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ON OBTAINING OF ERGODICITY RESULTS AND PERFORMANCE MEASURES OF HYPO-EXPONENTIAL DISTRIBUTED QUEUING MODEL WITH NEUTS MATRIX-GEOMETRIC METHOD

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Abstract

In this study, a queuing model with exponential distribution and inter-service hypo-exponential distribution with inter-arrival time λ parameter was studied. When the customer arrives in the system, if there are customers in the queue or service channels, they start to wait in the queue to get service, and the customer is not allowed to lose. In this model where FIFO method is used as queue discipline, the diagram shows the number of customers in the system and the transition rate diagram for the (k,i) showing which phase the client receiving the service is. Accordingly, the tridiagonal Q matrix and the sub-matrices of the matrix are constructed. The matrices V and W are calculated with the help of these submatrices. Neuts' R matrix is obtained by the iteration applied on the R_{l+1} sequence which is defined as a function of these matrices. The probability vector $\pi Q = 0$ from this R matrix π_k, which is the steady-state subvectors for the solution of the system of homogeneous linear equations, is computed as $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_k, \ldots)$, are obtained based on the initial value $\pi_0 = 1$ and depending on π_0. The actual probabilities of the system parameters are calculated by normalizing the solution and numerical analysis.

Key Words: *Hypo-Exponential, Neuts Matrix-Geometric, Ergodicity, Perfomance Measure*

HİZMET SÜRESİ HİPO-ÜSTEL DAĞILIMA UYAN KUYRUK MODELİNDE, NEUTS'UN MATRİS-GEOMETRİK YÖNTEMİNİN SİMULASYON YARDIMI İLE ERGODİKLİK SONUÇLARI VE PERFORMANS ÖLÇÜLERİNİN ELDE EDİLMESİ

Özet

Bu çalışmada, varış süresi λ parametreli üstel dağılım ve hizmet içi Hipo-Üstel dağılım içeren bir kuyruk modeli incelendi. Müşteri sisteme geldiğinde, kuyrukta veya hizmet kanalında müşteri varsa, hizmet almak için kuyrukta beklemeye başlar ve müşterinin kaybolmasına izin verilmez. FIFO yönteminin kuyruk disiplini olarak kullanıldığı bu modelde, diyagram sistemde müşteri sayısını ve hizmet alan istemcinin hangi aşaması olduğunu gösteren (k, i) geçiş hızı diyagramını gösterir. Buna göre, tridiagonal Q matris ve matris alt matrisleri inşa edilmiştir. V ve W matrisleri bu alt matrislerin yardımı ile hesaplanır. Neuts'un R matrisi, bu matrislerin bir fonksiyonu olan R_{l+1} sırası uygulanan yineleme tarafından elde edilir. Homojen doğrusal denklemler sisteminin çözümü için kararlı durumlu alt kümeler olan bu R matrisi π_k olasılık vektörü $\pi Q = 0$, $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_k, \ldots)$ olarak hesaplanır. $\pi_0 = 1$ başlangıç değerine ve π_0'a bağlı olarak elde edilir. Sistem parametrelerinin gerçek olasılıkları, çözüm ve sayısal analizlerin normalleştirilmesi ile hesaplanır.

Key Words: *Hipo-Üstel, Neuts'un Matris-Geometrik Yöntemi, Ergodiklik, Performans Ölçümleri*

1. INTRODUCTION

In order to model complex queuing systems, phase-type distributions can be created in which hypo-exponential and hyper-exponential distributions are used together. In this context, R. R. P. Jackson (1954), one of the first pioneers of phase-type studies, found the distribution of waiting times and the probabilities of the various numbers of customers with the average number of customers at each stage of the multi-stage poultry system with Poisson inputs and different service parameters. The steady state behavior of a discrete time, single channel, first come first served queueing problem wherein service phases at two consecutive time-marks (defined later) are correlated and the arrivals occur in General Stream with probabilities ai (i = 0, 1, 2 . . . .) at a time-mark, is investigated by R. K. Rana (1972). Ramaswami, V. and Neuts, M. F. (1980), study the duality of phase-type distributions. Neuts (1981), has solved the problem by using matrix-geometric method with algorithmic approaches in phase-stochastic models. J. R. Artalejo and G. Choudhury (2004), examined the steady state behavior of an M/G/1 queue with repeated attempts in which the server may provide an additional second phase of service. A novel approach for obtaining the response time in a discrete-time tandem-queue with blocking is presented by Houdt and Alfa (2005). Stewart (2009), analyzed some of the phase-type queuing systems using the Neuts Matrix-Geometric method. Smaili at al. (2013), studied on hypo-exponential distribution with different parameters. Zobu M. at al. (2013**)**, handled control of traffic intensity in hyper-exponential and mixed erlang queueing system with a method based on SPRT. Saglam at al. (2016**),** have made the simulation and control of traffic intensity in hypo-exponential and coxian queueing systems with a method based on sequential probability ratio tests. Michiel De M. at al. (2017), examined a non-classical discrete-time queueing model where customers demand variable amounts of work from a server that is able to perform this work at a varying rate.

In our study we also obtained a diagram of the transition rates for the (k,i) pair showing the number of customers in the system and the phase in which the client receiving the service is located. Accordingly, the tridiagonal Q matrix and the sub-matrices of the matrix are constructed. The matrices V and W are calculated with the help of these submatrices. Neuts R matrix is obtained by the iteration applied on the R_{l+1} sequence which is defined as a function of these matrices. The probability vector $\pi Q = 0$ from this R matrix π_k, which is the steady-state subvectors for the solution of the system of homogeneous linear equations, is computed as $\pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_k, \ldots)$, are obtained based on the initial value $\pi_0 = 1$ and depending on π_0. The actual probabilities of the system parameters are calculated by normalizing the solution.

2. NEUTS MATRİX-GEOMETRIC METHOD

The only probability law, exponential distribution, used in modeling distributions of inter-arrival times or service times in single service queues. When this is taken together, it reveals a system called birth-death processes. Transitions from any state in these systems are only neighboring states, and the structure of the transition matrix is diagonal. But sometimes exponential distribution may not be enough. In such cases, phase-type distributions provide the possibility to model more general cases.

The transition matrices of the arrival and service mechanisms in the phase queue queuing systems are expressed as block diagonal and half-birth (QBD) processes. On the other hand, in a simple birth-death process, the elements below the diagonal depart from the system and the elements above the diagonal represent client arrivals. In a QBD process, the lower crossover blocks involve a more complex separation process and a more complex process of arrival.

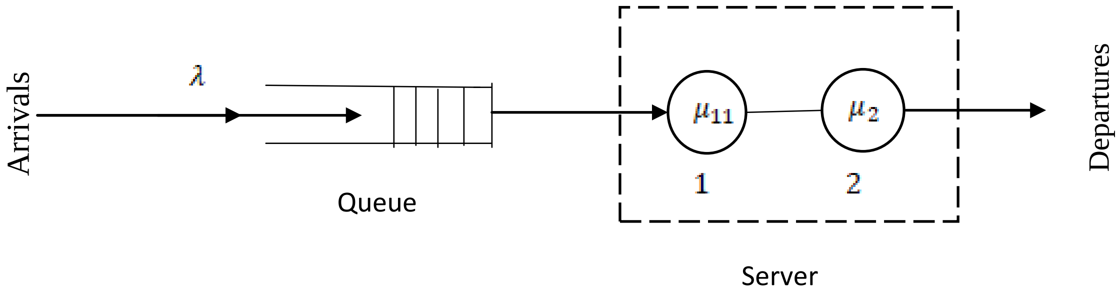
In the past, the mathematical techniques used to solve such queuing systems depended on the use of the z-transform. Today, Neuts matrix-geometric method is more widely used with high-speed computers and the emergence of efficient algorithms (Stewart, 2009).

In this work, we will discuss how the phase type distribution can be incorporated into single server queues and analyze the $M/Hypo_2/1$ queuing system using Neuts matrix-geometric method.

2.1. $M/Hypo_2/1$ Queue System

In this queue system, the number of arrivals at any time in the length of θ has Poisson with λ and exponential distribution (τ) between consecutive arrivals. Moreover, the service duration is two-phase and the service duration in each phase has different parameter exponential distribution, and the total service duration (η) has hypo-exponential distribution. This situation is shown in Fig 1.

Figure 1: $M/Hypo_2/1$ Queuing System.



Here, both service phases have different parameter exponential distributions, and the phases are completely independent of each other.

More than one customer cannot get service each time. One service provides each consecutive service phase to the customer, and then the customer removes it from the system. Probability density functions of inter-arrival times:

$$f_\tau(\theta) = \begin{cases} \lambda e^{-\lambda\theta} & , \quad \theta \geq 0 \\ 0 & , \quad d.d \end{cases} , \qquad (1)$$

it happens. Now ξ: the service period of the first phase and γ: let be two random variables showing the service period of the second phase. Suppose that ξ has an exponential distribution with the μ_1

parameter, γ has an exponential distribution with the μ_2 parameter, and these two random variables are independent. The sum of two independent and randomly distributed random variables is also a random change. For this reason, the probability density function of $\eta = \xi + \gamma$ random variable, which is defined as the total service duration, is defined by the convolution formula;

$$f_\eta(x) = \begin{cases} \frac{\mu_1 \mu_2}{\mu_1 - \mu_2}\left(e^{-\mu_2 x} - e^{-\mu_1 x}\right) & , \quad x \geq 0 \\ 0 & , \quad d.d \end{cases} \tag{2}$$

and is indicated by $\eta \sim Hypo_2$ and the distribution function of η,

$$F_\eta(x) = 1 - \frac{\mu_1}{\mu_1 - \mu_2} e^{-\mu_2 x} + \frac{\mu_2}{\mu_1 - \mu_2} e^{-\mu_1 x}, \quad x \geq 0 \tag{3}$$

it happens. Expected value, variance and relative variance of total service life distribution, respectively;

$$E(\eta) = \frac{1}{\mu_1} + \frac{1}{\mu_2} \tag{4}$$

$$Var(\eta) = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} \tag{5}$$

$$C_\eta^2 = \frac{\mu_1^2 + \mu_2^2}{(\mu_1 + \mu_2)^2} < 1 \tag{6}$$

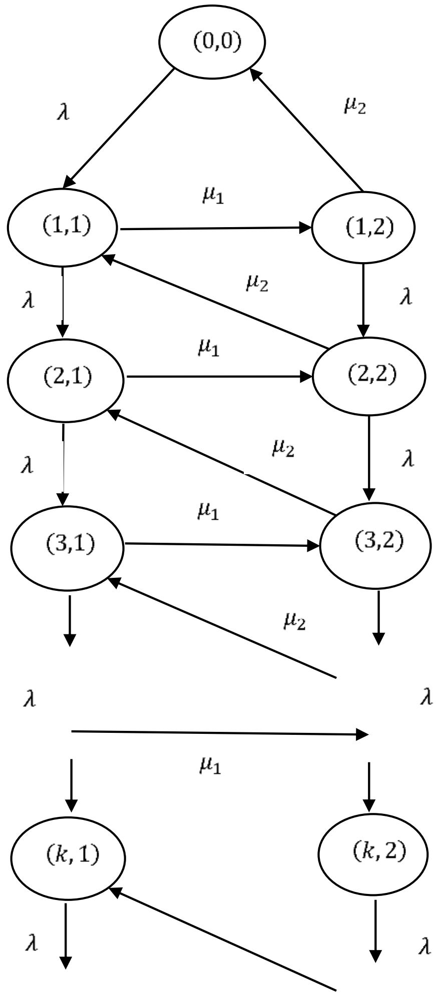
Laplace transformation is;

$$L_\eta(t) = \left(\frac{\mu_1}{t + \mu_1}\right)\left(\frac{\mu_2}{t + \mu_2}\right) \tag{7}$$

obtained in the form.

Now let's first create a status descriptor for the queue system $M/Hypo_2/1$. Then, in the queue system $M/Hypo_2/1$, let's record the number of existing customers and the current customer's servicing phase. Due to the exponential nature of the service period and the time between arrival of each of the two phases, information about the number of customers in the system and the current service phase is sufficient to capture the entire history of this system. A state of the system is defined by the (k, i) binary. Where $k(k \geq 0)$ represents the number of customers in a single-service system and $i(i = 1,2)$ represents the current service phase. Eğer $k = 0$ ise i'nin değeri ilişkili değildir. If $k > 0$, $2 - i + 1$ indicates the number of phases not yet completed by the customer in the service. The transition ratio λ, (k, i) to $(k + 1, i)$ when $k > 0$, when the transition from (k, i) to $(k, i + 1)$ is completed with μ_1 rate when $k > 0$ and $i < 2$, when $k > 0$, the transition from $(k, 2)$ to $(k-1, 1)$ is complemented by μ_2. The transition ratio diagram for $M/Hypo_2/1$ is shown in Figure 2:

Figure 2: State Transition Ratio Diagram of the $M/Hypo_2/1$ Queue System.



In this case it is clear from the viewpoint of the transition diagrams that the transition ratios have a triangular form with block matrix or QBD process. The transition rates matrix and the reduced sub-matrices for $M/Hypo_2/1$ are expressed as follows.

Here the matrices A_i, $i = 0,1,2$ are square matrices. Matrices A_0 represent service completions. The top-level diagonal matrix A_1 represents service completion with $k > 0$ and a rate of μ_1 at $i < 2$ (ie, phase 1). All other elements in A_1 are equal to zero. The matrices A_2 represent the number of arrivals at a rate of λ that can occur during service, where $k > 0$ and i in any phase at any level.

$$Q = \left(\begin{array}{c|cc|cc|cc|c}
-\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & \dots \\ \hline
0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & 0 & 0 & \dots \\
\mu_2 & 0 & -(\lambda+\mu_2) & 0 & \lambda & 0 & 0 & \dots \\ \hline
0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & \lambda & 0 & \dots \\
0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2) & 0 & \lambda & \dots \\ \hline
0 & 0 & 0 & 0 & 0 & -(\lambda+\mu_1) & \mu_1 & \dots \\
0 & 0 & 0 & \mu_2 & 0 & 0 & -(\lambda+\mu_2) & \dots \\ \hline
\dots & \dots & \dots & & & \dots & \dots &
\end{array}\right)$$

$$Q = \begin{pmatrix}
B_{00} & B_{01} & 0 & 0 & 0 & \dots \\
B_{10} & A_1 & A_2 & 0 & 0 & \dots \\
0 & A_0 & A_1 & A_2 & 0 & \dots \\
0 & 0 & A_0 & A_1 & A_2 & \dots \\
0 & 0 & 0 & A_0 & A_1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}$$

Thus sub-matrices;

$$A_0 = \begin{pmatrix} 0 & 0 \\ \mu_2 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -(\lambda+\mu_1) & \mu_1 \\ 0 & -(\lambda+\mu_2) \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \qquad (8)$$

$$B_{00} = (-\lambda), \quad B_{01} = (\lambda \quad 0), \quad B_{10} = \begin{pmatrix} 0 \\ \mu_2 \end{pmatrix} \tag{9}$$

is defined as. In the transition rate matrix, the B_{00} sub-matrix reflects that the sum of the first line must be zero. Using our matrix-geometric method,

$$\pi Q = 0 \tag{10}$$

to calculate the solution π constant probability vector of the homogeneous linear equation system. Constant probability vector;

$$\pi = (\pi_0, \pi_1, \pi_2, \dots, \pi_k, \dots) \tag{11}$$

is expressed as. Here, $\pi_0 : 1 \times 1$ is a vector which indicates the probability that the system is empty. Also for $k = 1,2, \dots$ is a row vector of dimension $\pi_k : 1 \times 2$, known as constant state subvectors. R is the ratio matrix of Neuts, for $k = 1,2, \dots$ between consecutive constant subvectors of π,

$$\pi_{k+1} = \pi_k R \tag{12}$$

there is a relationship. Thus, the first step in the implementation of the matrix-geometric approach will be the calculation of the R matrix. Equation (12) is used to find successive steady-state subvectors for the block triangular matrix Q:

$$R_{l+1} = -\left(V + R_l^2 W\right) \quad , \quad l = 0,1,2, \dots, \tag{13}$$

where V and W are respectively the matrices;

$$V = A_2 A_1^{-1}, \quad W = A_0 A_1^{-1} \tag{14}$$

using equations calculated. The R_l sequence is a monotone increasing sequence, with $l \to \infty$, converging to R. The initial value of $R_0 = 0$ is applied to find Neuts R matrix. However, the solution of equation (8) is not unique, so it should be normalized so that the sum of the components of the π constant probability vector is equal to 1. Normalization process,

$$1 = \pi_0 + \sum_{k=1}^{\infty} \pi_k e = \pi_0 + \sum_{k=0}^{\infty} \pi_1 R^k e = \pi_0 + \pi_1 (I - R)^{-1} e \tag{15}$$

by applying equality performed. Where $e : 2 \times 1$ is a vector of 1 elements.

2.2. Ergodic Results for $M/Hypo_2/1$ Queuing System

$M/Hypo_2/1$ queueing system is stable; ie the number of customers does not increase indefinitely. This means that the time between arrivals and service time are in balance. $E(A)$ mean arrival time and $E(S)$ mean service time when the inter-arrival time λ is smaller than the service time µ is the equilibrium state:

$$\frac{1}{E(A)} < \frac{1}{E(S)} \quad veya \quad E(S) < E(A) \qquad \mathbf{(16)}$$

can be written as. If $E(S) > E(A)$, then the number of customers in the queue grows forever and some customers can not get service. Therefore, the steady state probabilities of the system are undefined and performance measures can not be calculated. The customer must have a service guarantee, so the equilibrium state of equality (15) must be provided.

2.3. Performance Scales for $\mathbf{M/Hypo_2/1}$ Queuing System

The performance measures we have achieved for the $M/Hypo_2/1$ queuing system up to now are the stationary probabilities of Markov chains. It is possible to obtain very useful information directly from these. The probability of finding k customers in the queue system with $k \geq 1$, by adding the components of π_k steady state sub-vector, so,

$$p_k = \|\pi_k\|_1 = \left\|\pi_1 R^{k-1}\right\|_1, k = 1, 2, \dots \qquad \mathbf{(17)}$$

can easily be obtained by equality. Here, $\|\boldsymbol{x}\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty$ is defined as the norm.

Neuts R matrix can be used to calculate the average number of customers in the queue system $M/Hypo_2/1$. Thus, the average number of customers in the queue system $M/Hypo_2/1$,

$$E(N) = \sum_{k=1}^{\infty} k \|\pi_k\|_1 = \sum_{k=1}^{\infty} k \left\|\pi_1 R^{k-1}\right\|_1 = \left\|\pi_1 \sum_{k=1}^{\infty} \frac{d}{dR} R^k\right\|_1$$

$$= \left\|\pi_1 \frac{d}{dR}\left(\sum_{k=1}^{\infty} R^k\right)\right\|_1 = \left\|\pi_1 \frac{d}{dR}((I-R)^{-1} - I)\right\|_1$$

$$= \|\pi_1 (I-R)^{-2}\|_1 \qquad \mathbf{(18)}$$

obtained as.

The expected value $E(N_q)$ of the number of customers in the queueing is obtained from the standard formula $E(W)$ of a customer's stay time and the expected value $E(W_q)$ of a customer's queue.

$$E(N_q) = E(N) - \frac{\lambda}{\mu} \quad \textbf{(19)}$$

$$E(W) = \frac{E(N)}{\lambda} \quad \textbf{(20)}$$

$$E(W_q) = \frac{E(N_q)}{\lambda} \quad \textbf{(21)}$$

3. PRACTICE

Here, the numerical analysis of the $M/Hypo_2/1$ queueing system is carried out with the help of Neuts matrix-geometric method and the results obtained from these solutions are interpreted.

In this study, we give the average arrival time of the customers in the application $\lambda = 1$, the average service time in the first phase is $\mu_1 = 3.5$ min, and the average service time in the second phase is $\mu_2 = 2.5$ min. In this case, performance parameters will be calculated and evaluated using Neuts matrix-geometric method for given parameters $M/Hypo_2/1$ queuing system. $M/Hypo_2/1$ queue system transition rates matrix,

	1	0	0	0	0	0	...
0	-4.5	3.5	1	0	0	0	...
2.5	0	-3.5	0	1	0	0	...
0	0	0	-4.5	3.5	1	0	...
0	2.5	0	0	-3.5	0	1	...
0	0	0	0	0	-4.5	-3.5	...
0	0	0	2.5	0	0	-3.5	...
...	...	...			...	...	

is found as. According to the matrices obtained,

$$A_0 = \begin{pmatrix} 0 & 0 \\ 2.5 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -4.5 & 3.5 \\ 0 & -3.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B_{00} = (-1), \qquad B_{01} = (1 \quad 0), \quad B_{10} = \begin{pmatrix} 0 \\ 2.5 \end{pmatrix}$$

sub-matrices can be written. Later;

$$A_1^{-1} = \begin{pmatrix} -0.2222 & -0.2222 \\ 0 & -0.2857 \end{pmatrix}$$

to be of equation (13) the matrices V and W,

$$V = A_2 A_1^{-1} = \begin{pmatrix} -0.2222 & -0.2222 \\ 0 & -0.2857 \end{pmatrix}$$

$$W = A_0 A_1^{-1} = \begin{pmatrix} 0 & 0 \\ -0.5556 & -0.5556 \end{pmatrix}$$

obtained in the form. By using these repeated values given in Eq. (12), Matlab R2009 program was used to obtain Neuts ratio matrix R. R_0=0 is the starting value, the matrices obtained in the result of this $1\ st, 2\ nd, \ldots$, and 100 th repetitions are, respectively,

$$R_1 = \begin{pmatrix} 0.2222 & 0.2222 \\ 0 & 0.2857 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0.28487 & 0.28487 \\ 0.04533 & 0.33103 \end{pmatrix}, \ldots$$

$$R_{100} = \begin{pmatrix} 0.4000 & 0.4000 \\ 0.1143 & 0.4000 \end{pmatrix} = R$$

it was observed that the R matrix became stationary during the $100th$ step of the process.

The second step in using the matrix-geometric method is the calculation of the initial vector and successive constant sub-vectors. This requires π_0 and π_1 to be present. For this, the system of homogeneous linear equations given by Eq. (9) is considered and the following equation is written.

$$(\pi_0, \pi_1, \pi_2 \ldots, \pi_k, \ldots) \begin{pmatrix} B_{00} & B_{01} & 0 & 0 & 0 & \ldots \\ B_{10} & A_1 & A_2 & 0 & 0 & \ldots \\ 0 & A_0 & A_1 & A_2 & 0 & \ldots \\ 0 & 0 & A_0 & A_1 & A_2 & \ldots \\ 0 & 0 & 0 & A_0 & A_1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} = (0,0,0 \ldots, 0, \ldots)$$

As can be seen from the matrix, two linear equations of the π constant probability vector are obtained as follows.

$$\pi_0 B_{00} + \pi_1 B_{10} = 0$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_2 A_0 = 0$$

Then, to find the constant state sub-vectors of π from equation (11), π_2 is written as follows in $\pi_1 R$.

$$\pi_0 B_{00} + \pi_1 B_{10} = 0$$

$$\pi_0 B_{01} + \pi_1 A_1 + \pi_1 R A_0 = 0$$

$$(\pi_0, \pi_1)\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & A_1 + R A_0 \end{pmatrix} = (0,0)$$

Here,

$$A_1 + R A_0 = \begin{pmatrix} -3.5000 & 3.5000 \\ 1.000 & -3.5000 \end{pmatrix}$$

then, if all values known in the above equation are written instead,

$$(\pi_0, \pi_1)\left(\begin{array}{c|cc} -1 & 1 & 0 \\ \hline 0 & -3.5 & 3.5 \\ 2.5 & 1 & -3.5 \end{array}\right) = (0,0)$$

obtained. The solution of this equation is not the only one. Considering that π_0 is a scalar and $\pi_1 = [\pi_{1_1} \ \ \pi_{1_2}]$: 1×2 dimensional row vector, arbitrarily $\pi_0 = 1$ in the sense that for the initial value in the third column of the matrix of coefficients, If regulation is made, the equation system,

$$(\pi_0, \pi_{1_1}, \pi_{1_2})\left(\begin{array}{c|cc} -1 & 1 & 0 \\ \hline 0 & -3.5 & 3.5 \\ 2.5 & 1 & 0 \end{array}\right) = (0,0,1)$$

turn into a shape. In this case, the solution vector is easily calculated,

$$(\pi_0, \pi_{1_1}, \pi_{1_2}) = (1.0, 0.4, 0.4)$$

found as. Thus, π_k vectors with constant state sub-vectors are obtained depending on the initial condition $\pi_0 = 1$. For this reason, when the normalization process expressed in equation (14) is applied, the total value of all probabilities obtained from the steady-state subvectors,

$$\pi_0 + \pi_1 (I-R)^{-1} e = 1 + (0.4 \quad 0.4) \begin{pmatrix} 1.9091 & 1.2727 \\ 0.3636 & 1.9091 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 3.1818$$

obtained as. By dividing the total value of all possibilities by the constant case probabilities obtained above, the real possibilities of system parameters,

$$(\pi_0, \pi_{1_1}, \pi_{1_2}) = (1/3.1818, 0.4/3.1818, 0.4/3.1818)$$

$$= (0.3142, 0.1257, 0.1257)$$

is obtained as above. Thus, the only solution for π_0 and π_1;

$$\pi_0 = (0.3142)$$

$$\pi_1 = (0.1257 \quad 0.1257)$$

found as. The other $\pi_{k+1}, (k = 1,2,\dots)$ probabilities are also calculated for $k = 1,2,3$ using the recurrence formula given in equation (11).

$$\pi_2 = \pi_1 R = (0.1257 \quad 0.1257) \begin{pmatrix} 0.4000 & 0.4000 \\ 0.1143 & 0.4000 \end{pmatrix}$$

$$= (0.06464 \quad 0.10056)$$

$$\pi_3 = \pi_2 R = (0.03735 \quad 0.06608)$$

$$\pi_4 = \pi_3 R = (0.02249 \quad 0.04137) \dots\dots$$

By using this repetitive formula, there is also the fixed state sub-vectors in the case of $k > 3$. The likelihood of finding $k = 0, 1, 2, \dots$ in the queue system with equation (16) is found as follows with the addition of the components of this steady-state subvector.

$$p_0 = \pi_0 = 0.3142$$

$$p_1 = \|\pi_1\|_1 = \pi_{1_1} + \pi_{1_2} = 0.2514$$

$$p_2 = \|\pi_2\|_1 = \pi_{2_1} + \pi_{2_2} = 0.1652$$

$$p_3 = \|\pi_3\|_1 = \pi_{3_1} + \pi_{3_2} = 0.10348$$

.

.

. (v.b.)

For the probabilities of $p_k, k = 0,1,2,\dots$ found with the help of real probabilities of the system parameters obtained as a result of normalization,

As indicated by the equation (15) where $E(A)$ is the average arrival time and $E(S)$ is the average service time when the average arrival rate λ is smaller than the average service time μ, $E(A) = \frac{1}{\lambda} = \frac{1}{1} = 1$ and $E(S) = \frac{1}{\mu_1} + \frac{1}{\mu_2} = \frac{1}{3.5} + \frac{1}{2.5} = 0.6857$, $E(S) < E(A)$ equibilirum state is provided.

The average number of customers in the system is equal to (17),

$$E(N) = \|\pi_1(I - R)^{-2}\|_1 = 5.7851$$

as obtained.

The mean duration of service (μ) in the system, the expected value $[E(N_q)]$ of the number of customers in the queueing from the equation (18), (19) equals the expected value $[E(W)]$ of a customer's stay in the system and (20) equals, the expected value $[E(W_q)]$ of the duration of a customer's stay in the queue is obtained as follows.

$$\mu = \frac{1}{\frac{1}{\mu_1} + \frac{1}{\mu_2}} = 1.458333$$

$$E(N_q) = E(N) - \frac{\lambda}{\mu} = 5.0993$$

$$E(W) = \frac{E(N)}{\lambda} = 5.7851$$

$$E(W_q) = \frac{E(N_q)}{\lambda} = 5.0993$$

4. CONCLUSION

In this study, a stochastic queueing system consisting of two-stage, phase-type and serial service units was studied. In this system, it is assumed that the customers come to the system with the Poisson flow and λ parameter and after finishing the service with the μ_1 parameter in the first phase and after completing the service reception with the μ_2 parameter in the second phase. In addition, in this system, a customer can not go to service in the first and second phase without completing the service, that is, without leaving the whole system. This two-stage queuing system created under the given assumptions is known as the $M/Hypo_2/1$ queuing system. The two-stage queuing system that was created was analyzed by Neuts matrix-geometric method and the performance measures of the system were found and the theoretical results obtained by an application were supported.

It is assumed that the arrival rate of the customers in the queue system $M/Hypo_2/1$ is the same as $\lambda = 1$, the average service time in which the customers received in the first phase is $\mu_1 = 3.5$ min. and the average service time in which the customers are in the second phase, $\mu_2 = 2.5$ min. $M/Hypo_2/1$ sub-matrices belonging to the pass rate of the queueing system were created. Based on the matrices V and W obtained from these submatrices, the simulation values of the R_{l+1} sequence are obtained by Matlab R2009 program. Burada iterasyon sayısı 100 adım olarak alınmıştır; because after about 100 steps the same values have been reached. Later, R_{100} values of Neuts R matrix are obtained to form $\pi Q = 0$ homogeneous linear equations systems and the normalized sub-vectors π_k's are obtained by iterations by applying the necessary normalization operations to these equation systems. Depending on these π_k vectors, the probabilities of any number of customer in the system are found. According to these findings, the probability that there are no customers in the system is $p_0 = 0.3142$, the probability of being a customer is $p_1 = 0.2514$, the probability of having two customers is $p_2 = 0.1652$ and the probability of having three customers is $p_3 = 0.10348$ and so on.

The average time of arrival for the system was calculated as $E(A) = 1$ min and the average duration of service $E(S) = 0.6857$ min. Accordingly, it can be seen that $E(S) < E(A)$ provides the ergodic condition.

The performance measures of the system are as follows: the average number of customers in the system is $E(N) = 5.7851$; average number of customers in the queue E $E(N_q) = 5.0993$; the average waiting time of a customer in the system is $E(W) = 5.7851$; the average waiting time of a customer in the queue was calculated as $E(W_q) = 5.0993$. $E(N) = E(W)$ and $E(N_q) = E(W_q)$ are obtained because $\lambda = 1$ is accepted here.

In this study, it is shown that by using Neuts matrix-geometric method, $M/Hypo_2/1$ queuing system can be analyzed and simulated to obtain the probabilities and performance measures of customer numbers in the system. It is thought that in the following studies it is possible to record similar progress by increasing the number of phases in this queue model or applying this method to other queue models.

REFERENCES

Artalejo, J. R. & Choudhury, G. (2004). Steady-State analysis of an M/G/1 queue with repeated attempts and two-phase service. *Quality Technology & Quantative Management*, Vol.1, No.2, pp. 189-199.

De Muynck, M., Bruneel, H. & Wittevrongel, S. (2017). Anaysis of a discrete-time queue with general service demands and phase-type service capacities, Journal of Industrial and Management Optimisation, 2017, Doi: 10.3934/jimo.20170204, Volume: 13, Issue: 4, Pages: 1901-1926.

Jackson, R. R. P. (1954). Queueing systems with Phase-Typeservice. *Operat. Res. Quart*., 5, 109-120.

Neuts, M. F. (1981). *Matrix-Geometric solutions in stochastic models, an algoritthmic approach*, John Hopkins University Press, Baltimore.

Ramaswami, V. & Neuts, M. F. (1980). A duality theorem for phase type queues. *The Annals of Probability*, Vol.8, No.5, 974-985.

Rana, R. K. (1972). Queueing problems with arrivals in general stream and phase type service. *Metrika*, vol. 18, no. 1, pp. 69–80.

Sağlam, V. & Zobu, M. (2013). Control of Traffic Intensity in Hyperexponential and Mixed Erlang Queueing System with a Method Based on SPRT, MATHEMATICHAL PROBLEMS IN ENGINEERING, 2013, 241241, 6 / 2013, 10.1155.

Sağlam V., Zobu, M., Yücesoy, E. & Sağır, M. (2016). The simulation and control of traffic intensity in Hypoexponential and Coxian queueing systems with a method based on sequential probability ratio tests, International Journal of Sciences: Basic and Applied Research, No.4, 122-137.

Smaili, K., Kadri, T. & Kadry, S. (2013). Hypoexponential distribution with different parameters, Applied Mathematics, 624-631.

Stewart, W. J. (2009). *Probability, Markov Chains, Queues and Simulation*, Princeton University Press, United Kingdom.