|p|)}{-a\sqrt{|p|}+i(1-b|p|)}\right)=x\sqrt{|p|}+iy|p|$$

and so

$$\frac{i}{2}\operatorname{Log}\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)=x\sqrt{|p|}+iy|p|.$$

From here, the equalities

$$\begin{aligned}\frac{i}{2}\left(\ln\left|\frac{\left(pa^2-p^2b^2+1\right)+i\left(-2a\sqrt{|p|}\right)}{-pa^2+p^2b^2+1-2b|p|}\right|+iArg\left(\frac{\left(pa^2-p^2b^2+1\right)+i\left(-2a\sqrt{|p|}\right)}{-pa^2+p^2b^2+1-2b|p|}\right)\right) &= x\sqrt{|p|}+iy|p|\\ \frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2}+i\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2} &= x\sqrt{|p|}+iy|p|\end{aligned}$$

can be written. Thus we find the unique solutions for x and y as follows

$$x=\frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}},\quad y=\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}.$$

Therefore,

$$\Gamma_p=\frac{Arg\left(\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right)}{-2\sqrt{|p|}}+I\frac{\ln\left|\frac{pa^2-p^2b^2+1}{-pa^2+p^2b^2+1-2b|p|}+i\frac{-2a\sqrt{|p|}}{-pa^2+p^2b^2+1-2b|p|}\right|}{2|p|}$$

can be concluded.

6. By considering Lemma 2.2 and the theory of complex trigonometric functions, we can write

$$\begin{aligned}\csc_p(x+Iy)=a+Ib \quad &\Leftrightarrow \quad \sqrt{|p|}\csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)=a+Ib\\ &\Leftrightarrow \quad \csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)=\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}}\\ &\Leftrightarrow \quad \pi^*\left(\csc\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)\right)=\pi^*\left(\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}}\right)\\ &\Leftrightarrow \quad \csc\left(\pi^*\left(x\sqrt{|p|}+Iy\sqrt{|p|}\right)\right)=\pi^*\left(\frac{a}{\sqrt{|p|}}+I\frac{b}{\sqrt{|p|}}\right)\\ &\Leftrightarrow \quad \csc\left(x\sqrt{|p|}+iy|p|\right)=\frac{a}{\sqrt{|p|}}+ib\\ &\Leftrightarrow \quad \operatorname{arccsc}\left(\frac{a}{\sqrt{|p|}}+ib\right)=x\sqrt{|p|}+iy|p|\\ &\Leftrightarrow \quad -i\log\left(\sqrt{1-\frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}}+\frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)=x\sqrt{|p|}+iy|p|.\end{aligned}$$

The aim of us is to obtain unique solutions for x and y. For this reason, we use the principal value of arccosecant function. It is determined by employing the principal value of the logarithm function and the principal value of square-root function. By considering these cases, let us denote by $e+if$ the principal complex value derived from the expression $\left(\sqrt{1-\frac{1}{\left(\frac{a}{\sqrt{|p|}}+ib\right)^2}}+\frac{i}{\left(\frac{a}{\sqrt{|p|}}+ib\right)}\right)$. In this case, we have

$$-i\,\mathrm{Log}\,(e+if)=x\sqrt{|p|}+iy\,|p|\,.$$

This equation yields the followings

$$\begin{aligned}-i\left(\ln|e+if|+iArg\,(e+if)\right) &= x\sqrt{|p|}+iy\,|p|\,,\\ Arg\,(e+if)-i\ln|e+if| &= x\sqrt{|p|}+iy\,|p|\,.\end{aligned}$$

Then we get the unique solutions for x and y as

$$x=\frac{Arg\,(e+if)}{\sqrt{|p|}},\quad y=\frac{\ln|e+if|}{p}.$$

Thus, we can conclude

$$\Upsilon_p=\frac{Arg\,(e+if)}{\sqrt{|p|}}+I\,\frac{\ln|e+if|}{p}.\quad\square$$

By taking into account of Theorem 2.6, the following corollary can be given.

Corollary 2.7. *For any elliptical complex number $\psi_p=a+Ib\in\mathbb{C}_p{}^*$, the principal values of the inverse p-trigonometric functions:*

$$\begin{aligned}\mathrm{Arccos_p}\,(\psi_p) &= \lambda_p\\ \mathrm{Arcsin_p}\,(\psi_p) &= \chi_p\\ \mathrm{Arctan_p}\,(\psi_p) &= \Gamma_p\\ \mathrm{Arccot_p}\,(\psi_p) &= \Lambda_p\\ \mathrm{Arcsec_p}\,(\psi_p) &= \Delta_p\\ \mathrm{Arccsc_p}\,(\psi_p) &= \Upsilon_p\end{aligned}$$

can be expressed.

3. Conclusion

In this paper, the trigonometric and $p-$trigonometric functions of elliptical complex variables are considered. Also, the corresponding single-valued principle values of the inverse trigonometric and $p-$trigonometric functions are defined.

In the case $p=-1$, elliptical complex numbers correspond to complex numbers. As a result of this case, the elliptical complex valued trigonometric functions can be seen as generalized form of the complex valued trigonometric functions which have important roles in many areas of science.

In the future, the results obtained here may be used as a valuable tool in many areas of science just like in the case of complex valued trigonometric functions.

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