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Research Article

On Causal Characterization of Spherical Indicatrices of Timelike Curves in Minkowski 3-Space

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Abstract

The aim of this paper is to discuss the causal characterization of spherical indicatrices of timelike curves in Minkowski 3-space. Firstly, we focus on the introduction of curves in Minkowski 3-space with its Frenet equations. In the next section the concept of spherical indicatrix of tangent, normal and binormal vectors of timelike curves is provided structurally with their casual properties. In addition, numerical example is provided in the last section.

Keywords and 2010 Mathematics Subject Classification

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1. Introduction

The geometry in Minkowski space is very important for both mathematics and physics. Therefore, since the second mid of 20th century mathematicians and physicist actively studied about differential geometry of Riemannian manifold and its applications. Lipschutz [2] has explained in a good way about theory of curves and surface of geometry in Euclidean space. Many topics in classical differential geometry of Riemannian manifold are then extended into those of Lorentz-Minkowski manifolds. In Lorentz-Minkowski space, a curve can locally be timelike, spacelike or null depending of the casual character of the tangent vector of the curves. Some studies theory of curves in Minkowski space and its applications have been studied [3], [4], [5],[6].

The idea of spherical indicatrix goes back for long time to the time of Gauss. The idea is essentially simple. If there exists some group or set of lines in space in some organized relationship with one another, organized for examples as the joint axes of some spatial mechanism or as the member of mathematically determined rules surface, one might construct and examine the relevant spherical indicatrix [11]. Ali[1] has given a brief explanation about spherical indicatrices of curves in Euclidean space. The theory of spherical indicatrix of space curves in Riemannian geometry also has been studied by [2], [7] and [12] while the study of spherical indicatrix of curves in Minkowski space has can bee seen in [9], [8] and [10].

In this study, we focus on studying the causal characteristics if spherical indicatrices of timelike curves derived from theory in classical differential geometry. Firstly, a short brief of curve in Minkowski 3-space and their Frenet equations are given. In the next part, we derive the Frenet frame of the spherical indicatrices of timelike curve as well as their curvature and torsion. In addition, we give a numerical example of spherical indicatrix curves of timelike curves.

2. Preliminaries

Minkowski 3-space $\mathbb{E}_1^3$ is the real vector space $\mathbb{R}^3$ equipped with the standard indefinite Lorentzian metric $\langle , \rangle$ defined by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3, \tag{1}$$

for any vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$. The cross product in Minkowski 3-space is defined as

$$x \times y = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_3). \tag{2}$$

$V \in \mathbb{E}_1^3$ is said to be timelike if $\langle V,V\rangle < 0$, spacelike if $\langle V,V\rangle > 0$ or $V = 0$ and null(lightlike) if $\langle V,V\rangle = 0$ and $V \neq 0$. The norm of a vector in $\mathbb{E}_1^3$ is defined by $\|v\| = \sqrt{|\langle v,v\rangle|}$.

Let $\alpha : I \to \mathbb{E}_1^3$ be a curve in Minkowski space. Locally, α can be timelike, spacelike or null if its tangent vector is timelike, spacelike or null respectively. For non null curves, the arc length s is defined by $s = \int_0^t \sqrt{|\langle \alpha', \alpha'\rangle|}dt$. If $\langle \alpha', \alpha'\rangle = 1$ the non null curve is called curve parametrized by arc length. For null curves, since $\langle \alpha', \alpha'\rangle = 0$ then the pseudo-arc length is defined by $s = \int_0^t \langle \alpha'', \alpha''\rangle^{\frac{1}{4}} dt$ and if $\langle \alpha'', \alpha''\rangle = 1$ the null curve is parametrized by pseudo-arc length.

If α is a timelike curve then

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = -\tau N \tag{3}$$

where

$$\langle T,T\rangle = -1, \quad \langle N,N\rangle = \langle B,B\rangle = 1, \quad \langle T,N\rangle = \langle B,N\rangle = \langle T,B\rangle = 0$$

and

$$T \times N = -B \quad N \times B = T \quad B \times T = -N.$$

If α is a spacelike curve with spacelike principal normal N then

$$T' = \kappa N, \quad N' = -\kappa T + \tau B, \quad B' = \tau N, \tag{4}$$

where

$$\langle T,T\rangle = \langle N,N\rangle = 1, \quad \langle B,B\rangle = -1, \quad \langle T,N\rangle = \langle B,N\rangle = \langle T,B\rangle = 0$$

and

$$T \times N = B \quad N \times B = -T \quad B \times T = -N.$$

If α is a spacelike curve with timelike principal normal N then

$$T' = \kappa N, \quad N' = \kappa T + \tau B, \quad B' = \tau N, \tag{5}$$

where

$$\langle T,T\rangle = \langle B,B\rangle = 1, \quad \langle N,N\rangle = -1, \quad \langle T,N\rangle = \langle B,N\rangle = \langle T,B\rangle = 0$$

and

$$T \times N = -B \quad N \times B = -T \quad B \times T = N.$$

T, N, B, κ and τ are called the tangent, principal normal, binormal vector field, curvature and torsion of curve [3].

Let C be a regular curve. A curve defined by vector equation $T = T(s)$, where $T(s)$ is a unit tangent vector of C, lies on a unit sphere and is said to be a spherical indicatrix of tangents to the curve or its tangent spherical image. A curve defined by vector equation $N = N(s)$, where $N(s)$ is a unit principal normal vector of C, lies on a unit sphere and is said to be a spherical indicatrix of normals to the curve or its principal normal spherical image. A curve defined by vector equation $B = B(s)$, where $B(s)$ is a unit binormal vector of C, lies on a unit sphere and is said to be a spherical indicatrix of binormals to the curve or its binormal spherical image. All three indicatrices lie on the unit sphere, and, hence, are spherical curves.

In Euclidean space, spherical indicatrix of tangents of plane curve α is a great circle arc parallel to the plane where the curve lies. An osculating plane of a curve is parallel to a plane of the great circle which is tangent to the spherical indicatrix of tangents. A tangent vector of the spherical indicatrix of tangents of curve α and the unit principal normal vector of the curve have the same direction; that is, the spherical indicatrix of principal normals to the curve is at the same time a spherical indicatrix of tangents to the spherical indicatrix of tangents to curve α. A spherical indicatrix of tangents to curve α degenerates into a point if, and only if, the curve is a straight line. A spherical indicatrix of principal normals to a curve can not degenerate into a point. A spherical indicatrix of binormals to curve α degenerates into a point if and only if the curve is plane [7].

In case $C : I \to \mathbb{E}_1^3$ as a timelike curve parametrized by arc length with Frenet vectors T, N and B the curves generated by $T(s)$, $N(s)$ and $B(s)$ are respectively called spherical indicatrix of tangent, principal normal and binormal indicatrix of curve timelike curves.

3. Tangent Indicatrix of Timelike Curves

Theorem 1. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of timelike curve parametrized by arc length s with nonzero curvature and torsion. Then, α is a spacelike curve.*

Proof. Since $\alpha(s) = T(s)$, then $\alpha'(s) = \kappa N(s)$. Therefore, $\langle \alpha', \alpha' \rangle = \kappa^2 > 0$. Hence, α is a spacelike curve. ∎

Theorem 2. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of timelike curve parametrized by arc length s with non null principal normal vector. Let the curvature and torsion of timelike curve are nonzero and unequal. If $\{T^*, N^*, B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\kappa}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}} & 0 & \frac{\tau}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}} \\ \frac{\tau}{\sqrt{|-\kappa^2+\tau^2|}} & 0 & \frac{\kappa}{\sqrt{|-\kappa^2+\tau^2|}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{6}$$

Proof. Let $\alpha(s)$ have arc-length s^*, then

$$\frac{d\alpha}{ds^*}\frac{ds^*}{ds} = T'(s) = \kappa N$$

$$T^* \cdot \frac{ds^*}{ds} = \kappa N.$$

Since, T^* and N are unit vectors then $T^* = \pm N$. Assume that $T^* = N$ so that $\frac{ds^*}{ds} = \kappa$. Next,

$$\begin{aligned} &\frac{dT^*}{ds^*}\frac{ds^*}{ds} = N' \\ &\kappa^* N^* \kappa = \kappa T + \tau B. \end{aligned} \tag{7}$$

If we take the norm of equation (7) we have

$$\kappa^*\kappa = \varepsilon\sqrt{|-\kappa^2+\tau^2|}. \tag{8}$$

Since N^* can be spacelike or timelike for non null principal normal vector and $\varepsilon = 1$ or $\varepsilon = -1$ respectively. From equation (7), we also get

$$N^* = \frac{\kappa T + \tau B}{\kappa^*\kappa} = \frac{\kappa T + \tau B}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}}.$$

Since, $T^* \times N^* = \varepsilon B^*$ where $\varepsilon = 1$ for the case α is spacelike with spacelike principal normal and $\varepsilon = -1$ for the case of α is spacelike with timelike principal normal, then

$$\begin{aligned} B^* &= \varepsilon \cdot T^* \times N^* \\ &= \varepsilon \cdot N \times \left(\frac{\kappa T + \tau B}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}} \right) \\ &= \frac{\kappa(N \times T) + \tau(N \times B)}{\sqrt{|-\kappa^2+\tau^2|}} \\ &= \frac{\kappa B + \tau T}{\sqrt{|-\kappa^2+\tau^2|}}. \end{aligned}$$

This completes the proof. ∎

Corollary 3. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of timelike curve parametrized by arc length s. Let the curvature and torsion of timelike curve are nonzero and unequal. Suppose $\{T^*, N^*, B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

1. *If $|\kappa| < |\tau|$ then $\alpha(s)$ is a spacelike curve with spacelike principal normal and $\varepsilon = -1$.*

2. *If $|\kappa| > |\tau|$ then $\alpha(s)$ is a spacelike curve with timelike principal normal and $\varepsilon = 1$.*

Proof. It is clear from equation (6) that $\langle N^*, N^* \rangle = \frac{-\kappa^2+\tau^2}{|-\kappa^2+\tau^2|}$. ■

Theorem 4. *Let $\alpha(s) = T(s)$ be a tangent indicatrix of timelike curve parametrized by arc length s. Let the curvature and torsion of timelike curve are nonzero and unequal. Suppose κ^* and τ^* be the curvature and torsion of α, respectively, then*

$$\kappa^* = \frac{\varepsilon\sqrt{|-\kappa^2+\tau^2|}}{\kappa} \tag{9}$$

and

$$\tau^* = \frac{\kappa'\tau - \kappa\tau'}{\kappa(\tau^2-\kappa^2)}. \tag{10}$$

Proof. From equation (8), it is clear that

$$\kappa^* = \frac{\varepsilon\sqrt{|-\kappa^2+\tau^2|}}{\kappa}.$$

If we take the derivative of B^* toward s then,

$$\begin{aligned}
\frac{dB^*}{ds^*}\frac{ds^*}{ds} &= \frac{\kappa' B - \kappa\tau N + \tau' T + \kappa\tau N}{(-\kappa^2+\tau^2)^{1/2}} - \frac{(\kappa\kappa' + \tau\tau')(\kappa B + \tau T)}{(-\kappa^2+\tau^2)^{3/2}},\\
\frac{dB^*}{ds^*}\kappa &= \frac{(-\kappa^2+\tau^2)(\kappa' B + \tau' T) - (-\kappa\kappa' + \tau\tau')(\kappa B + \tau T)}{(-\kappa^2+\tau^2)^{3/2}},\\
\frac{dB^*}{ds^*}\kappa &= \frac{(-\kappa^2\tau' + \kappa\kappa'\tau)T + (\kappa'\tau^2 - \kappa\tau\tau')B}{(-\kappa^2+\tau^2)^{3/2}},\\
\frac{dB^*}{ds^*}\kappa &= \frac{(-\kappa\tau' + \kappa'\tau)(\kappa T + \tau B)}{(-\kappa^2+\tau^2)^{3/2}},\\
\frac{dB^*}{ds^*} &= \frac{(-\kappa\tau' + \kappa'\tau)(\kappa T + \tau B)}{\kappa(-\kappa^2+\tau^2)^{3/2}}.
\end{aligned}$$

Since $\tau^* = \|\frac{dB^*}{ds^*}\|$, then

$$\begin{aligned}
\tau^* &= \|\frac{dB^*}{ds^*}\|\\
&= \left(\frac{(-\kappa\tau' + \kappa'\tau)^2(-\kappa^2+\tau^2)}{\kappa^2(-\kappa^2+\tau^2)^3}\right)^{\frac{1}{2}}\\
&= \frac{-\kappa\tau' + \kappa'\tau}{\kappa(-\kappa^2+\tau^2)}.
\end{aligned}$$

■

4. Principal Normal Indicatrix of Timelike Curves

Theorem 5. *Let $\alpha(s) = N(s)$ be a principal normal indicatrix of timelike curve parametrized by arc length s. Then, α can be spacelike, timelike or null curve.*

Proof. Since $\alpha(s) = N(s)$, then $\alpha'(s) = \kappa(s)T(s) + \tau(s)B(s)$. Consequently,

$$\langle \alpha', \alpha' \rangle = \langle \kappa(s)T(s) + \tau(s)B(s), \kappa(s)T(s) + \tau(s)B(s) \rangle = -\kappa^2 + \tau^2.$$

Therefore, if $|\kappa| < |\tau|$ then $\langle \alpha', \alpha' \rangle = -\kappa^2 + \tau^2 > 0$ implying that α is a spacelike curve. If $|\kappa| > |\tau|$ then $\langle \alpha', \alpha' \rangle = -\kappa^2 + \tau^2 < 0$ implying that α is a timelike curve. If $|\kappa| = |\tau|$ then $\langle \alpha', \alpha' \rangle = -\kappa^2 + \tau^2 = 0$ implying that α is a null curve. ■

Theorem 6. *Let $\alpha(s)=N(s)$ be a non null principal normal indicatrix curve of timelike curve parametrized by arc length s. If $\{T^*,N^*,B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} \frac{\kappa}{\sqrt{|\lambda|}} & 0 & \frac{\tau}{\sqrt{|\lambda|}} \\ \frac{\tau\mu}{\sqrt{-\mu^2\lambda+\lambda^4}} & \frac{-\lambda^2}{\sqrt{-\mu^2\lambda+\lambda^4}} & \frac{\kappa\mu}{\sqrt{-\mu^2\lambda+\lambda^4}} \\ \frac{\tau\lambda}{\sqrt{-\mu^2+\lambda^3}} & \frac{-\mu}{\sqrt{-\mu^2+\lambda^3}} & \frac{\kappa\lambda}{\sqrt{-\mu^2+\lambda^3}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{11}$$

where $\lambda=-\kappa^2+\tau^2$ and $\mu=\kappa'\tau-\kappa\tau'$.

Proof. Let $\alpha(s)$ have arc-length s^*, then

$$\frac{d\alpha}{ds}=N'(s)=\kappa T+\tau B$$

and

$$\|\frac{d\alpha}{ds}\|=\sqrt{|\tau^2-\kappa^2|}.$$

So that,

$$T^*=\frac{\alpha'}{\|\alpha'\|}=\frac{\kappa T+\tau B}{\sqrt{|\tau^2-\kappa^2|}}.$$

Then, if we take the derivative of T^* we have

$$\begin{aligned}(T^*)'=&\frac{\kappa' T+\kappa^2 N+\tau' B-\tau^2 N}{(-\kappa^2+\tau^2)^{\frac{1}{2}}}-\frac{(-\kappa\kappa'+\tau\tau')(\kappa T+\tau B)}{(-\kappa^2+\tau^2)^{3/2}}\\ =&\frac{(-\kappa^2+\tau^2)(\kappa' T+(\kappa^2-\tau^2)N+\tau' B)-(-\kappa\kappa'+\tau\tau')(\kappa T+\tau B)}{(-\kappa^2+\tau^2)^{3/2}}\\ =&\frac{(\tau^2\kappa'-\kappa\tau\tau')T-(-\kappa^2+\tau^2)^2 N+(-\kappa^2\tau'-\kappa\kappa'\tau)B}{(-\kappa^2+\tau^2)^{3/2}}\\ =&\frac{(\kappa'\tau-\kappa\tau')(\tau T+\kappa B)-(-\kappa^2+\tau^2)^2 N}{(-\kappa^2+\tau^2)^{3/2}}.\end{aligned}$$

If we take the norm of the last equation, we have

$$\begin{aligned}\|(T^*)'\|^2=&\frac{(\kappa'\tau-\kappa\tau')^2(-\tau^2+\kappa^2)+(-\kappa^2+\tau^2)^4}{(-\kappa^2+\tau^2)^3}\\ =&\frac{-(\kappa'\tau-\kappa\tau')^2+(-\kappa^2+\tau^2)^3}{(-\kappa^2+\tau^2)^2}.\end{aligned}$$

So that,

$$\|(T^*)'\|=\frac{\sqrt{-(\kappa'\tau-\kappa\tau')^2+(-\kappa^2+\tau^2)^3}}{-\kappa^2+\tau^2} \tag{12}$$

and

$$N^*=\frac{(T^*)'}{\|(T^*)'\|}=\frac{(\kappa'\tau-\kappa\tau')(\tau T+\kappa B)-(-\kappa^2+\tau^2)^2 N}{\sqrt{-(\kappa'\tau-\kappa\tau')^2(-\kappa^2+\tau^2)+(-\kappa^2+\tau^2)^4}}.$$

Since, $T^* \times N^* = B^*$. Therefore,

$$
\begin{aligned}
B^* =& T^* \times N^* \\
=& \left(\frac{\kappa T + \tau B}{\sqrt{|\tau^2 - \kappa^2|}}\right) \times \left(\frac{(\kappa'\tau - \kappa\tau')(\tau T + \kappa B) - (-\kappa^2 + \tau^2)^2 N}{\sqrt{-(\kappa'\tau - \kappa\tau')^2(-\kappa^2 + \tau^2) + (-\kappa^2 + \tau^2)^4}}\right) \\
=& \left(\frac{(\kappa'\tau - \kappa\tau')\kappa^2 (T \times B) - (-\kappa^2 + \tau^2)^2 \kappa (T \times N)}{(-\kappa^2 + \tau^2)\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}\right) \\
&+ \left(\frac{(\kappa'\tau - \kappa\tau')\tau^2 (B \times T) - (-\kappa^2 + \tau^2)^2 \tau (B \times N)}{(-\kappa^2 + \tau^2)\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}\right) \\
=& \left(\frac{(\kappa'\tau - \kappa\tau')\kappa^2 (N) + (-\kappa^2 + \tau^2)^2 \kappa (B)}{(-\kappa^2 + \tau^2)\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}\right) \\
&+ \left(\frac{-(\kappa'\tau - \kappa\tau')\tau^2 (N) + (-\kappa^2 + \tau^2)^2 \tau (T)}{(-\kappa^2 + \tau^2)\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}\right) \\
=& \frac{(-\kappa^2 + \tau^2)^2 \tau (T) + (\kappa'\tau - \kappa\tau')(\kappa^2 - \tau^2)(N) + (-\kappa^2 + \tau^2)^2 \kappa (B)}{(-\kappa^2 + \tau^2)\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}} \\
=& \frac{(-\kappa^2 + \tau^2)\tau (T) - (\kappa'\tau - \kappa\tau')(N) + (-\kappa^2 + \tau^2)\kappa (B)}{\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}.
\end{aligned}
$$

Setting $\lambda = -\kappa^2 + \tau^2$ and $\mu = \kappa'\tau - \kappa\tau'$ completes the proof. ■

Corollary 7. *Let $\alpha(s) = N(s)$ be a non null principal normal indicatrix of timelike curve parametrized by arc length s. Suppose $\{T^*, N^*, B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

1. *If $|\kappa| < |\tau|$ then $\alpha(s)$ is a spacelike curve.*
2. *If $|\kappa| > |\tau|$ then $\alpha(s)$ is a timelike curve.*

Proof. It is clear from equation (13) that $\langle T', T'\rangle = -\kappa^2 + \tau^2$ in Theorem 5. ■

Theorem 8. *Let $\alpha(s) = N(s)$ be a non null principal normal indicatrix of timelike curve parametrized by arc length s. Let κ^* and τ^* be the curvature and torsion of α, respectively, then*

$$
\kappa^* = \frac{\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}{-\kappa^2 + \tau^2} \tag{13}
$$

and

$$
\tau^* = \sqrt{\left(\frac{2a(\tau'\lambda + \tau\lambda') - b\tau\lambda + 2\kappa\mu a}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2 + \left(-\frac{2a\mu' - b\mu}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2 + \left(\frac{2a(\kappa'\lambda + \kappa\lambda') - b\kappa\lambda + 2\kappa\mu a}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2}, \tag{14}
$$

where

$$
a = -\mu' + \lambda^3 \quad \textit{and} \quad b = -2\mu\mu' + 3\lambda^2\lambda'.
$$

Proof. From equation (12), it is clear that

$$
\kappa^* = \|(T^*)'\| = \frac{\sqrt{-(\kappa'\tau - \kappa\tau')^2 + (-\kappa^2 + \tau^2)^3}}{-\kappa^2 + \tau^2}.
$$

Then, from theorem 6, we have

$$
B^* = \frac{\tau\lambda}{\sqrt{-\mu^2 + \lambda^3}} T - \frac{\mu}{\sqrt{-\mu^2 + \lambda^3}} N + \frac{\kappa\lambda}{\sqrt{-\mu^2 + \lambda^3}} B.
$$

If we take the derivative of B^* respect to s, then

$$\begin{aligned}
\frac{dB^*}{ds} =&\frac{2(-\mu^2+\lambda^3)(\tau'\lambda+\tau\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\tau\lambda}{2(-\mu^2+\lambda^3)^{3/2}}T+\frac{\kappa\tau\lambda}{(-\mu^2+\lambda^3)^{1/2}}N\\
&-\frac{2(-\mu^2+\lambda^3)\mu'-(-2\mu\mu'+3\lambda^2\lambda')\mu}{2(-\mu^2+\lambda^3)^{3/2}}N+\frac{\mu}{\sqrt{-\mu^2+\lambda^3}}(\kappa T+\tau B)\\
&+\frac{2(-\mu^2+\lambda^3)(\kappa'\lambda+\kappa\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\kappa\lambda}{2(-\mu^2+\lambda^3)^{3/2}}B-\frac{\kappa\tau\lambda}{(-\mu^2+\lambda^3)^{1/2}}N\\
=&\frac{2(-\mu^2+\lambda^3)(\tau'\lambda+\tau\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\tau\lambda+2\kappa\mu(-\mu^2+\lambda^3)}{2(-\mu^2+\lambda^3)^{3/2}}T\\
&-\frac{2(-\mu^2+\lambda^3)\mu'-(-2\mu\mu'+3\lambda^2\lambda')\mu}{2(-\mu^2+\lambda^3)^{3/2}}N\\
&+\frac{2(-\mu'+\lambda^3)(\kappa'\lambda+\kappa\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\kappa\lambda+2\tau\mu(-\mu^2+\lambda^3)}{2(-\mu^2+\lambda^3)^{3/2}}B.
\end{aligned}$$

Note that by chain rule, we have

$$\begin{aligned}
\frac{d\alpha}{ds^*} =&\frac{d\alpha}{ds}\cdot\frac{ds}{ds^*}\\
T^* =&T'\cdot\frac{ds}{ds^*}\\
\frac{\kappa T+\tau B}{\sqrt{|\tau^2-\kappa^2|}} =&(\kappa T+\tau B)\cdot\frac{ds}{ds^*}\\
\frac{1}{\sqrt{|\tau^2-\kappa^2|}} =&\frac{ds}{ds^*}\\
\frac{1}{\sqrt{|\lambda|}} =&\frac{ds}{ds^*}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{dB^*}{ds^*} =&\frac{dB^*}{ds}\cdot\frac{ds}{ds^*}\\
=&\frac{2(-\mu^2+\lambda^3)(\tau'\lambda+\tau\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\tau\lambda+2\kappa\mu(-\mu^2+\lambda^3)}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}T\\
&-\frac{2(-\mu^2+\lambda^3)\mu'-(-2\mu\mu'+3\lambda^2\lambda')\mu}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}N\\
&+\frac{2(-\mu'+\lambda^3)(\kappa'\lambda+\kappa\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\kappa\lambda+2\kappa\mu(-\mu^2+\lambda^3)}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}B.
\end{aligned}$$

Hence,

$$\begin{aligned}
\tau^* =&\|\frac{dB^*}{ds^*}\|\\
=&[\left(\frac{2(-\mu^2+\lambda^3)(\tau'\lambda+\tau\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\tau\lambda+2\kappa\mu(-\mu^2+\lambda^3)}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}\right)^2\\
&\left(-\frac{2(-\mu^2+\lambda^3)\mu'-(-2\mu\mu'+3\lambda^2\lambda')\mu}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}\right)^2\\
&+\left(\frac{2(-\mu'+\lambda^3)(\kappa'\lambda+\kappa\lambda')-(-2\mu\mu'+3\lambda^2\lambda')\kappa\lambda+2\kappa\mu(-\mu^2+\lambda^3)}{2|\lambda|^{1/2}(-\mu^2+\lambda^3)^{3/2}}\right)^2]^{1/2}.
\end{aligned}$$

For more simple notation, we can write equation above as

$$\tau^* =\sqrt{\left(\frac{2a(\tau'\lambda+\tau\lambda')-b\tau\lambda+2\kappa\mu a}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2+\left(-\frac{2a\mu'-b\mu}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2+\left(\frac{2a(\kappa'\lambda+\kappa\lambda')-b\kappa\lambda+2\kappa\mu a}{2|\lambda|^{1/2}(a)^{3/2}}\right)^2},$$

where

$$a=-\mu'+\lambda^3 \quad \text{and} \quad b=-2\mu\mu'+3\lambda^2\lambda'.$$

■

5. Binormal Indicatrix of Timelike Curves

Theorem 9. *Let $\alpha(s)=B(s)$ be a binormal indicatrix of timelike curve parametrized by arc length s. Let the curvature and torsion of timelike curve are nonzero and unequal. Then α is a spacelike curve.*

Proof. Since $\alpha(s)=B(s)$, then $\alpha'(s)=-\tau N(s)$. Therefore, $\langle\alpha',\alpha'\rangle=\tau^2>0$. Hence, α is a spacelike curve. ■

Theorem 10. *Let $\alpha(s)=B(s)$ be a binormal indicatrix curve of timelike curve parametrized by arc length s with non null principal normal vector. Let the curvature and torsion of timelike curve are nonzero and unequal. If $\{T^*,N^*,B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

$$\begin{pmatrix} T^* \\ N^* \\ B^* \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \frac{\kappa}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}} & 0 & \frac{\tau}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}} \\ \frac{\tau}{\sqrt{|-\kappa^2+\tau^2|}} & 0 & \frac{\kappa}{\sqrt{|-\kappa^2+\tau^2|}} \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{15}$$

Proof. Let $\alpha(s)$ have arc-length s^*, then

$$\begin{aligned} &\frac{d\alpha}{ds^*}\frac{ds^*}{ds}=B'(s)=-\tau N \\ &T^*\cdot\frac{ds^*}{ds}=-\tau N. \end{aligned}$$

Since, T^* and N are unit vectors then $T^*=\pm N$. Assume that $T^*=N$ so that $\frac{ds^*}{ds}=-\tau$. Next,

$$\begin{aligned} &\frac{dT^*}{ds^*}\frac{ds^*}{ds}=N' \\ &-\kappa^*N^*\tau=\kappa T+\tau B. \end{aligned} \tag{16}$$

If we take the norm of equation (16) we have

$$\kappa^*\tau=\varepsilon\sqrt{|-\kappa^2+\tau^2|} \tag{17}$$

since N^* can be spacelike or timelike for non null principal normal vector and $\varepsilon=1$ or $\varepsilon=-1$ respectively. From equation (16), we also get

$$N^*=\frac{\kappa T+\tau B}{\kappa^*\tau}=\frac{\kappa T+\tau B}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}}.$$

Since, $T^*\times N^*=\varepsilon B^*$ where $\varepsilon=1$ if α is spacelike with spacelike principal normal and $\varepsilon=-1$ if α is spacelike with timelike principal normal. Therefore,

$$\begin{aligned} B^* =&\varepsilon\cdot T^*\times N^* \\ =&\varepsilon\cdot N\times\left(\frac{\kappa T+\tau B}{\varepsilon\sqrt{|-\kappa^2+\tau^2|}}\right) \\ =&\frac{\kappa(N\times T)+\tau(N\times B)}{\sqrt{|-\kappa^2+\tau^2|}} \\ =&\frac{\kappa B+\tau T}{\sqrt{|-\kappa^2+\tau^2|}} \\ =&\frac{\kappa B+\tau T}{\sqrt{|-\kappa^2+\tau^2|}}. \end{aligned}$$

This completes the proof. ■

Corollary 11. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of timelike curve parametrized by arc length s. Let the curvature and torsion of timelike curve are nonzero and unequal. Suppose $\{T^*, N^*, B^*\}$, κ^* and τ^* are the Frenet frame, curvature and torsion of α, respectively, then*

1. *If $|\kappa| < |\tau|$ then $\alpha(s)$ is a spacelike curve with spacelike principal normal and $\varepsilon = 1$.*
2. *If $|\kappa| > |\tau|$ then $\alpha(s)$ is a spacelike curve with timelike principal normal and $\varepsilon = -1$.*

Proof. It is clear from equation (15) that $\langle N^*, N^* \rangle = \frac{-\kappa^2+\tau^2}{|-\kappa^2+\tau^2|}$. ■

Theorem 12. *Let $\alpha(s) = B(s)$ be a binormal indicatrix of timelike curve parametrized by arc length s. Let the curvature and torsion of timelike curve are nonzero and unequal. Suppose κ^* and τ^* be the curvature and torsion of α, respectively, then*

$$\kappa^* = \frac{\varepsilon\sqrt{-\kappa^2+\tau^2}}{\tau} \tag{18}$$

and

$$\tau^* = \frac{\kappa'\tau - \kappa\tau'}{\tau(\tau^2-\kappa^2)}. \tag{19}$$

Proof. From equation (17), it is clear that

$$\kappa^* = \frac{\varepsilon\sqrt{-\kappa^2+\tau^2}}{\tau}.$$

If we take the derivative of B^* toward s then,

$$\begin{aligned}
\frac{dB^*}{ds^*}\frac{ds^*}{ds} &= \frac{\kappa' B - \kappa\tau N + \tau' T + \kappa\tau N}{(-\kappa^2+\tau^2)^{\frac{1}{2}}} - \frac{(-\kappa\kappa' + \tau\tau')(\kappa B + \tau T)}{(-\kappa^2+\tau^2)^{3/2}} \\
\frac{dB^*}{ds^*}\tau &= \frac{(-\kappa^2+\tau^2)(\kappa' B + \tau' T) - (-\kappa\kappa' + \tau\tau')(\kappa B + \tau T)}{(-\kappa^2+\tau^2)^{3/2}} \\
\frac{dB^*}{ds^*}\tau &= \frac{(-\kappa^2\tau' + \kappa\kappa'\tau)T + (\kappa'\tau^2 - \kappa\tau\tau')B}{(-\kappa^2+\tau^2)^{3/2}} \\
\frac{dB^*}{ds^*}\tau &= \frac{(-\kappa\tau' + \kappa'\tau)(\kappa T + \tau B)}{(-\kappa^2+\tau^2)^{3/2}} \\
\frac{dB^*}{ds^*} &= \frac{(-\kappa\tau' + \kappa'\tau)(\kappa T + \tau B)}{\tau(-\kappa^2+\tau^2)^{3/2}}.
\end{aligned}$$

Since $\tau^* = \|\frac{dB^*}{ds^*}\|$, then

$$\tau^* = \|\frac{dB^*}{ds^*}\| = \left(\frac{(-\kappa\tau' + \kappa'\tau)^2(-\kappa^2+\tau^2)}{\tau^2(-\kappa^2+\tau^2)^3}\right)^{\frac{1}{2}} = \frac{-\kappa\tau' + \kappa'\tau}{\tau(-\kappa^2+\tau^2)}.$$

■

6. Numerical Example

Example 13. *Let $C: I \to \mathbb{E}_1^3$ defined by*

$$C = \left(\sqrt{2}s, \sin s, -\cos s\right).$$

With simple calculation, we have the Frenet frame as follows:

$$\begin{aligned}
&T = \left(\sqrt{2}, \cos s, \sin s\right), \\
&N = (0, -\sin s, \cos s), \\
&B = \left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right), \\
&\kappa = 1 \quad \textit{and} \quad \tau = -\sqrt{2}.
\end{aligned}$$

1. *Tangential indicatrix of C*
Let $\alpha(s) = T(s)) = \left(\sqrt{2}, \cos s, \sin s\right)$ *be a tangential normal curve. Since* $|\kappa| < |\tau|$, N^* *is spacelike and* $\varepsilon = 1$. *Then, by applying Theorem 2, we have*

$$\begin{aligned} T^* =& N = (0, -\sin s, \cos s)\,, \\ N^* =& \frac{1}{\sqrt{(|-1+2|)}} T + \frac{-\sqrt{2}}{\sqrt{(|-1+2|)}} B = \left(\sqrt{2}, \cos s, \sin s\right) - \sqrt{2}\left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = (0, -\cos s, -\sin s)\,, \\ B^* =& \frac{-\sqrt{2}}{\sqrt{(|-1+2|)}} T + \frac{1}{\sqrt{(|-1+2|)}} B = -\sqrt{2}\left(\sqrt{2}, \cos s, \sin s\right) + \left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = (-1, 0, 0)\,. \end{aligned}$$

Then, by using Theorem 4, we have

$$\kappa^* = \frac{\sqrt{|-(1)^2 + (-\sqrt{2})^2|}}{1} = 1, \quad \tau^* = \frac{0 \cdot (-\sqrt{2}) - 1 \cdot 1}{1 \cdot \sqrt{|-(1)^2 + (-\sqrt{2})^2|}} = 0.$$

From the above equations, it can be seen that $\alpha(s)$ *is a spacelike plane curve with spacelike principal normal.*

2. *Principal normal indicatrix of C*
Let $\alpha(s) = N(s) = (0, -\sin s, \cos s)$ *be a principal normal curve. Applying Theorem 6 we have*

$$\lambda = |-(1)^2 + (-\sqrt{2})^2| = 1 \quad \mu = 0 \cdot (-\sqrt{2}) - 1 \cdot 0 = 0, \quad a = 0 + 1 = 1, \quad b = -2(0) + 3(1)(0) = 0.$$

So that,

$$\begin{aligned} T^* =& T - \sqrt{2}B = \left(\sqrt{2}, \cos s, \sin s\right) - \sqrt{2}\left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = (0, -\cos s, -\sin s)\,, \\ N^* =& -\frac{1}{\sqrt{1}} N = (0, \sin s, -\cos s)\,, \\ B^* =& -\sqrt{2}T + B = -\sqrt{2}\left(\sqrt{2}, \cos s, \sin s\right) + \left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = (-1, 0, 0)\,. \end{aligned}$$

Next, by using Theorem 8, we get

$$\begin{aligned} \kappa^* =& \frac{\sqrt{(0)^2 + (1)^2}}{1} = 1, \\ \tau^* =& \sqrt{\left(\frac{2(1)(0) - 0 \cdot \sqrt{2} \cdot 1 + 2 \cdot 1 \cdot 0 \cdot 1}{2|1|^{1/2}(1)^{3/2}}\right)^2 + \left(-\frac{2 \cdot 1 \cdot 0}{2|1|^{1/2}(1)^{3/2}}\right)^2 + \left(\frac{2(1)(0) - 0 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 0 \cdot 1}{2|1|^{1/2}(1)^{3/2}}\right)^2} = 0. \end{aligned}$$

From the above equations, it can be seen that $\alpha(s)$ *is a spacelike plane curve with spacelike principal normal.*

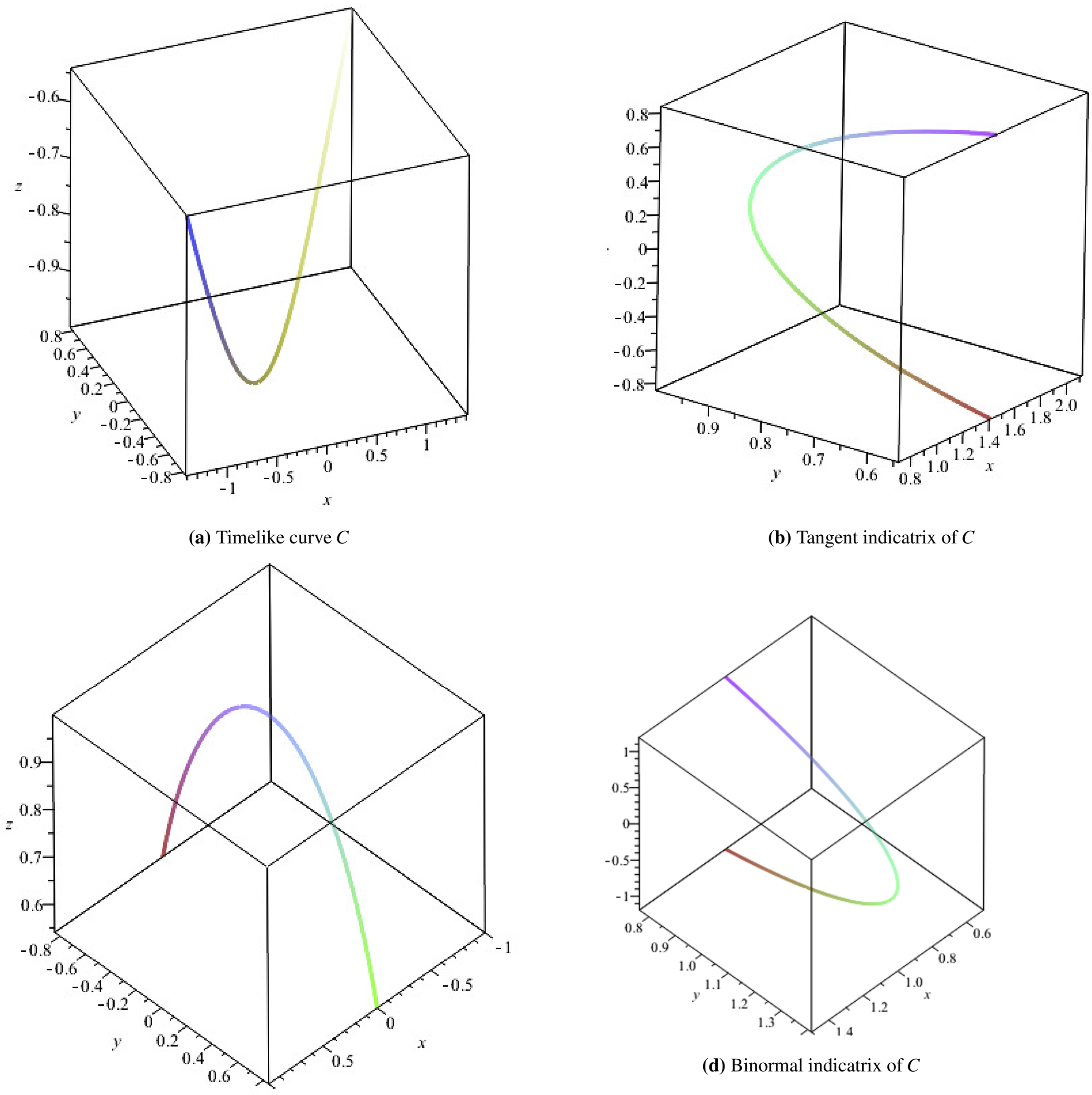
3. *Binormal indicatrix of C Let* $\alpha(s) = B(s) = \left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right)$ *be a binormal indicatrix curve. Since* $|\kappa| < |\tau|$, N^* *is spacelike and* $\varepsilon = 1$. *Then applying Theorem 10 yield,*

$$\begin{aligned} T^* =& -N = (0, \sin s, -\cos s)\,, \\ N^* =& \frac{-1}{\sqrt{(|-1+2|)}} T + \frac{\sqrt{2}}{\sqrt{(|-1+2|)}} B = \left(-\sqrt{2}, -\cos s, -\sin s\right) + \sqrt{2}\left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = (0, \cos s, \sin s)\,, \\ B^* =& \frac{\sqrt{2}}{\sqrt{(|-1+2|)}} T + \frac{1}{\sqrt{(|-1+2|)}} B = \sqrt{2}\left(\sqrt{2}, \cos s, \sin s\right) + \left(1, \sqrt{2}\cos s, \sqrt{2}\sin s\right) = \left(3, 2\sqrt{2}\cos s, 2\sqrt{2}\sin s\right). \end{aligned}$$

Then, by using Theorem 12, we have

$$\kappa^* = \frac{\sqrt{2}\sqrt{|-(1)^2 + \sqrt{2}^2|}}{\sqrt{2}} = 1, \quad \tau^* = \frac{0 \cdot \sqrt{2} - 1 \cdot 1}{\sqrt{2} \cdot \sqrt{|-(1)^2 + \sqrt{2}^2|}} = 0,$$

since the length of the speed vector is $\sqrt{2}$. *From the above equations, it can be seen that* $\alpha(s)$ *is a spacelike plane curve with spacelike principal normal.*



(a) Timelike curve C

(b) Tangent indicatrix of C

(c) Principal normal indicatrix of C

(d) Binormal indicatrix of C

Fig. 1. Plot of timelike curve C and their spherical indicatrix images

7. Conclusions

Based on the definition and theorems derived in the previous sections, we find that the curve generated by tangent and binormal vectors of timelike curves are spacelike curves while the curve generated by principal normal vector of timelike curves can be spacelike, timelike or null. This work can be useful for further research such as constructing and analyzing surfaces generated by spherical indicatrices of timelike curves.

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