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On the Chlodowsky variant of Jakimovski-Leviatan-Păltănea Operators

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Highlights

- This paper focuses on a generalization of Păltănea operators.
- Approximation properties of the operators are established with the help of Korovkin's theorem.
- The convergence of the operators are examined in the weighted space of functions.
- Weighted modulus of continuity is used for the rate of convergence of the operators.

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Abstract

In the present paper, our purpose is to generalize the Jakimovski-Leviatan-Păltănea operators in the sense of Chlodowsky. After introducing the new operators we first obtain the moments of these operators in order to establish the convergency properties with the help of Korovkin's theorem. After that, we give the local approximation result and the Voronovskaya type theorem. We also examine the convergence properties of the operators in the weighted space of functions. Lastly we determine the rate of convergence of the operators with the aid of the weighted modulus of continuity.

1. INTRODUCTION

In 1969, Jakimovski and Leviatan [1] introduced a sequence of linear positive operators in terms of the Appell polynomials, which, indeed, is a generalization of the Szász [2] operators. The generalized operators are known as the Jakimovski-Leviatan operators in the literature and are of the form

$$(L_n f)(x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \tag{1}$$

provided that $p_k(nx) \geq 0$ for $x \in [0,\infty)$ and $g(1) \neq 0$. Here $\{p_k(nx)\}$ are the Appell polynomials generated by

$$g(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x) t^k$$

where $g(t) = \sum_{r=0}^{\infty} a_r t^r, a_0 \neq 0$ is a function analytic in the disc $|t| < R (\mathrm{R} > 1)$. For the special case $g(t) = 1$ the operators given in Equation (1) reduces to the Szász operators defined by

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$$(S_n f)(x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right). \tag{2}$$

One can find several studies on Jakimovski-Leviatan operators as searching the literature about the Korovkin's theory. We refer the readers to, for example [3-7].
Another generalization of the Szász operators was given by Phillips [8] almost 70 years ago. Phillips operators have the form

$$(P_n f)(x) = e^{-nx} f(0) + n \sum_{k=1}^{\infty} \mu_{n,k}(x) \int_0^{\infty} \mu_{n,k-1}(t) f(t) dt \tag{3}$$

where $\mu_{n,k}(x) = e^{-nx} \frac{(nx)^k}{k!}$. These operators have also been handled by many authors since they are defined.
In order to generalize the Phillips operators, Păltănea [9] constructed the following operators,

$$L_{\alpha,\rho}(f;x) = \sum_{k=1}^{\infty} \mu_{\alpha,k}(x) \int_0^{\infty} \Theta_{\alpha,k}^{\rho}(x) f(t) dt + f(0) e^{-\alpha x} \tag{4}$$

based on the parameters $\rho > 0, \alpha > 0$ where

$$\mu_{n,k}(x) = e^{-\alpha x} \frac{(\alpha x)^k}{k!} \quad \text{and} \quad \Theta_{\alpha,k}^{\rho}(x) = \frac{\alpha\rho(\alpha\rho t)^{k\rho-1}}{\Gamma(k\rho)} e^{-\alpha\rho t}. \tag{5}$$

These operators give a connection with the Szász operators in the limiting case. i.e., $\lim_{\rho\to\infty} L_{\alpha,\rho}(f;x) = S_\alpha(f;x)$, uniformly for all $x \in [0,\infty)$. Here $S_\alpha(f,x)$ is the Szász operators constructed by replacing $n \in \mathbb{N}$ by a continuous parameter $\alpha > 0$. Note that for $\rho = 1$ the Phillips operators $(P_n f)(x)$ are obtained.
Verma and Gupta [10] proposed the Jakimovski-Leviatan-Păltănea operators for $f \in C[0,\infty)$. These operators are the generalization of the operators given in Equation (4) and defined as

$$M_{n,\rho}(f,x) = \sum_{k=1}^{\infty} l_{n,k}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(x) f(t) dt + l_{n,0}(x) f(0) \tag{6}$$

where

$$l_{n,k} = \frac{e^{-nx}}{g(1)} p_k(nx). \tag{7}$$

In this case, for $g(z) = 1$ and $\rho = 1$ the operators in Equation (6) reduce to Phillips operators. Very recently, inspired by the study of Verma and Gupta, Mursaleen et. al [11] gave a generalized form of the Jakimovski-Leviatan-Păltănea operators by using Sheffer polynomials. They constructed the operators as

$$T_{n,\rho}^{*}(f;x) = \sum_{k=1}^{\infty} L_{n,k}^{*}(x) \int_0^{\infty} \Theta_{n,k}^{\rho}(x) f(t) dt + L_{n,k}^{*}(0) f(0) \tag{8}$$

where $L_{n,k}^{*}(x) = \frac{e^{-nxH(1)}}{A(1)} \tilde{p}_k(nx)$. Here $\tilde{p}_k$ are the Sheffer polynomials defined by $A(u)e^{xH(u)} = \sum_{k=0}^{\infty} \tilde{p}_k(x) u^k$ and $A(z) = \sum_{n=1}^{\infty} a_n z^n$ and $H(z) = \sum_{z=1}^{\infty} h_n z^n$ are analytic functions in the disc $|z| < R, (R > 1)$. For $H(t) = t$, the operators $T_{n,\rho}^{*}(f;x)$ turns into $M_{n,\rho}(f,x)$ given by Equation (6).

In the present work, we construct the Chlodowsky-type generalization of the Jakimovski-Leviatan-Păltănea operators given by Equation (6). The Chlodowsky generalization of the linear positive operators has been studied by several authors for the last two decades. For some examples related to this subject, one can take a look at the papers [12-15].
The rest of the paper have been systemized in the following way: We constructed the operators, derived the moments and established the Korovkin type theorem in Section 2. Some direct results related to the operators including local approximation and Voronovskaya type theorem were given in Section 3. We also gave weighted approximation results in this section. By the help of the weighted modulus of continuity we estimated the rate of convergence of our new operators.
Before passing on our main results, we here give some auxiliary lemmas, theorems and definitions we used in our study.

Lemma 1 [4] For all $x \in [0;1)$ the following equalities hold.

$$\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) = g(1)e^{\frac{n}{b_n}x},$$

$$\sum_{k=0}^{\infty} k\, p_k\left(\frac{n}{b_n}x\right) = \left\{\frac{n}{b_n}g(1)x + g'(1)\right\}e^{\frac{n}{b_n}x},$$

$$\sum_{k=0}^{\infty} k^2\, p_k\left(\frac{n}{b_n}x\right) = \left\{\left(\frac{n}{b_n}\right)^2 g(1)x^2 + \left(g(1)+2g'(1)\right)\left(\frac{n}{b_n}\right)x + g'(1) + g''(1)\right\}e^{\frac{n}{b_n}x},$$

$$\sum_{k=0}^{\infty} k^3\, p_k\left(\frac{n}{b_n}x\right) = \{\left(\frac{n}{b_n}\right)^3 g(1)x^3 + \left(4g(1)+3g'(1)\right)\left(\frac{n}{b_n}\right)^2 x^2 + \left(g(1)+8g'(1) + 3g''(1)\right)\left(\frac{n}{b_n}\right)x + g'(1) + 4g''(1) + g'''(1)\}e^{\frac{n}{b_n}x},$$

$$\sum_{k=0}^{\infty} k^4\, p_k\left(\frac{n}{b_n}x\right) = \left\{\left(\frac{n}{b_n}\right)^4 g(1)x^4 + \left(10g(1)+4g'(1)\right)\left(\frac{n}{b_n}\right)^3 x^3 + \left(14g(1)+30g'(1) + 6g''(1)\right)\left(\frac{n}{b_n}\right)^2 x^2 + \left(g(1)+28g'(1) + 30g''(1)+4g'''(1)\right)\left(\frac{n}{b_n}\right)x + g'(1) + 14g''(1) + 10g'''(1) + g^{(4)}(1)\right\}e^{\frac{n}{b_n}x}.$$

In order to examine the local approximation properties of the operators, we need the following definitions:

Let $f \in C_B[0,\infty)$ where $C_B[0,\infty)$ is the set containing all continuous and bounded functions on $[0,\infty)$. For $\delta > 0$, the modulus of continuity of f is defined by

$$\omega(f;\delta) = \sup_{|t-x|\leq\delta} |f(t) - f(x)|,\ t \in [0,\infty).$$

Recall that the second modulus of function $f \in C_B[0,\infty)$ is defined by

$$\omega_2(f;\delta) = \sup_{0\leq h\leq\delta}\sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

The Peetre's $K-$functional of the function $f \in C_B[0,\infty)$ is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\|\}.$$

Here $C_B^2[0,\infty)$ is the space of functions $f, f', f'' \in C_B[0,\infty)$. The norm on C_B^2 is defined as

$$\|g\|_{C_B^2} = \|g\|_{C_B} + \|g'\|_{C_B} + \|g''\|_{C_B}.$$

Note that, Theorem 2.4 in [16] implies that there exists a positive constant $C > 0$ such that

$$K_2(f;\delta) \leq C\omega_2(f;\delta). \tag{9}$$

Lastly, for further discussions, we also recall the weight function and the weighted spaces of functions on $[0,\infty)$. Gadjiev [17] proved an analogues Korovkin's Theorem for such kind of functions**.**

Let $\eta(x) = 1 + \varphi^2(x)$ be the weight function defined on $[0,\infty)$ such that $\varphi(x)$ is both increasing and continuous on the real axis. Let us consider the following spaces related to the function $\eta(x)$.

$$B_\eta[0,\infty) = \left\{f: [0,\infty) \to \mathbb{R}: |f(x)| \leq M_f \eta(x),\ M_f \text{ is constant depending on } f\right\}$$

$$\|f\|_\eta = \sup_{x\geq 0} \frac{|f(x)|}{\eta(x)}, \qquad f \in B_\eta[0,\infty),$$

$$C_\eta[0,\infty) = \left\{f: [0,\infty) \to \mathbb{R}: f \in B_\eta[0,\infty) \text{ and } f \in C[0,\infty)\right\},$$

$$C_\eta{}^*[0,\infty) = \left\{f: [0,\infty) \to \mathbb{R}: f \in C_\eta[0,\infty) \text{ and } \lim_{x\to\infty} \frac{|f(x)|}{\eta(x)} \text{ exists finitely}\right\}.$$

Theorem 2 [17]

The sequence of positive linear operators $(L_n)_{n\geq 1}$ act from $C_\eta[0,\infty)$ to $B_\eta[0,\infty)$ if and only if there exists a positive constant k such that $\|L_n(\eta)\|_\eta \leq k$.

Theorem 3 [17]

i) There exists a sequence of linear positive operators L_n acting from $C_\eta[0,\infty)$ to $B_\eta[0,\infty)$ such that

$$\lim_{n\to\infty} \left\|L_n(\varphi^k;.) - \varphi^k\right\|_\eta = 0 \quad (k = 0,1,2) \tag{10}$$

and a function $f^* \in C_\eta \backslash C_\eta{}^*$ with $\lim_{n\to\infty} \|L_n(f^*;.) - f^*\|_\eta \geq 1$.

ii) If a sequence of linear positive operators L_n acting from $C_\eta[0,\infty)$ to $B_\eta[0,\infty)$ satisfies the conditions in Equation (10), then

$$\lim_{n\to\infty} \|L_n(f;.) - f\|_\eta = 0$$

for every $f \in C_\eta{}^*[0,\infty)$.

2. MAIN RESULTS

2.1. Korovkin Type Approximation:

In this section we consider the Chlodowsky variant of the operators given by Equation (6). With the help of the Korovkin's theorem which is one of the most influential theorem in approximation theory, we will examine the approximation process of the constructed operators.

For $f \in C[0,\infty)$, we define the Chlodowsky type Jakimovski-Leviatan-Păltănea operators as follows:

$$M_{n,\rho}^{*}(f,x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\int_{0}^{\infty}\Theta_{\alpha,k}^{\rho}(x)f(t)dt+\ell_{n,0}(x)f(0) \tag{11}$$

where

$$\ell_{n,k}=\frac{e^{-\frac{n}{b_n}x}}{g(1)}p_k\left(\frac{n}{b_n}x\right) \text{ and } \Theta_{n,k}^{\rho}(x)=\frac{\frac{n}{b_n}\rho\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}}{\Gamma(k\rho)}e^{-\frac{n}{b_n}\rho t}. \tag{12}$$

Here (b_n) is a positive increasing sequence such that

$$\lim_{n\to\infty}b_n=\infty \text{ and } \lim_{n\to\infty}\frac{b_n}{n}=0. \tag{13}$$

For our main theorem we need the Lemma given below.

Lemma 4 The operators defined in Equation (11) satisfy next equalities.

$$M_{n,\rho}^{*}(1;x)=1,$$

$$M_{n,\rho}^{*}(t;x)=x+\frac{b_n}{n}\frac{g'(1)}{g(1)},$$

$$M_{n,\rho}^{*}(t^2;x)=x^2+\frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1)+2g'(1)}{g(1)}\right)x+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1)+g''(1)}{g(1)}\right),$$

$$M_{n,\rho}^{*}(t^3;x)=x^3+\frac{b_n}{n}\left(\frac{\left(4+\frac{3}{\rho}\right)g(1)+3g'(1)}{g(1)}\right)x^2+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{3}{\rho}+\frac{2}{\rho^2}\right)g(1)+\left(8+\frac{6}{\rho}\right)g'(1)+3g''(1)}{g(1)}\right)x+$$
$$\left(\frac{b_n}{n}\right)^3\left(\frac{\left(1+\frac{3}{\rho}+\frac{2}{\rho^2}\right)g'(1)+\left(4+\frac{3}{\rho}\right)g''(1)+g'''(1)}{g(1)}\right),$$

$$M_{n,\rho}^{*}(t^4;x)=x^4+\frac{b_n}{n}x^3\left(\frac{10g(1)+4g'(1)}{g(1)}+\frac{6}{\rho}\right)+\left(\frac{b_n}{n}\right)^2\left(\frac{14g(1)+30g'(1)+6g''(1)}{g(1)}+\frac{6}{\rho}\frac{4g(1)+3g'(1)}{g(1)}+\right.$$
$$\left.\frac{11}{\rho^2}\right)x^2+\left(\frac{b_n}{n}\right)^3\left\{\frac{g(1)+28g'(1)+30g''(1)+4g'''(1)}{g(1)}+\frac{6}{\rho}\frac{g(1)+8g'(1)+3g''(1)}{g(1)}+\frac{11}{\rho^2}\left(\frac{g(1)+4g'(1)}{g(1)}\right)+\frac{6}{\rho^3}\right\}x+$$
$$\left(\frac{b_n}{n}\right)^4\left\{\frac{g'(1)+14g''(1)+10g'''(1)+g^{(4)}(1)}{g(1)}+\frac{6}{\rho}\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}+\frac{11}{\rho^2}\left(\frac{g'(1)+4g''(1)+g'''(1)}{g(1)}\right)+\frac{6}{\rho^3}\frac{g'(1)}{g(1)}\right\}.$$

Proof In the view of the equalities given in Lemma 1,

$$M_{n,\rho}^{*}(1;x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\frac{\frac{n}{b_n}\rho}{\Gamma(k\rho)}\int_{0}^{\infty}\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}e^{-\frac{n}{b_n}\rho t}dt+\ell_{n,0}(x)$$
$$=\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}p_k\left(\frac{n}{b_n}x\right)=1.$$

$$M_{n,\rho}^{*}(t;x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\frac{\frac{n}{b_n}\rho}{\Gamma(k\rho)}\int_{0}^{\infty}\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}e^{-\frac{n}{b_n}\rho t}tdt$$
$$=\frac{b_n}{n}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{k=0}^{\infty}kp_k\left(\frac{n}{b_n}x\right)=x+\frac{b_n}{n}\frac{g'(1)}{g(1)}.$$

$$M_{n,\rho}^*(t^2;x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\frac{\frac{n}{b_n}\rho}{\Gamma(k\rho)}\int_0^{\infty}\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}e^{-\frac{n}{b_n}\rho t}t^2dt$$

$$=\left(\frac{b_n}{n}\right)^2\frac{1}{\rho^2}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{\mathrm{k}=0}^{\infty}(k^2\rho^2+k\rho)p_k\left(\frac{n}{b_n}x\right)$$

$$=x^2+\frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1)+2g'(1)}{g(1)}\right)x+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1)+g''(1)}{g(1)}\right).$$

Similarly writing $M_{n,\rho}^*(t^3;x)$ and $M_{n,\rho}^*(t^4;x)$ in the forms

$$M_{n,\rho}^*(t^3;x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\frac{\frac{n}{b_n}\rho}{\Gamma(k\rho)}\int_0^{\infty}\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}e^{-\frac{n}{b_n}\rho t}t^3dt$$

$$=\left(\frac{b_n}{n}\right)^3\frac{1}{\rho^3}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{\mathrm{k}=0}^{\infty}(k^3\rho^3+3k^2\rho^2+2k\rho)p_k\left(\frac{n}{b_n}x\right)$$

and

$$M_{n,\rho}^*(t^4;x)=\sum_{k=1}^{\infty}\ell_{n,k}(x)\frac{\frac{n}{b_n}\rho}{\Gamma(k\rho)}\int_0^{\infty}\left(\frac{n}{b_n}\rho t\right)^{k\rho-1}e^{-\frac{n}{b_n}\rho t}t^4dt$$

$$=\left(\frac{b_n}{n}\right)^4\frac{1}{\rho^4}\frac{e^{-\frac{n}{b_n}x}}{g(1)}\sum_{\mathrm{k}=0}^{\infty}(k^4\rho^4+6k^3\rho^3+11k^2\rho^2+6k\rho)p_k\left(\frac{n}{b_n}x\right)$$

then using the equalities in Lemma 1, we get the desired results.

Remark 5 One can obtain from Lemma 4 that the operators in Equation (11) satisfy

$$M_{n,\rho}^*\big((t-x);x\big)=\frac{b_n}{n}\frac{g'(1)}{g(1)},\tag{14}$$

$$M_{n,\rho}^*((t-x)^2;x)=\frac{b_n}{n}x\left(1+\frac{1}{\rho}\right)+\left(\frac{b_n}{n}\right)^2\left(\left(1+\frac{1}{\rho}\right)\frac{g'(1)}{g(1)}+\frac{g''(1)}{g(1)}\right),\tag{15}$$

$$M_{n,\rho}^*((t-x)^4;x)=\left(\frac{b_n}{n}\right)^2x^2\left\{\left(10+\frac{12}{\rho}+\frac{3}{\rho^2}\right)+4\frac{g'(1)}{g(1)}\right\}+\left(\frac{b_n}{n}\right)^3x\left\{\left(1+\frac{6}{\rho}+\frac{11}{\rho^2}+\frac{6}{\rho^3}\right)-\left(24+\frac{36}{\rho}+\frac{14}{\rho^2}\right)\frac{g'(1)}{g(1)}+\left(14+\frac{6}{\rho}\right)\frac{g''(1)}{g(1)}\right\}+\left(\frac{b_n}{n}\right)^4\left\{\left(1+\frac{6}{\rho}+\frac{11}{\rho^2}+\frac{6}{\rho^3}\right)\frac{g'(1)}{g(1)}+\left(14+\frac{24}{\rho}+\frac{44}{\rho^2}\right)\frac{g''(1)}{g(1)}+\left(10+\frac{6}{\rho}+\frac{11}{\rho^2}\right)\frac{g'''(1)}{g(1)}+\frac{g^{(4)}(1)}{g(1)}\right\}.\tag{16}$$

Consider the space

$$C_{1+x^\zeta}[0,\infty):=\left\{f\in C[0,\infty):|f(x)|\le M_f\left(1+x^\zeta\right),\text{ for some }M_f>0,\zeta\ge 2\right\}$$

endowed with the norm

$$\|f\|_{1+x^\zeta}=\sup_{x\ge 0}\frac{|f(x)|}{1+x^\zeta}$$

Now we will prove the Korovkin type theorem in the space

$$E_\zeta := \left\{ f \in C_{1+x^\zeta}[0,\infty) : \lim_{x\to\infty} \frac{|f(x)|}{1+x^\zeta} < \infty \right\}.$$

Theorem 6 Let $n \in \mathbb{N}$, $M^*_{n,\rho}(f;.)$ is given by Equation (11). For any $f \in E_\zeta$ the uniform convergence is satisfied on any compact subset $K \subset [0,\infty)$.

Proof From Lemma 4 and Equation (13) one can verify that

$$\lim_{n\to\infty} M^*_{n,\rho}(1;x) = 1,$$

$$\lim_{n\to\infty} M^*_{n,\rho}(t;x) = x,$$

$$\lim_{n\to\infty} M^*_{n,\rho}(t^2;x) = x^2.$$

Hence, the desired result follows from the Korovkin type theorem in the book of Altomare and Campiti, 1994 [18].

The Lemma given below is needed for the proof of the Voronovskaya theorem:

Lemma 7 For the operators $M^*_{n,\rho}(f;.)$, $n \in \mathbb{N}$ given by Equation (11) one can obtain the following identities.

$$\lim_{n\to\infty} \frac{n}{b_n} M^*_{n,\rho}\big((t-x);x\big) = \frac{g'(1)}{g(1)}, \tag{17}$$

$$\lim_{n\to\infty} \frac{n}{b_n} M^*_{n,\rho}((t-x)^2;x) = \left(1+\frac{1}{\rho}\right)x, \tag{18}$$

$$\lim_{n\to\infty} \frac{n^2}{b_n^2} M^*_{n,\rho}((t-x)^4;x) = \left[\left(1+\frac{12}{\rho}+\frac{3}{\rho^2}\right)+4\frac{g'(1)}{g(1)}\right]x^2. \tag{19}$$

2.2. Voronovskaya Theorem

Theorem 8 (Voronovskaya Type Theorem) Let $M^*_{n,\rho}(f;.)$, $n\epsilon\mathbb{N}$ is given by Equation (11) with the conditions in Equation (13) and let $f \in C_{1+x^\zeta}[0,\infty)$ such that f' and $f'' \in C_{1+x^\zeta}[0,\infty)$. Then we get

$$\lim_{n\to\infty} \frac{n}{b_n}\left[M^*_{n,\rho}(f;x)-f(x)\right] = \frac{g'(1)}{g(1)}f'(x) + \frac{1}{2}x\left(1+\frac{1}{\rho}\right)f''(x).$$

Proof By using the Taylor's formula we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}f''(x)(t-x)^2 + \lambda(t,x)(t-x)^2 \tag{20}$$

where $\lambda(t,x)$ belongs to $C_{1+x^\zeta}[0,\infty)$ and $\lim_{t\to x}\lambda(t,x) = 0$. Applying $M^*_{n,\rho}$ to the both side of the Equation (20) and then multiplying by $\frac{n}{b_n}$, we have

$$\begin{aligned}\frac{n}{b_n}\left[M^*_{n,\rho}(f;x)-f(x)\right] &= \frac{n}{b_n}f'(x)M^*_{n,\rho}\big((t-x);x\big) + \frac{1}{2}\frac{n}{b_n}f''(x)M^*_{n,\rho}((t-x)^2;x) \\ &\quad + \frac{n}{b_n}M^*_{n,\rho}((t-x)^2\lambda(t,x);x).\end{aligned} \tag{21}$$

For the third term on the right hand side of the above equality, we can write

$$M_{n,\rho}^{*}((t-x)^{2}\lambda(t,x);x) \leq \sqrt{M_{n,\rho}^{*}(\lambda^{2}(t,x);x)}\sqrt{M_{n,\rho}^{*}((t-x)^{4};x)} \tag{22}$$

by applying the Cauchy-Schwarz inequality. Since $\lambda^{2}(x,x)=0$; Theorem 6 implies that

$$\lim_{n\to\infty} M_{n,\rho}^{*}(\lambda^{2}(t,x);x) = \lambda^{2}(x,x)=0. \tag{23}$$

Combining the Equation (19) with the Equation (23) and using them in the Equation (22) one can observe that

$$\lim_{n\to\infty}\frac{n}{b_n}M_{n,\rho}^{*}((t-x)^{2}\lambda(t,x);x)=0.$$

If we take the limit of both sides of the Equation (21) as $n\to\infty$, and taking Equation (17) and Equation (18) into account, one can easily obtain

$$\lim_{n\to\infty}\frac{n}{b_n}\left[M_{n,\rho}^{*}(f;x)-f(x)\right]=\frac{g'(1)}{g(1)}f'(x)+\frac{1}{2}x\left(1+\frac{1}{\rho}\right)f''(x)$$

from which the proof is completed.

2.3. Local Approximation

Theorem 9 Let $f\in C_B[0,\infty)$. For every $x\in[0,\infty)$

$$\left|M_{n,\rho}^{*}(f;x)-f(x)\right|\leq C\omega_{2}\left(f;\sqrt{\delta_n(x)}\right)+\ \omega(f;\beta_n)$$

where

$$\delta_n(x)=\frac{b_n}{n}\,x\left(1+\frac{1}{\rho}\right)+\left(\frac{b_n}{n}\right)^{2}\left(\left(1+\frac{1}{\rho}\right)\frac{g'(1)}{g(1)}+\left(\frac{g'(1)}{g(1)}\right)^{2}+\frac{g''(1)}{g(1)}\right)$$

and

$$\beta_n(x)=\frac{b_n}{n}\frac{g'(1)}{g(1)}.$$

Proof For $x\in[0,\infty)$ let us take the following $\widetilde{M}_{n,\rho}^{*}\colon C_B[0,\infty)\to C_B[0,\infty)$ auxiliary operators into account:

$$\widetilde{M}_{n,\rho}^{*}(f;x)=M_{n,\rho}^{*}(f;x)+f(x)-f\left(x+\frac{b_n}{n}\frac{g'(1)}{g(1)}\right). \tag{24}$$

$\widetilde{M}_{n,\rho}^{*}$ reproduce linear functions, i.e.

$$\widetilde{M}_{n,\rho}^{*}(t-x;x)=0.$$

Let $x\in[0,\infty)$ and $h\in C_B^{2}[0,\infty)$. By the Taylor's Theorem we can write

$$h(t)=h(x)+h'(x)(t-x)+\int_x^t(t-u)h''(u)du.$$

Applying $\widetilde{M}_{n,\rho}^{*}$ to the both side of the above equality, we get

$$\widetilde{M}^*_{n,\rho}(h(t);x) - h(x) = \widetilde{M}^*_{n,\rho}\left(\int_x^t (t-u)h''(u)du\,;x\right)$$
$$= M^*_{n,\rho}\left(\int_x^t (t-u)h''(u)du\,;x\right) - \int_x^{x+\frac{b_n g'(1)}{n\, g(1)}} \left(x + \frac{b_n}{n}\frac{g'(1)}{g(1)} - u\right) h''(u)\, du.$$

So,

$$\left|\widetilde{M}^*_{n,\rho}(h(t);x) - h(x)\right| = \|h''\| \left\{ M^*_{n,\rho}\left(\int_x^t (t-u)du\,;x\right) + \int_x^{x+\frac{b_n g'(1)}{n\, g(1)}} \left|x + \frac{b_n}{n}\frac{g'(1)}{g(1)} - u\right| du \right\}$$
$$\leq \frac{\|h''\|}{2}\left\{ M^*_{n,\rho}((t-x)^2;x) + \left(\frac{b_n}{n}\frac{g'(1)}{g(1)}\right)^2 \right\}$$
$$= \frac{\|h''\|}{2}\left\{ \frac{b_n}{n} x\left(1+\frac{1}{\rho}\right) + \left(\frac{b_n}{n}\right)^2 \left(\left(1+\frac{1}{\rho}\right)\frac{g'(1)}{g(1)} + \left(\frac{g'(1)}{g(1)}\right)^2 + \frac{g''(1)}{g(1)}\right) \right\}.$$

On the other hand from Equation (24) we have

$$\left|\widetilde{M}^*_{n,\rho}(f;x)\right| \leq \left|M^*_{n,\rho}(f;x)\right| + |f(x)| + \left|f\left(x + \frac{b_n}{n}\frac{g'(1)}{g(1)}\right)\right| \leq 3\|f\|.$$

Thus, we can write

$$\left|M^*_{n,\rho}(f;x) - f(x)\right| \leq \left|\widetilde{M}^*_{n,\rho}(f;x) - f(x)\right| + \left|f(x) - f\left(x + \frac{b_n}{n}\frac{g'(1)}{g(1)}\right)\right|$$
$$\leq \left|\widetilde{M}^*_{n,\rho}(f-h;x)\right| + |(f-h)(x)| + \left|\widetilde{M}^*_{n,\rho}(h;x) - h(x)\right| + \left|f(x) - f\left(x + \frac{b_n}{n}\frac{g'(1)}{g(1)}\right)\right|$$

$$\leq 4\|f-h\| + \delta_n(x)\frac{\|h''\|}{2} + \omega\left(f;\frac{b_n}{n}\frac{g'(1)}{g(1)}\right).$$

Taking infimum of both side of the above inequality over all $h \in C_B^2[0,\infty)$, we get

$$\left|M^*_{n,\rho}(f;x) - f(x)\right| \leq 4K_2(f,\delta_n) + \omega\left(f;\frac{b_n}{n}\frac{g'(1)}{g(1)}\right)$$

from which we have the desired result by the inequality in Equation (9).

2.4. Weighted Approximation

Lemma 10 Let $\eta(x) = 1 + x^2$ be a weight function. If $f \in C_\eta[0,\infty)$, then

$$\left\|M^*_{n,\rho}(\eta)\right\|_\eta \leq k$$

where k is a positive constant.

Proof From Lemma 4

$$M^*_{n,\rho}(\eta) = 1 + x^2 + \frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1) + 2g'(1)}{g(1)}\right)x + \left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1) + g''(1)}{g(1)}\right).$$

Then we get

$$\left\|M^*_{n,\rho}(\eta)\right\|_\eta = \sup_{x\geq 0}\frac{1}{1+x^2}\left\{\left(1 + x^2 + \frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1) + 2g'(1)}{g(1)}\right)x\right) + \left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1) + g''(1)}{g(1)}\right)\right\}$$

$$\leq 1+\frac{b_n}{2n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1)+2g'(1)}{g(1)}\right)+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1)+g''(1)}{g(1)}\right).$$

Since $\lim_{n\to\infty}\frac{b_n}{n}=0$, we can write

$$\left\|M_{n,\rho}^{*}(\eta)\right\|_{\eta}\leq k.$$

Lemma 10 implies that the operators $M_{n,\rho}^{*}$ maps $C_{\eta}[0,\infty)$ into $B_{\eta}[0,\infty)$.

Theorem 11 Let the sequence of operators $M_{n,\rho}^{*}(f;.)$ defined by Equation (11). For any $f\in C_{\eta}{}^{*}[0,\infty)$, one gets

$$\lim_{n\to\infty}\left\|M_{n,\rho}^{*}(f;.)-f\right\|_{\eta}=0.$$

Proof It is sufficient to show that the sequence of operators $M_{n,\rho}^{*}(f;.)$ satisfy three criterions of the weighted Korovkin Theorem. From Lemma 4,

$$\lim_{n\to\infty}\left\|M_{n,\rho}^{*}(1;.)-1\right\|_{\eta}=\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|M_{n,\rho}^{*}(1;x)-1\right|}{1+x^2}=0.$$

$$\lim_{n\to\infty}\left\|M_{n,\rho}^{*}(t;.)-x\right\|_{\eta}=\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{\left|M_{n,\rho}^{*}(t;x)-x\right|}{1+x^2}$$

$$=\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{1}{1+x^2}\frac{b_n}{n}\frac{g'(1)}{g(1)}$$

From Equation (13), we get

$$\lim_{n\to\infty}\left\|M_{n,\rho}^{*}(t;.)-x\right\|_{\eta}=0.$$

Lastly, we have,

$$\sup_{x\in[0,\infty)}\frac{\left|M_{n,\rho}^{*}(t^2;x)-x^2\right|}{1+x^2}$$

$$\leq\frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1)+2g'(1)}{g(1)}\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1)+g''(1)}{g(1)}\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2}$$

$$\leq\frac{b_n}{n}\left(\frac{\left(1+\frac{1}{\rho}\right)g(1)+2g'(1)}{g(1)}\right)+\left(\frac{b_n}{n}\right)^2\left(\frac{\left(1+\frac{1}{\rho}\right)g'(1)+g''(1)}{g(1)}\right).$$

So, we have $\lim_{n\to\infty}\left\|M_{n,\rho}^{*}(t^2;.)-x^2\right\|_{\eta}=0$ and hence the proof is completed

2.5. Rate of Convergence

Here, we will verify the rate of weighted convergence of the sequence $M_{n,\rho}^*(f;.)$. It is known that, on an infinite interval the limit of classical modulus of continuity as $\delta \to 0$ is not zero. Yüksel and İspir [19] defined the following weighted modulus of continuity $\Omega(f,\delta)$, tending to zero as $\delta \to 0$ on infinite interval.

$$\Omega(f;\delta) = \sup_{\substack{x\geq 0\\ 0<h\leq\delta}} \frac{|f(x+h)-f(x)|}{(1+x^2)(1+h^2)} \tag{25}$$

where $f \in C_\eta{}^*[0,\infty)$.

Theorem 12 Let the operators $M_{n,\rho}^*(f;.), n \in \mathbb{N}$ is given by Equation (11). If $f \in C_\eta{}^*[0,\infty)$ then we have

$$\left\|M_{n,\rho}^*(f;.)-f\right\|_{\bar{\eta}} \leq K\Omega\left(f;\sqrt{\frac{b_n}{n}}\right)$$

where $\bar{\eta} = (1+x^2)^2$, n is sufficiently large and K is a constant which does not depend on n.

Proof We will use the following property of the weighted modulus of continuity $\Omega(f,\delta)$ given by Equation (25).

$$|f(t)-f(x)| \leq 2(1+x^2)(1+\delta^2)\left(1+\frac{|t-x|}{\delta}\right)(1+(t-x)^2)\Omega(f;\delta).$$

Applying $M_{n,\rho}^*(f;.)$ to the both side of the above inequality, one can write

$$\begin{aligned}\left|M_{n,\rho}^*(f;x)-f(x)\right| &\leq 2(1+x^2)(1+\delta^2)\Omega(f;\delta)\\ &\times\left\{1+\frac{1}{\delta}M_{n,\rho}^*(|t-x|;x)+M_{n,\rho}^*((t-x)^2;x)+\frac{1}{\delta}M_{n,\rho}^*(|t-x|(t-x)^2;x)\right\}.\end{aligned}$$

Cauchy-Schwarz inequality implies

$$\begin{aligned}&\left|M_{n,\rho}^*(f;x)-f(x)\right| \leq 2(1+x^2)(1+\delta^2)\Omega(f;\delta)\\ &\times\left\{1+\frac{1}{\delta}\left(M_{n,\rho}^*((t-x)^2;x)\right)^{\frac{1}{2}}+M_{n,\rho}^*((t-x)^2;x)+\frac{1}{\delta}\left(M_{n,\rho}^*((t-x)^2;x)\right)^{\frac{1}{2}}\left(M_{n,\rho}^*((t-x)^4;x)\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

From Equations (15) and (16) we can write

$$M_{n,\rho}^*((t-x)^2;x) \leq \frac{b_n}{n}\mu(\rho)(x+1),$$

$$M_{n,\rho}^*((t-x)^4;x) \leq \left(\frac{b_n}{n}\right)^2\bar{\mu}(\rho)(x^2+x+1),$$

where $\mu(\rho)$ and $\bar{\mu}(\rho)$ are the constants depending on the constant ρ. Hence we have,

$$\begin{aligned}\left|M_{n,\rho}^*(f;x)-f(x)\right| &\leq 2(1+x^2)(1+\delta^2)\Omega(f;\delta)\left\{1+\frac{1}{\delta}\left(\frac{b_n}{n}\mu(\rho)(x+1)\right)^{\frac{1}{2}}\right.\\ &\left.+\frac{b_n}{n}\mu(\rho)(x+1)+\frac{1}{\delta}\left(\frac{b_n}{n}\mu(\rho)(x+1)\right)^{\frac{1}{2}}\left(\left(\frac{b_n}{n}\right)^2\bar{\mu}(\rho)(x^2+x+1)\right)^{\frac{1}{2}}\right\}\end{aligned}$$

If we choose $\delta := \delta_n = \sqrt{\frac{b_n}{n}}$, we have

$$\left|M^*_{n,\rho}(f;x) - f(x)\right| \leq 2(1+x^2)\left(1+{\delta_n}^2\right)$$
$$\Omega(f;\delta_n)\left\{1 + \left(\mu(\rho)(x+1)\right)^{\frac{1}{2}} + \delta_n^2\mu(\rho)(x+1) + \delta_n^2\left(\mu(\rho)(x+1)\right)^{\frac{1}{2}}\left(\bar{\mu}(\rho)(x^2+x+1)\right)^{\frac{1}{2}}\right\}.$$

Since $\delta_n^2 < 1$, for sufficiently large n and K which is a constant not depending on n, we obtain

$$\sup_{\substack{x\geq 0\\ 0<h\leq\delta}} \frac{\left|M^*_{n,\rho}(f;x)-f(x)\right|}{(1+x^2)^2} \leq K\Omega\left(f;\sqrt{\frac{b_n}{n}}\right)$$

from which the proof is completed.

3. CONCLUSION

As we mentioned in the introduction part of the paper, our primary purpose here is to investigate the approximation properties of the Chlodowsky form of Jakimovski-Leviatan-Păltănea operators. We used Korovkin's Theorem in order to investigate the uniform convergence of the operators on a compact subset of $[0,\infty)$. We also give Voronovskaya theorem and examine local approximation properties of the operators. Since the Korovkin's theorem does not hold for unbounded intervals, we studied in weighted spaces and used Theorem 1 and Theorem 2 for approximating unbounded functions on the unbounded interval $[0,\infty)$. Rate of convergence of the operators are also analyzed with the help of weighted modulus of continuity $\Omega(f,\delta)$.

For the last two decades q-calculus has been developed very rapidly and it has been used very frequently in approximation theory, especially in Korovkin type approximation. Several linear and positive operators are reconstructed via q −integers. In this direction our new question may be "Can we construct the q-analogue of the Chlodowsky variant of Jakimovski-Leviatan-Păltănea operators and examine the approximation properties?"

Another way of generalizing operators is to define operators by using different polynomials (such as Brenke, Sheffer, Boas-Buck polynomials, multiple Appell polynomials). So, one can generalize the mentioned operators with the help of polynomials and then examine their approximation properties. The convergence of the operators may be investigated in the statistical sense.

CONFLICTS OF INTEREST

No conflict of interest was declared by the authors.

REFERENCES

[1] Jakimovski, A., Leviatan, D., "Generalized Sźasz operators for the approximation in the infinite interval", Mathematica (Cluj), 11: 97-103, (1969).

[2] Szász, O., "Generalization of S. Bernsteins polynomials to the infinite interval", J. Res. Natl. Bur. Stand., 97: 239-245, (1950).

[3] Atakut, Ç., Büyükyazıcı, İ., "Approximation by modified integral type Jakimovski-Leviatan operators", Filomat, 30(1): 29-39, (2016).

[4] Büyükyazıcı, İ., Tanberkan, H., Serenbay, S.K., Atakut, Ç., "Approximation by Chlodowsky type Jakimovski-Leviatan operators", Journal of Computational and Applied Mathematics, 259: 153-163, (2014).

[5] Dalmanoglu, Ö. and Serenbay, S.K., "Approximation by Chlodowsky type q-Jakimovski-Leviatan operators", Communications Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics, 65(1): 157-169, (2016).

[6] Gupta, P., Agrawal, P.N., "Jakimovski-Leviatan operators of Durrmeyer type involving Appell polynomials", Turkish Journal of Mathematics, 42(3): 1457-1470, (2018).

[7] Karaisa, A., "Approximation by Durrmeyer type Jakimoski Leviatan operators", Mathematical Methods and Applied Sciences, 39(9): 2401-2410, (2016).

[8] Phillips, R.S., "An inversion formula for Laplace transforms and semi-groups of linear operators", Annals of Mathematics, 325-356, (1954).

[9] Păltănea, R., "Modified Szász-Mirakjan operators of integral form", Carpathian Journal of Mathematics, 24(3): 378-385, (2018).

[10] Verma, D.K., Gupta, V., "Approximation for Jakimovski-Leviatan-Păltănea operators", Annali dell' Università di Ferrara, 61(3): 367-380, (2015).

[11] Mursaleen, M., AL-Abeid, A.A.H., Ansari, K.J., "Approximation by Jakimovski-Leviatan-Păltănea operators involving Sheffer polynomials", Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113(2): 1251-1265, (2019).

[12] Gadjiev, A. D., Efendiev, R. O., İbikli, E., "Generalized Bernstein-Chlodowsky polynomials", Rocky Mountain Journal of Mathematics, 28(4): 1267-1277, (1998).

[13] Mursaleen, M., Ansari, K.J., "On Chlodowsky variant of Szász operators by Brenke type polynomials", Applied Mathematics and Computation, 271: 991-1003, (2015).

[14] Mursaleen, M., AL-Abied A., Ansari, K.J., "Rate of convergence of Chlodowsky type Durrmeyer Jakimovski-Leviatan operators", Tbilisi Mathematical Journal, 10(2): 173-184, (2017).

[15] Neer, T., Acu, A.M., Agrawal, P.N., "Degree of approximation by Chlodowsky variant of Jakimovski-Leviatan-Durrmeyer type operators", Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, 113, 3445-3459, (2019).

[16] DeVore R. A., Lorentz, G. G., "Constructive Approximation", Springer-Verlag, Berlin, (1993).

[17] Gadjiev, A.D., "The convergence problem for a sequence of positive linear operators on bounded sets and theorems analogous to that of P.P. Korovkin", Dokl. Akad. Nauk SSSR, 218(5): 1433-1436, (1974). (transl.in Sov. Math. Dokl. 15(5), 1974)

[18] Altomare, F., Campiti, M., "Korovkin-type Approximation Theory and its Applications", De Gruyter Studies in Mathematics 17, W. De Gruyter, Berlin-New York, (1994).

[19] Yüksel, İ., İspir, N., "Weighted Approximation by a Certain Family of Summation Integral-Type Operators", Computers and Mathematics with Applications, 52(10-11): 1463-1470, (2006).