Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 2, No. 2, 48-55, 2020
Research Article

Geometric Elements of Constant Precession Curve

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Abstract
In this paper, we determine the geodesic curvature and the geodesic torsion of the constant precession curve, and the normal curvature of the circular hyperboloid of one-sheet in the direction of tangent vector of the constant precession curve, through the Darboux frame. We give the causal character of the constant precession curve in Minkowski space and we state the constant angle that its principal normal makes with fixed direction. Moreover, we give some angles just as, the angle between the osculating plane of the constant precession curve and the tangent plane to the circular hyperboloid of one-sheet; the angle between principal unit normal of the constant precession curve and unit normal vector of the circular hyperboloid of one-sheet, in terms of curvatures of the curve.

Keywords and 2010 Mathematics Subject Classification
Keywords: Constant precession curve — slant helix — hyperboloid of one sheet
MSC: 53A04, 14H45, 14H50

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Article History: Received 8 October 2020; Accepted 15 November 2020

1. Introduction

The concept of the slant helix is firstly introduced by Izumiya and Takeuchi [5]. They characterized the slant helices by the following proposition.

Proposition 1. *Let γ be a unit speed curve with $\kappa(s) \neq 0$. Then γ is a slant helix if and only if*

$$\sigma(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}} \left(\frac{\tau}{\kappa}\right)' \right)(s)$$

is a constant function.

A curve of constant precession is defined by the property that as the curve is traversed with unit speed, its centrode revolves about a fixed axis with constant angle and constant speed. Scofield [9] obtained an arclength parametrized closed-form solution of the natural equations for curves of constant precession through the direct geometric analysis.

In [6], Kula and Yaylı investigated the spherical images, the tangent indicatrix and the binormal indicatrix of a slant helix. They obtained the spherical images of the slant helices as spherical helices. Also they showed that the curve of constant precession is a slant helix.

In [1], the position vector of a slant helix with respect to standard frame in Euclidean space $\mathbb{R}^3$ is studied in terms of Frenet equations. Ali determined the position vector of an arbitrary slant helix and the parametric representation of the slant helices from the intrinsic equations. Then he applied this method to find the parametric representation of a curve of constant precession, as examples of a slant helices, by means of intrinsic equations.

In [2], Ali and Turgut introduced the type-2 harmonic curvatures of the slant helices by meanings of the Frenet-Serret formulae of the curve. They expressed the constant angle between the principal normal vector of the slant helices and fixed direction in terms of the type-2 harmonic curvatures.

In this paper, we determine the geodesic curvature and the geodesic torsion of the constant precession curve (lies on the circular hyperboloid of one-sheet) by the help of the Darboux frame. We find the normal curvature of the circular hyperboloid of one-sheet, in the direction of tangent vector of the constant precession curve. It is known that the principal normal of the slant helices makes the constant angle any unit fixed direction (see [5]). It is shown in [6, Theorem 2.5] that the constant precession curve is a slant helix. So, this leads to such a question,

"What is the constant angle between the principal normal vector of the constant precession curve and the fixed line?"

By the help of [1, Lemma 3.2] we show that the principal normal of the constant precession curve makes the constant angle $\widehat{\xi} = \cot^{-1}(\mu/\omega)$ with the fixed direction. In terms of [6, Theorem 2.5] this angle is equal to $\cot^{-1}\sigma(s)$. Moreover, we get the expression of the angle between the osculating plane of the constant precession curve and the tangent plane to the circular hyperboloid of one-sheet, depends on the curvature of the curve $\kappa(s)$ and the type-2 harmonic curvature. Let us give the inner product of two vectors in $n-$dimensional Euclidean space by

$$u \cdot v = \sum_{i=1}^{n} u_i v_i,$$

where $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$. Let $\gamma : I \to \mathbb{R}^3$ be a regular curve in $\mathbb{R}^3$ Euclidean space (i.e. $\|\gamma'\|$ is nowhere zero), where I is an interval in $\mathbb{R}$. For each unit speed curve there occur an associated orthonormal $3-$frame $\{T, N, B\}$ along γ, the Frenet $3-$frame, and functions $\kappa, \tau : I \longrightarrow \mathbb{R}$, the Frenet curvatures, such that

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{1}$$

Here κ and τ is called the curvature and the torsion of the curve, respectively. The principal normal vector of the curve $N(s) = T'(s)/\|T'(s)\|$ and the binormal vector $B = T \times N$.

We introduce some useful concepts, with the help of [3]. Let us consider a biregular curve $\gamma(s)$ on a given surface M. There are two natural orthonormal frames at a point in $\gamma(s)$, the Frenet frame and the frame whose first two vectors are the tangent to $\gamma(s)$ and the normal to M. If we call the corresponding normals N and U, respectively, the difference between the two bases is given just by the angle ϕ between N and U. This angle is sure to be something interesting, as is its derivative $d\phi/ds$, where s is the arclength along γ. The right invariant turns out to be the function

$$\tau_g(s) = \tau(s) - \frac{d\phi}{ds} \tag{2}$$

called the geodesic torsion of $\gamma(s)$. We note that, the geodesic torsion of γ on M depends only on the tangent to γ, and not, as the curvature in general does, on the normal to γ. In fact, we have the formula

$$\tau_g = (k_2 - k_1)\sin\varphi\cos\varphi,$$

where φ is the angle that the tangent to γ makes with the principal direction associated with k_1. As a consequence, curves of constant geodesic torsion coincide with lines of curvature. Additionally, we can give the geodesic torsion, in terms of [[4], p.153] such that

$$\tau_g = \frac{dn}{ds}(0) \cdot h,$$

where n is the unit normal of the surface, $\gamma(s)$ is the unit speed curve on hypersurface M, $\gamma(0) = p$, $\gamma'(s) = T$, and $\{T, h\}$ is orthonormal basis of T_pM.

In the light of [8], we define the curvatures. For a unit vector $u \in T_pM$, the normal curvature of M in the $u-$direction is

$$k(u) = S_p(u) \cdot u,$$

where S_p is the shape operator of the surface at point p. Let γ be a curve with $\gamma(0) = p$, $\gamma'(0) = u$. Then

$$k(u) = \kappa(0)\cos\theta, \tag{3}$$

where N is the Frenet normal to the curve γ and κ is the curvature of γ. The angle θ is the angle between $N(0)$ and $U(p)$. In this case, the geodesic curvature $\kappa_g = \kappa(0)\sin\theta$ which leads to $\kappa^2 = k^2(u) + \kappa_g^2$.

In the light of [7], we can associate above. Let $\{T(s), V(s), U(s)\}$ be the moving Darboux frame on the curve γ, where $U(s)$ is the surface normal restricted to γ and $V(s) = U(s) \times T(s)$. Then, we have

$$\begin{pmatrix} T' \\ V' \\ U' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ V \\ U \end{pmatrix}, \tag{4}$$

where κ_n, κ_g and τ_g are the normal curvature of the surface in the direction of T, the geodesic curvature and the geodesic torsion of the curve γ, respectively.

In [1], slant helix characterized by following.

Lemma 2. *Let $\psi : I \longrightarrow \mathbb{R}^3$ be a curve that is parametrized by arclength with intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$. The curve is a slant helix (its normal vectors make a constant angle, $\theta = \pm \arccos(n)$ with a fixed straight line in the space) if and only if*

$$\tau(s) = \pm \frac{m\kappa(s) \int \kappa(s)ds}{\sqrt{1 - m^2 \left(\int \kappa(s)ds\right)^2}},$$

where $m = \frac{n}{\sqrt{1-n^2}}$.

In [2], Ali and Turgut defined type-2 harmonic curvatures for slant helix by the help of the harmonic curvatures and they gave the following theorem:

Theorem 3. *Let $\alpha : I \longrightarrow \mathbb{R}^n$ be a unit speed curve in $n-$dimensional Euclidean space. Let $\{V_1, V_2, ..., V_n\}, \{G_1, G_2, ..., G_n\}$ denote the Frenet frame and the higher ordered type-2 harmonic curvatures of the curve, respectively. Then, if α is a slant helix the following condition is satisfied*

$$\sum_{i=1}^{n} G_i^2 = C,$$

where C is a non-zero constant. Moreover, the constant $C = \sec^2\theta$, θ being the angle that V_2 makes with the fixed direction U that determines α.

Finally in [6], the following result is given.

Theorem 4. *A curve of constant precession is a slant helix.*

Throughout this article, we consider the curvature and the torsion of the constant precession curve with respect to [9], as $\kappa(s) = -\omega \sin(\mu s)$ and $\tau(s) = \omega \cos(\mu s)$, $(\omega > 0)$ respectively, unless otherwise mentioned.

2. Main Results

In this section, in the light of [9], we determine the geodesic curvature and the geodesic torsion of the constant precession curve by the help of the Darboux frame. Moreover, we find the normal curvature of the circular hyperboloid of one-sheet, in the direction of tangent vector of the constant precession curve and we get the constant angle that the principal normal of the constant precession curve makes with the fixed direction.

Let us call the constant precession curve $\gamma(s) = (x(s), y(s), z(s))$ on the circular hyperboloid of one-sheet

$$\begin{aligned} x(s) &= \frac{\alpha+\mu}{2\alpha}\frac{\sin[(\alpha-\mu)s]}{\alpha-\mu} - \frac{\alpha-\mu}{2\alpha}\frac{\sin[(\alpha+\mu)s]}{\alpha+\mu}, \\ y(s) &= -\frac{\alpha+\mu}{2\alpha}\frac{\cos[(\alpha-\mu)s]}{\alpha-\mu} + \frac{\alpha-\mu}{2\alpha}\frac{\cos[(\alpha+\mu)s]}{\alpha+\mu}, \\ z(s) &= \frac{\omega}{\mu\alpha}\sin(\mu s), \end{aligned} \tag{5}$$

where $\alpha^2 = \omega^2 + \mu^2$ and ω, μ are constant. The equation of the circular hyperboloid of one-sheet M, in cartesian coordinates in the ambient space $\mathbb{R}^3$, is given by

$$x^2 + y^2 - \frac{\mu^2}{\omega^2}z^2 = \frac{4\mu^2}{\omega^4}. \tag{6}$$

Here, the constant precession curve $\gamma(s)$ lies on M and it is closed if and only if μ/α is rational. Let we give our main results.

Theorem 5. *Let $\gamma(s)$ be a constant precession curve that lies on the circular hyperboloid of one-sheet M and $T(s)$ be a unit tangent vector field of the curve. The normal curvature of the circular hyperboloid of one-sheet, in the direction of T is given by*

$$\kappa_n(s) = -\frac{\kappa^2(s)}{\sqrt{4\mu^2+\kappa^2(s)}} \quad . \tag{7}$$

Proof. We use the derivative formulas (4) of Darboux frame to determine the normal curvature of the circular hyperboloid of one-sheet, in the direction of T. From (5), the unit speed tangent vector field of $\gamma(s)$ is given by $T(s) = (T_1(s), T_2(s), T_3(s))$ such that

$$\begin{aligned}
T_1(s) &= \frac{\alpha+\mu}{2\alpha}\cos[(\alpha-\mu)s] - \frac{\alpha-\mu}{2\alpha}\cos[(\alpha+\mu)s], \\
T_2(s) &= \frac{\alpha+\mu}{2\alpha}\sin[(\alpha-\mu)s] - \frac{\alpha-\mu}{2\alpha}\sin[(\alpha+\mu)s], \\
T_3(s) &= \frac{\omega}{\alpha}\cos(\mu s).
\end{aligned} \tag{8}$$

In cartesian coordinates, we determine the unit normal U of M as

$$U = \left(\frac{\omega^2 x}{\mu\sqrt{4+(\mu^2+\omega^2)z^2}}, \frac{\omega^2 y}{\mu\sqrt{4+(\mu^2+\omega^2)z^2}}, -\frac{\mu z}{\mu\sqrt{4+(\mu^2+\omega^2)z^2}}\right). \tag{9}$$

It follows from (5) and (9), the restricted unit normal on the curve γ is $U(\gamma(s)) = (U_1(s), U_2(s), U_3(s))$ such that

$$\begin{aligned}
U_1(s) &= \frac{\omega^2\left(\frac{\alpha+\mu}{2\alpha}\frac{\sin[(\alpha-\mu)s]}{\alpha-\mu} - \frac{\alpha-\mu}{2\alpha}\frac{\sin[(\alpha+\mu)s]}{\alpha+\mu}\right)}{\sqrt{4\mu^2+\omega^2\sin^2(\mu s)}}, \\
U_2(s) &= \frac{\omega^2\left(-\frac{\alpha+\mu}{2\alpha}\frac{\cos[(\alpha-\mu)s]}{\alpha-\mu} + \frac{\alpha-\mu}{2\alpha}\frac{\cos[(\alpha+\mu)s]}{\alpha+\mu}\right)}{\sqrt{4\mu^2+\omega^2\sin^2(\mu s)}}, \\
U_3(s) &= \frac{\mu\omega\sin(\mu s)}{\sqrt{\mu^2+\omega^2}\sqrt{4\mu^2+\omega^2\sin^2(\mu s)}}.
\end{aligned} \tag{10}$$

The equations of (8), (10), and $\kappa_n(s) = T'(s)\cdot U(\gamma(s))$ gives the intended. ∎

Theorem 6. *Let $\gamma(s)$ be a constant precession curve that lies on the circular hyperboloid of one-sheet. The geodesic curvature and the geodesic torsion of constant precession curve is given by*

$$\kappa_g(s) = \frac{2\mu\kappa(s)}{\sqrt{4\mu^2+\kappa^2(s)}} \tag{11}$$

and

$$\tau_g(s) = \left(\frac{2\mu^2+\kappa^2(s)}{4\mu^2+\kappa^2(s)}\right)\tau(s), \tag{12}$$

respectively.

Proof. Let us determine the tangent-normal vector $V(s)$ of the Darboux frame of the curve by $V(s) = U(s)\times T(s) = (V_1(s), V_2(s), V_3(s))$. By straightforward calculations we get

$$\begin{aligned}
V_1(s) &= -\frac{\omega\sqrt{4\mu^2+\omega^2\sin^2(\mu s)}\{\mu(3+\cos(2\mu s))\cos(\alpha s) + \alpha\sin(2\mu s)\sin(\alpha s)\}}{2\alpha\left(4\mu^2+\omega^2\sin^2(\mu s)\right)}, \\
V_2(s) &= \frac{\omega\sqrt{4\mu^2+\omega^2\sin^2(\mu s)}\{\alpha\cos(\alpha s)\sin(2\mu s) - \mu(3+\cos(2\mu s))\sin(\alpha s)\}}{2\alpha\left(4\mu^2+\omega^2\sin^2(\mu s)\right)}, \\
V_3(s) &= \frac{\left(4\mu^2+\omega^2-\omega^2\cos(2\mu s)\right)\sqrt{8\mu^2+\omega^2-\omega^2\cos(2\mu s)}}{2\sqrt{2}\alpha\left(4\mu^2+\omega^2\sin^2(\mu s)\right)}.
\end{aligned} \tag{13}$$

It is obvious from (4) that the geodesic curvature and the geodesic torsion of the curve is given by

$$\kappa_g(s) = T'(s) \cdot V(s) \tag{14}$$

and

$$\tau_g(s) = V'(s) \cdot U(s), \tag{15}$$

respectively. The equations of (8), (10) and (13) gives the intended. ■

Corollary 7. *The angle θ between principal unit normal of the constant precession curve and the unit normal vector of the circular hyperboloid of one-sheet satisfies*

$$\cos(\theta(s)) = \frac{\omega \sin(\mu s)}{\sqrt{4\mu^2 + \omega^2 \sin^2(\mu s)}}.$$

Proof. This result seen from (3) and (7). ■

Corollary 8. *Let $\gamma(s)$ be a constant precession curve. The causal character of γ is given by following:*

$$\begin{cases} \gamma \text{ is spacelike, } \sigma^2(s) > \cos(2\mu s) \\ \gamma \text{ is timelike, } \sigma^2(s) < \cos(2\mu s) \\ \gamma \text{ is lightlike, } \sigma^2(s) = \cos(2\mu s) \end{cases} \tag{16}$$

where $\sigma(s) = \dfrac{\mu}{\omega}$.

Proof. Let $\mathbb{R}_1^3$ be the 3−dimensional Minkowski space endowed with the canonical flat metric $ds^2 = dx_1^2 + dx_2^2 - dx_3^2$, where $x = (x_1, x_2, x_3) \in \mathbb{L}^3$. From (8), we get

$$T(s) \cdot T(s) = \frac{\mu^2 - \omega^2 \cos(2\mu s)}{\mu^2 + \omega^2}.$$

Since $\sigma(s) = \dfrac{\mu}{\omega}$ for constant precession curve, (16) holds. ■

Theorem 9. *Let $\gamma(s)$ be a constant precession curve that lies on the circular hyperboloid of one-sheet M, ϕ be the angle between the osculating plane of the constant precession curve and the tangent plane of the surface M. The corresponding angle ϕ is given by*

$$\phi(s) = -\tan^{-1}\left(\frac{\kappa(s)}{2\mu}\right).$$

Proof. It follows from (2) and (12) that

$$\tau(s) - \frac{d\phi}{ds} = \left(\frac{2\mu^2 + \kappa^2(s)}{4\mu^2 + \kappa^2(s)}\right)\tau(s),$$

where $\tau(s) = \omega \cos(\mu s)$. Hence, we get

$$\phi(s) = \int \frac{2\mu^2 \omega \cos(\mu s)}{4\mu^2 + \omega^2 \sin^2(\mu s)} ds = \tan^{-1}\left(\frac{\omega \sin(\mu s)}{2\mu}\right) = -\tan^{-1}\left(\frac{\kappa(s)}{2\mu}\right)$$

and this completes the proof. ■

Corollary 10. *Let θ be the angle between principal unit normal of the constant precession curve and the unit normal vector of the circular hyperboloid of one-sheet; ϕ be the angle between the osculating plane of the curve and the tangent plane of the surface. These angles satisfy*

$$\sec^2\theta = 1 + \cot^2\phi,$$

that is θ and ϕ are complementary angles.

Proof. This result seen from Corollary 7 and Theorem 9 . ■

The alternative way for determine the Frenet curvatures of the constant precession curve is given by the following.

Corollary 11. *The curvature and the torsion of the curve of constant precession is given by*

$$\kappa(s) = -2\mu\tan(\phi(s))$$

and

$$\tau(s) = \sqrt{\omega^2 - 4\mu^2\tan^2(\phi(s))},$$

respectively. Here, ϕ is the angle between the osculating plane of the constant precession curve and the tangent plane to the surface.

Proof. It is obvious from Theorem 9 that $\kappa(s) = -2\mu\tan(\phi(s))$. This leads to $\sin(\mu s) = 2\mu\tan(\phi(s))/\omega$ and so $\cos(\mu s) = \sqrt{\omega^2 - 4\mu^2\tan^2(\phi(s))}/\omega$, that is $\tau(s) = \sqrt{\omega^2 - 4\mu^2\tan^2(\phi(s))}$. ■

Theorem 12. *Let the principal normal vector of the constant precession curve makes the constant angle $\widehat{\xi}$ with fixed direction. This angle is given by*

$$\widehat{\xi} = \cot^{-1}\left(\frac{\mu}{\omega}\right). \tag{17}$$

Proof. It is known from Theorem 4 that the constant precession curve is a slant helix. Thus, it follows from Lemma 2 that the torsion of the constant precession curve is given by

$$\tau(s) = \pm\frac{\omega m\sin(\mu s)\cos(\mu s)}{\mu\sqrt{1 - \omega^2 m^2\frac{\cos^2(\mu s)}{\mu^2}}},$$

where $m = \dfrac{n}{\sqrt{1-n^2}}$ and $\widehat{\xi} = \pm\cos^{-1}(n)$. Besides, the torsion of the constant precession curve is $\tau(s) = \omega\cos(\mu s)$, which leads to

$$\omega^2\cos^2(\mu s) = \frac{\omega^2\cot^2\widehat{\xi}\sin^2(\mu s)\cos^2(\mu s)}{\mu^2\left(1 - \frac{\omega^2\cot^2\widehat{\xi}}{\mu^2}\cos^2(\mu s)\right)}.$$

By straightforward calculations, we get $\cot^2\widehat{\xi} = \left(\frac{\mu}{\omega}\right)^2$ which is intended. ■

Corollary 13. *The constant angle between the principal normal vector of the constant precession curve and the fixed direction is given by*

$$\widehat{\xi} = \cot^{-1}\sigma(s),$$

where $\sigma(s) = \left(\dfrac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\dfrac{\tau}{\kappa}\right)'\right)(s)$.

Proof. It seen from proof of Theorem 4 that $\sigma(s) = \dfrac{\mu}{\omega}$. Thus, (17) completes the proof. ■

In the light of [2], we give the type-2 harmonic curvatures of the constant precession curve as

$$G_1(s) = \frac{\omega}{\mu}\cos(\mu s),$$

$$G_2(s) = 1,$$

$$G_3(s) = -\frac{\omega}{\mu}\sin(\mu s)$$

It seen that $G_1^2(s)+G_2^2(s)+G_3^2(s)=1+\left(\frac{\omega}{\mu}\right)^2$ which leads to

$$G_1^2(s)+G_2^2(s)+G_3^2(s)=\sec^2\widehat{\xi}.$$

Thus, Theorem 12 coincides with Theorem 3.

Corollary 14. *The constant angle between the principal normal vector of the constant precession curve and the fixed direction satisfies*

$$\cot\widehat{\xi}=\sqrt{\frac{1}{G_1^2(s)+G_3^2(s)}}, \tag{18}$$

where $G_1(s)$ and $G_3(s)$ are the type-2 harmonic curvatures of the constant precession curve.

Corollary 15. *In terms of the type-2 harmonic curvatures, the angle ϕ between the osculating plane of the curve of constant precession and the tangent plane of the surface is given by*

$$\phi(s)=-\tan^{-1}\left(\frac{G_3(s)}{2}\right).$$

Theorem 16. *γ is a constant precession curve if and only if $\alpha(s)=(\alpha_1(s),\alpha_2(s))$ is a circle with radius of $\frac{\omega}{\mu}$, where $\alpha_1(s)=\int\kappa(s)ds$ and $\alpha_2(s)=\int\tau(s)ds$.*

Proof. If γ is a constant precession curve then $\kappa(s)=-\omega\sin(\mu s)$ and $\tau(s)=\omega\cos(\mu s)$. Hence

$$\alpha_1(s)=-\omega\int\sin(\mu s)\,ds=\frac{\omega}{\mu}\cos(\mu s)$$

and

$$\alpha_2(s)=\omega\int\cos(\mu s)\,ds=\frac{\omega}{\mu}\sin(\mu s)$$

which leads to $\alpha(s)=\left(\frac{\omega}{\mu}\cos(\mu s),\frac{\omega}{\mu}\sin(\mu s)\right)$. Thus, α is a circle with radius of $\frac{\omega}{\mu}$. Conversely, without losing the orientation, we can consider $\alpha(s)=\left(\frac{\omega}{\mu}\cos(\mu s),\frac{\omega}{\mu}\sin(\mu s)\right)$ which leads to

$$\alpha_1(s)=\frac{\omega}{\mu}\cos(\mu s)=\int\kappa(s)ds \tag{19}$$

and

$$\alpha_2(s)=\frac{\omega}{\mu}\sin(\mu s)=\int\tau(s)ds. \tag{20}$$

From (19) and (20) we get $\kappa(s)=-\omega\sin(\mu s)$ and $\tau(s)=\omega\cos(\mu s)$, respectively. Moreover, the curvature of α,

$$\kappa_\alpha=\frac{\alpha_1'(s)\alpha_2''(s)-\alpha_1''(s)\alpha_2'(s)}{\left(\alpha_1'(s)^2+\alpha_2'(s)^2\right)^{3/2}}=\frac{\kappa^2(s)}{\left(\kappa^2(s)+\tau^2(s)\right)^{3/2}}\left(\frac{\tau}{\kappa}\right)'(s)=\frac{\mu}{\omega}$$

is constant. Thus, $\sigma(s)=\frac{\mu}{\omega}$ is constant and this completes the proof. ■

3. Conclusions

In the present study, we examined the curvatures of the Darboux frame of the constant precession curve and we gave some important angles that characterize the relevant pair of curve-surface. The causal character of the constant precession curve is given in order to its geodesic curvature of the spherical image of the principal normal indicatrix. In the further studies, we hope to get curvatures of the Darboux frame of the Salkowski and anti-Salkowski curve. Also we can study on another slant helix or any special curve in the ambient spaces that have different metrics from the present study.

4. Acknowledgment

The author would like to thank the referee for carefully reading which helped to improve the manuscript.

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