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On Some Special Functions for Conformable Fractional Integrals

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Abstract

In this paper, we introduce the (α,k)-gamma function, (α,k)-beta function, Pochhammer symbol $(x)_{n,k}^{\alpha}$ and Laplace transforms for conformable fractional integrals. We prove several properties generalizing those satisfied by the classical gamma function, beta function and Pochhammer symbol. The results presented here would provide generalizations of those given in earlier works.

Keywords: *Gamma function, beta function, Pochhammer symbol, confromable fractional integrals.*
2010 Mathematics Subject Classification: *26D15, 26A51, 26A33, 26A42.*

1. Introduction

The classical Euler gamma function or Euler integral of the second kind is given by

$$\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt,\ x>0$$

and the beta function or Eulerian integral of the first kind with two variables is defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\ x,\ y>0.$$

Therefore, the classical beta function in terms of gamma function is defined in [3] as

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x,\ y>0.$$

The rising factorial $x^{(n)}$, sometimes also denoted $(x)_n$([5, p. 6]) or x^n ([9, p. 48]), is defined by

$$x^{(n)}=x(x+1)\dots(x+n-1).$$

This function is also known as the rising factorial power ([9, p. 48]) and frequently called the Pochhammer symbol in the theory of special functions. The rising factorial is implemented in the Wolfram Language as Pochhammer $[x,n]$. In recently, Diaz and Pariguan give a new definition for the function of variable x as follows

$$(x)_{n,k}=x(x+k)(x+2k)\dots(x+(n-1)k)$$

and they called the Pochhammer k-symbol. Setting $k=1$ one obtains the usual Pochhammer symbol $(x)_n$. Recently, in a series of research publications, Diaz et al. ([6]-[8]) have introduced k-gamma and k-beta functions and proved a number of their properties. They have also studied k-zeta function and k-hypergeometric functions based on Pochhammer k-symbols for factorial functions. The k-gamma function is defined by

$$\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(kn)^{\frac{x}{k}-1}}{(x)_{n,\alpha}},\ k>0.$$

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It has been shown that the Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k-gamma function, explicitly given by

$$\Gamma_k(x)=\int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt.$$

Clearly, $\Gamma(x)=\lim_{k\to 1}\Gamma_k(x), \Gamma_k(x)=k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$ and $\Gamma_k(x+k)=x\Gamma_k(x)$. This gives rise to k-beta function defined by

$$B_k(x,y)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt$$

so that $B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right)$ and $B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}$.

The purpose of this paper is to introduce (α,k)-gamma function and (α,k)-beta function for conformable fractional integrals and obtain some of their properties. When $(\alpha,k)\to(1,1)$, it turns out to be the usual gamma function and beta function.

2. Definitions and Properties of Conformable Fractional Derivative and Integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [1], [2], [10], [12]-[17].

Definition 2.1. *(**Conformable fractional derivative**) Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \tag{2.1}$$

for all $t>0,\ \alpha\in(0,1)$. If f is $\alpha-$differentiable in some $(0,a),\ \alpha>0,\ \lim_{t\to 0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to 0^+}f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable. For $2\le n\in N$, we denote $D_\alpha^n(f)(t)=D_\alpha D_\alpha^{n-1}(f)(t)$.

Theorem 2.2. *Let $\alpha\in(0,1]$ and f,g be α-differentiable at a point $t>0$. Then*
i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,
ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,
iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,
iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{gD_\alpha(f)-fD_\alpha(g)}{g^2}$.
v. If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \tag{2.3}$$

Definition 2.3 (Conformable fractional integral)**.** *Let $\alpha\in(0,1]$ and $0\le a<b$. A function $f:[a,b]\to\mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\,d_\alpha x:=\int_a^b f(x)x^{\alpha-1}dx \tag{2.4}$$

exists and is finite.

Remark 2.4.

$$I_\alpha^a(f)(t)=I_1^a\left(t^{\alpha-1}f\right)=\int_a^t\frac{f(x)}{x^{1-\alpha}}dx,$$

where the integral is the usual Riemann improper integral, and $\alpha\in(0,1]$.

Theorem 2.5. *Let $f:(a,b)\to\mathbb{R}$ be differentiable and $0<\alpha\le 1$. Then, for all $t>a$ we have*

$$I_\alpha^a D_\alpha^a f(t)=f(t)-f(a). \tag{2.5}$$

Theorem 2.6. *(**Integration by parts**) Let $f,g:[a,b]\to\mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\int_a^b f(x)D_\alpha^a(g)(x)\,d_\alpha x=fg|_a^b-\int_a^b g(x)D_\alpha^a(f)(x)\,d_\alpha x. \tag{2.6}$$

Theorem 2.7. *[17] **(Inverse property)** Assume that $a\ge 0$, and $\alpha\in(0,1)$, and also let f be a continuous function such that $I_a^\alpha f$ exists. Then, for all $t>a$ we have*

$$D_\alpha^a I_\alpha^a f(t)=f(t).$$

In this paper, we firstly introduce the (α,k)-gamma function, (α,k)-beta function, Pochhammer symbol $(x)_{n,k}^\alpha$ and we prove several properties generalizing those satisfied by the classical gamma function, beta function and Pochhammer symbol. Then, we give a new definition of Laplace transform for conformable fractional integrals and we prove several properties of generalized Laplace transform. The results presented here would provide generalizations of those given in earlier works.

3. Gamma and Beta Functions for Conformable fractional integral

Definition 3.1. *(Pochhammer symbol) Let $p\in(0,\infty)$, $k>0$, $\alpha\in(0,1]$, and $n\in N^{+}$ Pochhammer symbol $(p)_{n,k}^{\alpha}$ is given by*

$$(p)_{n,k}^{\alpha}=(p+\alpha-1)(p+\alpha-1+\alpha k)(p+\alpha-1+2\alpha k)\dots(p+\alpha-1+(n-1)\alpha k).$$

Proposition 3.2. *Let $\alpha\in(0,1]$ and $\Gamma_k^{\alpha}:(0,\infty)\to\mathbb{R}$. For $0<p<\infty$, Conformable gamma function Γ_k^{α} is given by*

$$\Gamma_k^{\alpha}(p)\quad=\int_0^{\infty}t^{p-1}e^{-\frac{t^{\alpha k}}{\alpha k}}d_{\alpha}t=\lim_{n\to\infty}\frac{n!\alpha^n k^n(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}}{(p)_{n,k}^{\alpha}}.$$

Proof. We will give two different proofs. Firstly, we take

$$\Gamma_k^{\alpha}(p)\quad=\int_0^{\infty}t^{p-1}e^{-\frac{t^{\alpha k}}{\alpha k}}d_{\alpha}t=\lim_{n\to\infty}\int_0^{(nk)^{\frac{1}{k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^n t^{p-1}d_{\alpha}t.$$

Let $A_{n,i}(p)$, $i=0,\dots,n$, be given by $A_{n,i}(p)=\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\dfrac{t^{\alpha k}}{n\alpha k}\right)^i t^{p-1}d_{\alpha}t$. The following recursion formula is proven using integration by parts

$$\begin{aligned}
A_{n,i}(p)\quad&=\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^i t^{p-1}d_{\alpha}t\\
&=\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^i t^{p+\alpha-2}dt\\
&=\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^i\frac{t^{p+\alpha-1}}{p+\alpha-1}\Bigg|_0^{(n\alpha k)^{\frac{1}{\alpha k}}}+\frac{i}{n(p+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{i-1}t^{p+\alpha+\alpha k-2}dt\\
&=\frac{i}{n(p+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{i-1}t^{p+\alpha k-1}d_{\alpha}t\\
&=\frac{i}{n(p+\alpha-1)}A_{n,i-1}(p+\alpha k).
\end{aligned}$$

Also,

$$A_{n,0}(p)\quad=\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}t^{p-1}d_{\alpha}t=\frac{(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}}}{p+\alpha-1}.$$

Therefore, integrating by parts

$$\begin{aligned}
A_{n,n}(p)\quad&=\quad\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^n t^{p-1}d_{\alpha}t\\
&=\quad\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^n\frac{t^{p+\alpha-1}}{p+\alpha-1}\Bigg|_0^{(n\alpha k)^{\frac{1}{\alpha k}}}+\frac{n}{n(p+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{n-1}t^{p+\alpha k-1}d_{\alpha}t\\
&=\quad\frac{n}{n(p+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{n-1}t^{p+\alpha k-1}d_{\alpha}t\\
&=\quad\frac{n}{n(p+\alpha-1)}\left\{\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{n-1}\frac{t^{p+\alpha k+\alpha-1}}{p+\alpha k+\alpha-1}\Bigg|_0^{(n\alpha k)^{\frac{1}{\alpha k}}}+\frac{n-1}{n(p+\alpha k+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{n-1}t^{p+2\alpha k-1}d_{\alpha}t\right\}\\
&=\quad\frac{n}{n(p+\alpha-1)}\left\{\frac{n-1}{n(p+\alpha k+\alpha-1)}\int_0^{(n\alpha k)^{\frac{1}{\alpha k}}}\left(1-\frac{t^{\alpha k}}{n\alpha k}\right)^{n-1}t^{p+2\alpha k-1}d_{\alpha}t\right\}\\
&\ \ \vdots\\
&=\quad\frac{n(n-1)(n-2)\dots(n-(n-1))(n\alpha k)^{\frac{p+\alpha+n\alpha k-1}{\alpha k}}}{n(p+\alpha-1)n(p+\alpha k+\alpha-1)n(p+2\alpha k+\alpha-1)\dots n(p+(n-1)\alpha k+\alpha-1)(p+\alpha+n\alpha k-1)}\\
&=\quad\frac{n!n^n.\alpha^n.k^n(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}}}{n^n(p)_{n,k}^{\alpha}n\alpha k\left(1+\frac{p+\alpha-1}{n\alpha k}\right)}\\
&=\quad\frac{n!\alpha^n k^n(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}}{(p)_{n,k}^{\alpha}\left(1+\frac{p+\alpha-1}{n\alpha k}\right)}
\end{aligned}$$

and

$$\Gamma_k^\alpha(p) = \lim_{n\to\infty} A_{n,n}(p) = \lim_{n\to\infty} \frac{n!\alpha^n k^n (n\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}}{(p)_{n,k}^\alpha}.$$

which completes the proof.
Secondly, for proof of proposition, we first prove that

$$\int_0^1 \left(1-t^{\alpha k}\right)^n t^{p-1} d_\alpha t = \frac{n!\alpha^n k^n}{(p)_{n+1,k}^\alpha} \tag{3.1}$$

for $p>0$ and $n=0,1,2,\ldots$. In order to prove (3.1) by induction we first take $n=0$ to obtain for $p>0$

$$\int_0^1 t^{p-1} d_\alpha t = \frac{1}{p+\alpha-1} = \frac{1}{(p)_{1,k}^\alpha}.$$

Now we assume that (3.1) holds for $n=m$. Then we have

$$\begin{aligned}
\int_0^1 \left(1-t^{\alpha k}\right)^{m+1} t^{p-1} d_\alpha t &= \int_0^1 \left(1-t^{\alpha k}\right)\left(1-t^{\alpha k}\right)^m t^{p-1} d_\alpha t \\
&= \int_0^1 \left(1-t^{\alpha k}\right)^m t^{p-1} d_\alpha t - \int_0^1 \left(1-t^{\alpha k}\right)^m t^{p+\alpha k-1} d_\alpha t \\
&= \frac{m!\alpha^m k^m}{(p)_{m+1,k}^\alpha} - \frac{m!\alpha^m k^m}{(p+\alpha k)_{m+1,k}^\alpha} \\
&= \frac{m!\alpha^m k^m}{(p)_{m+2,k}^\alpha}\left(p+\alpha-1+(m+1)\alpha k-p-\alpha+1\right) \\
&= \frac{(m+1)!\alpha^{m+1} k^{m+1}}{(p)_{m+2,k}^\alpha}
\end{aligned}$$

which shows that (3.1) holds for $n=m+1$. This proves that (3.1) holds for all $n=0,1,2,\ldots$. Now we set $t=u(n\alpha k)^{-\frac{1}{\alpha k}}$ into (3.1) to find that

$$\frac{1}{(n\alpha k)^{\frac{p+\alpha-1}{\alpha k}}} \int_0^{(n\alpha k)^{1/\alpha k}} \left(1-\frac{u^{\alpha k}}{n\alpha k}\right)^n u^{p-1} d_\alpha u = \frac{n!\alpha^n k^n}{(p)_{n,k}^\alpha\, n\alpha k\left(1+\frac{p+\alpha-1}{n\alpha k}\right)}$$

and then

$$\int_0^{(n\alpha k)^{1/\alpha k}} \left(1-\frac{u^{\alpha k}}{n\alpha k}\right)^n u^{p-1} d_\alpha u = \frac{n!\alpha^n k^n (n\alpha k)^{\frac{p+\alpha-1}{k}-1}}{(p)_{n,k}^\alpha \left(1+\frac{p+\alpha-1}{n\alpha k}\right)}.$$

Since we have

$$\lim_{n\to\infty}\left(1-\frac{u^{\alpha k}}{n\alpha k}\right)^n = e^{-\frac{u^{\alpha k}}{\alpha k}},$$

we conclude that

$$\Gamma_k^\alpha(p) = \int_0^\infty u^{p-1} e^{-\frac{u^{\alpha k}}{\alpha k}} d_\alpha u = \lim_{n\to\infty} \frac{n!\alpha^n k^n (n\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}}{(p)_{n,k}^\alpha}.$$

□

Proposition 3.3. *The (α,k)-Gamma function $\Gamma_k^\alpha(p)$ satisfies the following identities*
(1) $\Gamma_k^\alpha(p+k) = (p+\alpha-1)\Gamma_k^\alpha(p)$
(2) $\Gamma_k^\alpha(p+n\alpha k) = (p)_{n,k}^\alpha \Gamma_k^\alpha(p)$
(3) $\Gamma_k^\alpha(p) = (\alpha k)^{\frac{p+\alpha-1}{\alpha k}-1}\Gamma\left(\frac{p+\alpha-1}{\alpha k}\right)$
(4) $\Gamma_k^\alpha(p) = (\alpha)^{\frac{p+\alpha-1}{\alpha k}-1}\Gamma_k\left(\frac{p+\alpha-1}{\alpha}\right)$
(5) $\Gamma_k^\alpha(\alpha k+1-\alpha) = 1$
(6) $\Gamma_k^\alpha(p) = a^{\frac{p+\alpha-1}{\alpha k}} \int_0^\infty t^{p-1} e^{-a\frac{t^{\alpha k}}{\alpha k}} d_\alpha t.$

Proof. *(1) Using the integration by parts, we have*

$$\begin{aligned}\Gamma_k^{\alpha}(p+\alpha k) &= \int_0^{\infty} t^{p+\alpha k-1} e^{-\frac{t^{\alpha k}}{\alpha k}} d_{\alpha}t \\ &= (p+\alpha-1)\int_0^{\infty} t^{p-1} e^{-\frac{t^{\alpha k}}{\alpha k}} d_{\alpha}t \\ &= (p+\alpha-1)\Gamma_k^{\alpha}(p).\end{aligned}$$

(2) Integrating the by parts for n-times we get

$$\begin{aligned}\Gamma_k^{\alpha}(p+n\alpha k) &= \int_0^{\infty} t^{p+n\alpha k-1} e^{-\frac{t^{\alpha k}}{\alpha k}} d_{\alpha}t \\ &= \int_0^{\infty} t^{p+\alpha-2+n\alpha k} e^{-\frac{t^{\alpha k}}{\alpha k}} dt \\ &= (p+\alpha-1+(n-1)\alpha k)\int_0^{\infty} t^{p+\alpha-2+(n-1)\alpha k} e^{-\frac{t^{\alpha k}}{\alpha k}} dt \\ &= (p+\alpha-1+(n-1)\alpha k)(p+\alpha-1+(n-2)\alpha k)\int_0^{\infty} t^{p+\alpha-2+(n-3)k} e^{-\frac{t^{\alpha k}}{\alpha k}} dt \\ &\ \vdots \\ &= (p+\alpha-1+(n-1)\alpha k)(p+\alpha-1+(n-2)\alpha k)\ldots(p+\alpha-1)\int_0^{\infty} t^{p+\alpha-2} e^{-\frac{t^{\alpha k}}{\alpha k}} dt \\ &= (p)_{n,k}^{\alpha}\Gamma_k^{\alpha}(p).\end{aligned}$$

(*3*) By definition (α,k)-Gamma function $\Gamma_k^{\alpha}(p)$,

$$\Gamma_k^{\alpha}(p) = \int_0^{\infty} u^{p-1} e^{-\frac{u^{\alpha k}}{\alpha k}} d_{\alpha}u$$

and by changing the variable $t = \frac{u^{\alpha k}}{\alpha k}$, we obtain the result (*3*). The proof of the properties (*4*), (*5*) and (*6*) are obvious from the definition of (α,k)-Gamma function Γ_k^{α}. □

Definition 3.4. *Let $\alpha \in (0,1]$. The (α,k)-Beta function $B_k^{\alpha}(p,q)$ is given the by formula*

$$B_k^{\alpha}(p,q) = \frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-1}(1-t)^{\frac{q}{\alpha k}-1} d_{\alpha}t,\ p,\ q,\ k>0.$$

Proposition 3.5. *The (α,k)-Beta function $B_k^{\alpha}(p,q)$ satisfies the following identities*
1) $B_k^{\alpha}(p,\alpha k) = \frac{1}{p+\alpha k(\alpha-1)}$,
2) $B_k^{\alpha}(\alpha k(2-\alpha),q) = \frac{1}{q}$.

Proof. From the definition of the (α,k)-Beta function $B_k^{\alpha}(p,q)$, we have

$$B_k^{\alpha}(p,\alpha k) = \frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-1} d_{\alpha}t = \frac{1}{p+\alpha k(\alpha-1)}$$

and similarly,

$$B_k^{\alpha}(\alpha k(2-\alpha),q) = \frac{1}{\alpha k}\int_0^1 t^{1-\alpha}(1-t)^{\frac{q}{\alpha k}-1} d_{\alpha}t = \frac{1}{q}.$$

This completes the proof. □

Remark 3.6. *From the Proposition 3.5, we have*

$$B_k^{\alpha}(\alpha k,\alpha k) = \frac{1}{k\alpha^2}.$$

Remark 3.7. *By the Proposition 3.5 with $\alpha = 1$, we have the following properties for k-Beta function*

$$B_k(p,k) = \frac{1}{p},\quad B_k(k,q) = \frac{1}{q}.$$

Proposition 3.8. *The following property holds for (α,k)-Beta function $B_k^{\alpha}(p,q)$*

$$B_k^{\alpha}(p,q) = \frac{p+\alpha k(\alpha-2)}{p+q+\alpha k(\alpha-2)} B_k^{\alpha}(p-\alpha k,q).$$

Proof. Integrating the by parts, we have

$$\begin{aligned}
B_k^{\alpha}(p,q) &= \frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-1}(1-t)^{\frac{q}{\alpha k}-1}d_{\alpha}t \\
&= \frac{1}{\alpha k}\frac{\alpha k}{q}t^{\frac{p}{\alpha k}+\alpha-2}(1-t)^{\frac{q}{\alpha k}}\Big|_0^1 \\
&\quad +\frac{1}{\alpha k}.\frac{\alpha k}{q}\left(\frac{p}{\alpha k}+\alpha-2\right)\int_0^1 t^{\frac{p}{\alpha k}-2}(1-t)^{\frac{q}{\alpha k}}d_{\alpha}t \\
&= \frac{p+\alpha k(\alpha-2)}{q}\frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-2}(1-t)(1-t)^{\frac{q}{\alpha k}-1}d_{\alpha}t \\
&= \frac{p+\alpha k(\alpha-2)}{q}\left[\frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-2}(1-t)^{\frac{q}{\alpha k}-1}d_{\alpha}t-\frac{1}{\alpha k}\int_0^1 t^{\frac{p}{\alpha k}-1}(1-t)^{\frac{q}{\alpha k}-1}d_{\alpha}t\right] \\
&= \frac{p+\alpha k^2(\alpha-2)}{q}\left[B_k^{\alpha}(p-\alpha k,q)-B_k^{\alpha}(p,q)\right].
\end{aligned}$$

That is,

$$B_k^{\alpha}(p,q)=\frac{p+\alpha k(\alpha-2)}{q}\left[B_k^{\alpha}(p-\alpha k,q)-B_k^{\alpha}(p,q)\right]$$

which completes the proof. □

Proposition 3.9. *The following identity holds*

$$B_k^{\alpha}(p,q)=B_k(p+\alpha k(\alpha-1),q)=\frac{1}{\alpha k}B\left(\frac{p}{\alpha k}+\alpha-1,\frac{q}{\alpha k}\right)$$

where $B_k(x,y)$ is k-Beta function and $B(x,y)$ is classical Beta function.

Proof. The proof is follows directly from the definitions of (α,k)-Beta function and conformable integral. □

Proposition 3.10. *The following property holds for (α,k)-Beta function in terms of (α,k)-gamma function*

$$B_k^{\alpha}(p+\alpha k(1-\alpha),q)=\frac{\Gamma_k^{\alpha}(p)\Gamma_k^{\alpha}(q)}{\Gamma_k^{\alpha}(p+q+1-\alpha)}.$$

Proof. By using definition of (α,k)-gamma function, we get

$$\begin{aligned}
\Gamma_k^{\alpha}(p)\Gamma_k^{\alpha}(q) &= \int_0^{\infty} t^{p-1}e^{-\frac{t^{\alpha k}}{\alpha k}}d_{\alpha}t\int_0^{\infty} s^{q-1}e^{-\frac{s^{\alpha k}}{\alpha k}}d_{\alpha}s \\
&= \int_0^{\infty}\int_0^{\infty} e^{-\frac{t^{\alpha k}+s^{\alpha k}}{\alpha k}}t^{p-1}s^{q-1}d_{\alpha}t d_{\alpha}s.
\end{aligned}$$

Now we apply the change of variables $t^{\alpha k}=x^{\alpha k}y$ and $s^{\alpha k}=x^{\alpha k}(1-y)$ to this double integral. Note that $t^{\alpha k}+s^{\alpha k}=x^{\alpha k}$ and that $0<t<\infty$ and $0<s<\infty$ imply that $0<x<\infty$ and $0<y<1$. The Jacobian of this transformation (see [4]) is

$$\begin{aligned}
\begin{pmatrix} \frac{\partial^{\alpha} f}{\partial x^{\alpha}} & \frac{\partial^{\alpha} f}{\partial y^{\alpha}} \\ \frac{\partial^{\alpha} g}{\partial x^{\alpha}} & \frac{\partial^{\alpha} g}{\partial y^{\alpha}} \end{pmatrix} &= \begin{pmatrix} x^{1-\alpha}\frac{\partial f}{\partial x} & y^{1-\alpha}\frac{\partial f}{\partial y} \\ x^{1-\alpha}\frac{\partial g}{\partial x} & y^{1-\alpha}\frac{\partial g}{\partial y} \end{pmatrix} \\
&= \begin{pmatrix} x^{1-\alpha}y^{\frac{1}{\alpha k}} & \frac{1}{\alpha k}y^{1-\alpha}xy^{\frac{1}{\alpha k}-1} \\ x^{1-\alpha}(1-y)^{\frac{1}{\alpha k}} & -\frac{1}{\alpha k}y^{1-\alpha}x(1-y)^{\frac{1}{\alpha k}-1} \end{pmatrix} \\
&= -\frac{1}{\alpha k}x^{2-\alpha}y^{\frac{1}{\alpha k}-\alpha}(1-y)^{\frac{1}{\alpha k}-1}.
\end{aligned}$$

Since $x,\ y,\ k>0$, we conclude that $d_{\alpha}td_{\alpha}s=\left|\begin{pmatrix} \frac{\partial^{\alpha} f}{\partial x^{\alpha}} & \frac{\partial^{\alpha} f}{\partial y^{\alpha}} \\ \frac{\partial^{\alpha} g}{\partial x^{\alpha}} & \frac{\partial^{\alpha} g}{\partial y^{\alpha}} \end{pmatrix}\right|d_{\alpha}xd_{\alpha}y$. Hence we have

$$\begin{aligned}
\Gamma_k^{\alpha}(p)\Gamma_k^{\alpha}(q) &= \int_0^1\int_0^{\infty} e^{-\frac{x^{\alpha k}}{\alpha k}}x^{p-1}y^{\frac{p-1}{\alpha k}}x^{q-1}(1-y)^{\frac{q-1}{\alpha k}}\frac{1}{\alpha k}x^{2-\alpha}y^{\frac{1}{\alpha k}-\alpha}(1-y)^{\frac{1}{\alpha k}-1}d_{\alpha}xd_{\alpha}y \\
&= \left(\int_0^{\infty} e^{-\frac{x^{\alpha k}}{\alpha k}}x^{p+q-\alpha}d_{\alpha}x\right)\left(\frac{1}{\alpha k}\int_0^1 y^{\frac{p}{\alpha k}-\alpha}(1-y)^{\frac{q}{\alpha k}-1}d_{\alpha}y\right) \\
&= \Gamma_k^{\alpha}(p+q+1-\alpha)B_k^{\alpha}(p+\alpha k(1-\alpha),q).
\end{aligned}$$

□

Remark 3.11. *By the Proposition 3.10 with $\alpha=1$, we have the following properties*

$$B_k(p,q)=\frac{\Gamma_k(p)\Gamma_k(q)}{\Gamma_k(p+q)}.$$

4. Laplace Transform for Conformable Fractional Integral

In Abbeljawad give the definition of the Laplace transform for conformable left fractional integral of order $0<\alpha\leq 1$. In this section, we will generalize the definition of the Laplace transform for conformable fractional integral and use it to soleve prove some properties.

Definition 4.1. *Let $\alpha\in(0,1]$, $k>0$, and $f:[0,\infty)\to\mathbb{R}$ be a function. Then the fractional Laplace transform of order α of f defined by*

$$L_k^\alpha\{f(t)\}(s)=F_k^\alpha(s)=\int_0^\infty e^{-s\frac{t^{\alpha k}}{\alpha k}}f(t)d_\alpha t \tag{4.1}$$

which is called (α,k)-Laplace transform.

Some properties of the (α,k)-Laplace Transform
1) $L_k^\alpha\{0\}(s)=0$
2) $L_k^\alpha\{f(t)+g(t)\}(s)=L_k^\alpha\{f(t)\}(s)+L_k^\alpha\{g(t)\}(s)$
3) $L_k^\alpha\{cf(t)\}(s)=cL_k^\alpha\{f(t)\}(s)$, c is a constant.
Properties 2) and 3) together means that the Laplace transform is linear.

Theorem 4.2. *Let $\alpha\in(0,1]$, $k>0$, and $f:(0,\infty)\to\mathbb{R}$ be differentiable function. Then*

$$L_k^\alpha\{D_\alpha f(t)\}(s)=sL_k^\alpha\left\{t^{\alpha(k-1)}f(t)\right\}(s)-f(0). \tag{4.2}$$

Proof. By definition (α,k)-Laplace transform and using the (2.6), we have (4.2). □

It is easy to see from definiton of the (α,k)-Laplace transform that we have rather unusual results given in the following theorem.

Theorem 4.3. *Let $\alpha\in(0,1]$, $c\in\mathbb{R}$ and $k>0$. Then we have the following results*
i) $L_k^\alpha\{1\}(s)=s^{-\frac{1}{k}}\Gamma_k^\alpha(1)$,
ii) $L_k^\alpha\{t\}(s)=s^{-\frac{1+\alpha}{\alpha k}}\Gamma_k^\alpha(2)$,
iii) $L_k^\alpha\{t^p\}(s)=s^{-\frac{p+\alpha}{\alpha k}}\Gamma_k^\alpha(p+1)$,
iv) $L_k^\alpha\left\{e^{c\frac{t^{\alpha k}}{\alpha k}}\right\}(s)=(s-c)^{-\frac{1}{k}}\Gamma_k^\alpha(1)$,
v) $L_k^\alpha\{f(t)\}(s)=F_k^\alpha(s)\Rightarrow L_k^\alpha\left\{f(t).e^{c\frac{t^{\alpha k}}{\alpha k}}\right\}(s)=F_k^\alpha(s-c)$,
vi) $L_k^\alpha\{f(t)\}(s)=F_k^\alpha(s)\Rightarrow L_k^\alpha\{f(ct)\}(s)=\frac{1}{c^\alpha}F_k^\alpha\left(\frac{s}{c^{\alpha k}}\right)$.

Example 4.4. *Let us consider the function $f(t)=\sin w\frac{t^\alpha}{\alpha}$, then by using the property $D_\alpha\left(\cos w\frac{t^\alpha}{\alpha}\right)=-w\sin w\frac{t^\alpha}{\alpha}$, we can write*

$$L_k^\alpha\left\{\sin w\frac{t^\alpha}{\alpha}\right\}(s)=\int_0^\infty e^{-s\frac{t^{\alpha k}}{\alpha k}}\sin w\frac{t^\alpha}{\alpha}d_\alpha t=-\frac{1}{w}\int_0^\infty e^{-s\frac{t^{\alpha k}}{\alpha k}}D_\alpha\left(\cos w\frac{t^\alpha}{\alpha}\right)d_\alpha t.$$

Therefore, using integration by part for conformable integral, we have

$$\begin{aligned}-\frac{1}{w}\int_0^\infty e^{-s\frac{t^{\alpha k}}{\alpha k}}D_\alpha\left(\cos w\frac{t^\alpha}{\alpha}\right)d_\alpha t &= -\frac{1}{w}\left\{e^{-s\frac{t^{\alpha k}}{\alpha k}}\cos w\frac{t^\alpha}{\alpha}\Big|_0^\infty-\int_0^\infty \cos w\frac{t^\alpha}{\alpha}D_\alpha\left(e^{-s\frac{t^{\alpha k}}{\alpha k}}\right)d_\alpha t\right\}\\ &= \frac{1}{w}-\frac{s}{w}\int_0^\infty t^{\alpha k-\alpha}e^{-s\frac{t^{\alpha k}}{\alpha k}}\cos w\frac{t^\alpha}{\alpha}d_\alpha t\\ &= \frac{1}{w}-\frac{s}{w^2}\int_0^\infty t^{\alpha k-\alpha}e^{-s\frac{t^{\alpha k}}{\alpha k}}D_\alpha\left(\sin w\frac{t^\alpha}{\alpha}\right)d_\alpha t.\end{aligned}$$

Similarly, we get

$$L_k^\alpha\left\{\sin w\frac{t^\alpha}{\alpha}\right\}(s)=\frac{1}{w}+\frac{s(k-\alpha)}{w^2}L_k^\alpha\left\{t^{k-2\alpha}\sin w\frac{t^\alpha}{\alpha}\right\}(s)-\frac{s^2}{w^2}L_k^\alpha\left\{t^{k-\alpha}\sin w\frac{t^\alpha}{\alpha}\right\}(s). \tag{4.3}$$

If we take $k=\alpha$ in (4.3), we have

$$L_\alpha^\alpha\left\{\sin w\frac{t^\alpha}{\alpha}\right\}(s)=\frac{w}{1+s^2}$$

which is proved by Abdeljawad in [1].

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