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On the Resolution of the Acceleration Vector According to Bishop Frame†

Kahraman Esen Özen[1*] and Murat Tosun[2]

[1]*Sakarya, Turkey*
[2]*Department of Mathematics, Faculty of Science and Arts, Sakarya University, Sakarya, Turkey*
[*]*Corresponding author*

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Abstract

In the second half of the 19th century, Siacci investigated the motion of a particle in space under the influence of any forces (Atti R Accad Sci. Torino **14**(1879)). In this study, Siacci obtained a resolution of the acceleration vector which is very useful when the angular momentum is conserved. On the other hand, Bishop introduced the Bishop frame which is well defined for every curves and so very convenient for mathematical researches in the third quarter of the 20th century (Am Math Monthly **82**(1975)). In this study, we discuss the Siacci's resolution of the acceleration vector according to Bishop frame of the trajectory of the moving particle. Also, we provide an illustrative example for the obtained results.

1. Introduction

In kinematics, the change in velocity of a moving particle in 3-dimensional Euclidean space with respect to the time parameter gives the acceleration. Since the force acting on a particle is concerned with its acceleration through the equation $\mathbf{F} = m\mathbf{a}$, the acceleration vector has an important place in kinematics and Newtonian physics.

The acceleration vector is usually written as the sum of its tangent and normal components. This writing style is useful in many applications. But we can not say this in movements where angular momentum is conserved. In this case, it is more useful to write the acceleration vector as the sum of its tangent and radial components. The success of obtaining the acceleration vector along tangent and radial components belongs to the Italian mathematician Francesco Siacci. The acceleration vector is stated in the aforementioned form by Siacci in the study [1]. In this study performed by Siacci, the motion of the particle is restricted to the plane. Also, Siacci performed a similar study for a moving particle in space [2].

Siacci's theorem has been studied widely by many authors. Whittaker was the first person to deal with this issue after Siacci. Whittaker proved the Siacci's theorem in the plane geometrically in his work carried out in 1937 [3]. Grossman succeeded in providing a more modern proof than Whittaker's in 1996 [4]. Afterwards, Casey discussed the Siacci's theorem in space which is based on the Serret-Frenet formulas to simplify the mathematical expressions in the theorem [5]. One of the most recent studies has been carried out by Kucukarslan et al [6]. The authors expressed and proved the Siacci's theorem for the curves lying on the Finsler manifold in this study. Then, Ozen studied on the Siacci's theorem for Bishop and Type-2 Bishop frames in his master's thesis [7] under the supervision of M. Tosun (The present article is derived from this master's thesis). Also Ozen et al. [8] researched the Siacci's theorem in view of the Darboux frame for the motion of a particle along the regular surface curve. Afterwards, Ozen et al [9] discussed the Siacci's theorem in the space endowed with the modified orthogonal frame. Finally, Ozen expressed and proved the Siacci's theorem for Frenet curves in 3-dimensional Minkowski space [10].

In the theory of curves, Serret-Frenet frame is a moving frame which is very useful and has an important place. To ride along a curve and illustrate the typical properties of this curve, e.g. the curvatures is possible thanks to this frame. But this frame has a disadvantage. For the curves which have vanishing second derivatives, it is not well defined. Hence an alternative frame, that is more convenient for mathematical investigations, was required. The discovery of Bishop frame finished this requirement in 1975 [11].

†This article is derived from the master's thesis titled "Siacci's theorem for the first and the second Bishop frame" which was carried out by K. E. Özen under the supervision of M. Tosun at Sakarya University / Graduate School of Natural and Applied Sciences.

***Email addresses and ORCID numbers*:** **kahraman.ozen1@ogr.sakarya.edu.tr, 0000-0002-3299-6709 (K. E. Özen), tosun@sakarya.edu.tr, 0000-0002-4888-1412 (M. Tosun)**

This frame is well defined for every curves. As a result of this, it has been studied by a lot of researchers to deal various concepts. Today, the studies on the Bishop frame have expanded to areas such as Biology and Computer graphics. Bishop's framework is used in predicting the structural information of DNA in biology and controlling virtual cameras in the field of Computer Graphics. The readers are referred to the studies [12–18] which are related to Bishop frame.
This article is organized as follows. In Section 2, we have given a short knowledge on the fundamental concepts to ensure understanding the ensuing sections. In Section 3, for a moving particle in space, we give Siacci's theorem in terms of Bishop elements of the trajectory. Moreover, an illustrative example is given for the aforementioned theorem.

2. Preliminaries

Let us consider the 3−dimensional Euclidean space E^3 with the standard scalar product:

$$\langle \mathbf{Q},\mathbf{R}\rangle = q_1r_1+q_2r_2+q_3r_3, \tag{2.1}$$

where $\mathbf{Q}=(q_1\,,\,q_2\,,\,q_3)\,,\ \mathbf{R}=(r_1\,,\,r_2\,,\,r_3)$ are arbitrary vectors in this space. The norm of the vector $\mathbf{Q}$ is given by $\|\mathbf{Q}\| = \sqrt{\langle \mathbf{Q},\mathbf{Q}\rangle}$. If a curve $\sigma=\sigma(s): I\subset\mathbb{R}\to E^3$ satisfies the equality $\left\|\frac{d\sigma}{ds}\right\|=1$ for all $s\in I$, this curve is said to be a unit speed curve and s is said to be arc-length parameter of this unit speed curve.
The moving Serret-Frenet frame of $\sigma(s)$ is showed with $\{\mathbf{T}(s)\,,\mathbf{N}(s)\,,\mathbf{B}(s)\}$. In this frame, the vector $\mathbf{T}(s)$ is called the unit tangent vector, the vector $\mathbf{N}(s)$ is called the unit principal normal vector. Also, the vector $\mathbf{B}(s)$ is called the unit binormal vector and it is obtained by vectorial product of $\mathbf{T}(s)$ and $\mathbf{N}(s)$. Another thing that can be of importance is that this frame satisfies the following formulas:

$$\begin{aligned}\frac{d\mathbf{T}}{ds}&=\kappa\mathbf{N}\\ \frac{d\mathbf{N}}{ds}&=-\kappa\mathbf{T}+\tau\mathbf{B}\\ \frac{d\mathbf{B}}{ds}&=-\tau\mathbf{N},\end{aligned} \tag{2.2}$$

where $\kappa=\left\|\frac{d\mathbf{T}}{ds}\right\|$ and $\tau=-\left\langle\frac{d\mathbf{B}}{ds},\mathbf{N}(s)\right\rangle$ represent the curvature function and the torsion function, respectively [19].
We know that the unit tangent vector $\mathbf{T}(s)$ of a given curve is determined uniquely. The Bishop frame of this given curve comprises the unique tangent vector $\mathbf{T}(s)$ and two normal vectors $\mathbf{N_1}(s)$ and $\mathbf{N_2}(s)$, that are obtained by applying the circular rotation to the vectors $\mathbf{N}(s)$ and $\mathbf{B}(s)$ in the instantaneous normal plane $\mathbf{T}(s)^{\perp}$ such that $\mathbf{N_1}'(s)$ and $\mathbf{N_2}'(s)$ are collinear with $\mathbf{T}(s)$ [11]. Consequently, we have the Bishop frame $\{\mathbf{T},\mathbf{N_1},\mathbf{N_2}\}$ which satisfies the derivative formulas:

$$\begin{aligned}\frac{d\mathbf{T}}{ds}&=k_1\mathbf{N_1}+k_2\mathbf{N_2}\\ \frac{d\mathbf{N_1}}{ds}&=-k_1\mathbf{T}\\ \frac{d\mathbf{N_2}}{ds}&=-k_2\mathbf{T},\end{aligned} \tag{2.3}$$

where k_1 and k_2 indicate the Bishop curvatures. As a result of the aforementioned circular rotation, there is a relation between the Serret-Frenet frame and Bishop frame as follows:

$$\begin{aligned}\mathbf{T}&=\mathbf{T}\\ \mathbf{N_1}&=\cos\varphi\,\mathbf{N}-\sin\varphi\,\mathbf{B}\\ \mathbf{N_2}&=\sin\varphi\,\mathbf{N}+\cos\varphi\mathbf{B},\end{aligned} \tag{2.4}$$

where φ represents the aforementioned rotation angle. On the other hand, the equalities

$$\begin{aligned}\varphi(s)&=\arctan\frac{k_2(s)}{k_1(s)}\\ \kappa(s)&=\sqrt{{k_1}^2(s)+{k_2}^2(s)}\\ k_1(s)&=\sqrt{{k_1}^2(s)+{k_2}^2(s)}\cos\varphi(s)\\ k_2(s)&=\sqrt{{k_1}^2(s)+{k_2}^2(s)}\sin\varphi(s)\\ \tau(s)&=\frac{d\varphi}{ds}\end{aligned} \tag{2.5}$$

hold [16, 20].
In E^3, let us assume that a particle P moves along a curve γ endowed with the Bishop frame. At time t, let us show the position vector of P relative to the origin O with $\mathbf{x}$. Denote by s the arc-length parameter of γ which is a function of the time t. Then the equality

$$\mathbf{T}=\frac{d\mathbf{x}}{ds}$$

is immediately obtained. This equality yields the velocity of P as follows [5]:

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{x}}{dt}\\ &= \frac{d\mathbf{x}}{ds}\frac{ds}{dt}\\ &= \frac{ds}{dt}\mathbf{T}.\end{aligned}$$

Similarly above, the acceleration

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt}\\ &= \frac{d}{dt}\left(\frac{ds}{dt}\mathbf{T}\right)\\ &= \frac{d}{dt}\left(\frac{ds}{dt}\right)\mathbf{T}+\frac{ds}{dt}\frac{d\mathbf{T}}{dt}\\ &= \frac{d^2s}{dt^2}\mathbf{T}+\frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\\ &= \frac{d^2s}{dt^2}\mathbf{T}+\left(\frac{ds}{dt}\right)^2 k_1\mathbf{N_1}+\left(\frac{ds}{dt}\right)^2 k_2\mathbf{N_2}\end{aligned}$$

is found. With the help of (2.5), $\mathbf{a}$ can be written as in the following form:

$$\mathbf{a} = \frac{d^2s}{dt^2}\mathbf{T}+\sqrt{{k_1}^2+{k_2}^2}\left(\frac{ds}{dt}\right)^2(\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}). \tag{2.6}$$

Then we conclude that the acceleration vector lies in the instantaneous plane $S_{\mathbf{p}}\{\mathbf{T},\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}\}$. The instantaneous vector $-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}$ is the normal vector of this instantaneous plane and the system $\{\mathbf{T},\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2},-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}\}$ is a right-handed orthonormal system [7].

3. Alternative Resolution of Acceleration Vector According to Bishop Frame

In this section, we express Siacci's theorem according to Bishop Frame and give an example for the application of this theorem (see [7] for more details). We continue to take into account of the aforementioned particle P.
Suppose that the position vector of the particle P is resolved as

$$\mathbf{x} = a\mathbf{T}-b(\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2})+c(-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}), \tag{3.1}$$

where

$$\begin{aligned}a &= \langle\mathbf{x},\mathbf{T}\rangle\\ b &= \langle\mathbf{x},-\cos\varphi\,\mathbf{N_1}-\sin\varphi\,\mathbf{N_2}\rangle\\ c &= \langle\mathbf{x},-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}\rangle.\end{aligned} \tag{3.2}$$

Denote by $\mathbf{r}$ the vector

$$\mathbf{r} = a\mathbf{T}-b(\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}), \tag{3.3}$$

lying in the instantaneous plane $Sp\{\mathbf{T},\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}\}$. Where r symbolizes the length of $\mathbf{r}$

$$r^2 = \langle\mathbf{r},\mathbf{r}\rangle = a^2+b^2 \tag{3.4}$$

can be written easily (see Figure 3.1).
On the other hand, the angular momentum vector of P about the origin O is obtained as

$$\mathbf{H}^O = mc\frac{ds}{dt}(\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2})+mb\frac{ds}{dt}(-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}) \tag{3.5}$$

by vector product of $\mathbf{x}$ and $m\frac{ds}{dt}\mathbf{T}$.
Now we try to resolve the acceleration vector in (2.6) along the radial direction BP and tangential direction in the instantaneous plane $Sp\{\mathbf{T},\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}\}$. To do so, let us state the vector $\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}$ in terms of $\mathbf{r}$ and $\mathbf{T}$. Due to (3.3), that can be possible when $b\neq 0$. If we assume that the component of angular momentum along the vector $-\sin\varphi\,\mathbf{N_1}+\cos\varphi\,\mathbf{N_2}$ never vanishes, we can ensure that b never equals to zero. Considering this assumption, we can write the following equalities

$$\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2} = \frac{1}{b}(-\mathbf{r}+a\mathbf{T}) \tag{3.6}$$

and

$$\mathbf{e_r} = \frac{1}{r}\mathbf{r}. \tag{3.7}$$

Here $\mathbf{e_r}$ indicates the unit vector in direction of $\mathbf{r}$. By means of (3.7), we get

$$\cos\varphi\,\mathbf{N_1}+\sin\varphi\,\mathbf{N_2}=\frac{1}{b}\left(-r\,\mathbf{e_r}+a\mathbf{T}\right). \tag{3.8}$$

Considering (3.8) in (2.6), we obtain the fundamental form of the acceleration vector as in the following:

$$\mathbf{a}=\left[\frac{d^2s}{dt^2}+\frac{a}{b}\sqrt{{k_1}^2+{k_2}^2}\left(\frac{ds}{dt}\right)^2\right]\mathbf{T}+\left[-\frac{r}{b}\sqrt{{k_1}^2+{k_2}^2}\left(\frac{ds}{dt}\right)^2\right]\mathbf{e_r}=S_t\,\mathbf{T}+S_r\,\mathbf{e_r}. \tag{3.9}$$

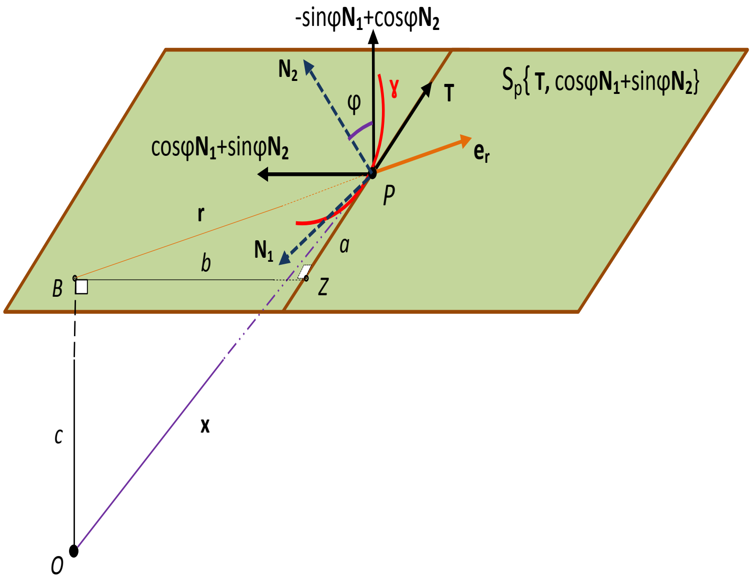

Figure 3.1: An illustration for explaining the motion of the moving particle P and the components of its acceleration vector.

Now we want to obtain the acceleration components S_t and S_r of the particle P in various forms. Because of that, we need some preparation. Let us differentiate the right and left side of (3.1) with respect to s. Then we obtain

$$\begin{aligned}\mathbf{T}=&\left(\frac{da}{ds}+k_1c\sin\varphi+k_1b\cos\varphi-k_2c\cos\varphi+k_2b\sin\varphi\right)\mathbf{T}\\&+\left(k_1a-\frac{dc}{ds}\sin\varphi-\frac{d\varphi}{ds}c\cos\varphi-\frac{db}{ds}\cos\varphi+\frac{d\varphi}{ds}b\sin\varphi\right)\mathbf{N_1}\\&+\left(k_2a+\frac{dc}{ds}\cos\varphi-\frac{d\varphi}{ds}c\sin\varphi-\frac{db}{ds}\sin\varphi-\frac{d\varphi}{ds}b\cos\varphi\right)\mathbf{N_2}.\end{aligned} \tag{3.10}$$

Because the vectors $\mathbf{T}$, $\mathbf{N_1}$ and $\mathbf{N_2}$ compose an orthonormal system,

$$\begin{aligned}1&=\frac{da}{ds}+k_1c\sin\varphi+k_1b\cos\varphi-k_2c\cos\varphi+k_2b\sin\varphi\\0&=k_1a-\frac{dc}{ds}\sin\varphi-\frac{d\varphi}{ds}c\cos\varphi-\frac{db}{ds}\cos\varphi+\frac{d\varphi}{ds}b\sin\varphi\\0&=k_2a+\frac{dc}{ds}\cos\varphi-\frac{d\varphi}{ds}c\sin\varphi-\frac{db}{ds}\sin\varphi-\frac{d\varphi}{ds}b\cos\varphi\end{aligned} \tag{3.11}$$

can be written. By keeping the equalities $k_1=\sqrt{{k_1}^2+{k_2}^2}\cos\varphi$ and $k_2=\sqrt{{k_1}^2+{k_2}^2}\sin\varphi$ in mind, we get

$$\begin{aligned}\frac{da}{ds}&=1-b\sqrt{{k_1}^2+{k_2}^2}\\\frac{db}{ds}&=a\sqrt{{k_1}^2+{k_2}^2}-c\frac{d\varphi}{ds}\\\frac{dc}{ds}&=b\frac{d\varphi}{ds}.\end{aligned} \tag{3.12}$$

If we differentiate (3.4) and use (3.12), it is not difficult to see the followings:

$$\begin{aligned}r\frac{dr}{ds}&=a-cb\frac{d\varphi}{ds}\\r\frac{dr}{ds}&=a-c\frac{dc}{ds}.\end{aligned} \tag{3.13}$$

Let us use the notation

$$h=b\frac{ds}{dt}. \tag{3.14}$$

Then, we obtain S_r as in the following form:

$$S_r=-\frac{rh^2\sqrt{{k_1}^2+{k_2}^2}}{b^3}. \tag{3.15}$$

If (3.12) and (3.14) are taken into consideration, S_t can be written as

$$S_t=\frac{1}{2b^2}\left(\frac{d}{ds}\left(h^2\right)+\frac{h^2}{b^2}\frac{d}{ds}\left(c^2\right)\right). \tag{3.16}$$

Similar to above, we can easily get

$$S_t=\frac{1}{2}\frac{d}{ds}\left(\left(\frac{ds}{dt}\right)^2\right)+\sqrt{{k_1}^2+{k_2}^2}\left(\frac{ds}{dt}\right)^2\left(\frac{1}{2b}\frac{d}{ds}\left(r^2\right)+\frac{d\varphi}{ds}c\right) \tag{3.17}$$

by using the first equality in (3.13).
Finally, it is very easy to see the following:

$$S_t=\frac{1}{2}\frac{d}{ds}\left(\left(\frac{ds}{dt}\right)^2\right)+\frac{1}{2b}\left(\frac{ds}{dt}\right)^2\sqrt{{k_1}^2+{k_2}^2}\frac{d}{ds}\left(\langle\mathbf{x},\mathbf{x}\rangle\right) \tag{3.18}$$

from the second equality in (3.13), since $\langle\mathbf{x},\mathbf{x}\rangle=a^2+b^2+c^2=r^2+c^2$.
Consequently, if we consider the above derivation, we can state the following theorem and corollary for a particle moving along a space curve endowed with the Bishop frame.

Theorem 3.1 (Siacci's Theorem According to Bishop Frame)**.** *([7]) In E^3, let P be a particle moving on a curve γ endowed with Bishop frame. Suppose that the component of the angular momentum of P along the unit vector $-\sin\varphi\mathbf{N_1}+\cos\varphi\mathbf{N_2}$ never equals to zero. Then, the acceleration vector $\mathbf{a}$ of the particle P can be expressed as in (3.9). The component S_t, given in (3.9), lies along the tangent line of γ. The component S_r, given in (3.9), lies along the line which passes through P and the foot of the perpendicular that is from O to the instantaneous plane $Sp\{\mathbf{T},\cos\varphi\mathbf{N_1}+\sin\varphi\mathbf{N_2}\}$.*

Corollary 3.2. *([7]) S_t can be given as in (3.16), (3.17) and (3.18) except for the fundamental form, while S_r can be given as in (3.15) except for the fundamental form.*

Remark 3.3. *([7]) In Euclidean 3-space E^3, let the trajectory γ be restricted to the fixed plane $Sp\{\mathbf{T},\cos\varphi\mathbf{N_1}+\sin\varphi\mathbf{N_2}\}$ containing or not containing O. Then, it is obvious that the unit vector $-\sin\varphi\mathbf{N_1}+\cos\varphi\mathbf{N_2}$, that is the unit normal vector of this fixed plane, is constant along γ. This means that its derivative $\frac{d}{ds}(-\sin\varphi\mathbf{N_1}+\cos\varphi\mathbf{N_2})$ equals to zero for all s values of the parameter. If this derivative is calculated, one can easily conclude that $\frac{d\varphi}{ds}=0$. So, for this case, (3.16) and (3.17) reduce to*

$$S_t=\frac{1}{2b^2}\frac{d}{ds}\left(h^2\right) \tag{3.19}$$

and

$$S_t=\frac{1}{2}\frac{d}{ds}\left(\left(\frac{ds}{dt}\right)^2\right)+\frac{1}{2b}\sqrt{{k_1}^2+{k_2}^2}\left(\frac{ds}{dt}\right)^2\frac{d}{ds}\left(r^2\right), \tag{3.20}$$

respectively.

Example 3.4. *Assume that the helix curve $\delta(t)=\left(8\cos\frac{t}{17},8\sin\frac{t}{17},15\frac{t}{17}\right)$ is the trajectory of the moving particle P. Then we can easily write*

$$\mathbf{x}=\left(8\cos\frac{s}{17},8\sin\frac{s}{17},15\frac{s}{17}\right). \tag{3.21}$$

Firstly, let us note that

$$\langle\mathbf{x},\mathbf{x}\rangle=\left\langle\left(8\cos\frac{s}{17},8\sin\frac{s}{17},15\frac{s}{17}\right),\left(8\cos\frac{s}{17},8\sin\frac{s}{17},15\frac{s}{17}\right)\right\rangle=64+\frac{225}{289}s^2.$$

By differentiating (3.21) twice with respect to time t, we get

$$\mathbf{a}=\left(-\frac{8}{289}\left(\frac{ds}{dt}\right)^2\cos\frac{s}{17}-\frac{8}{17}\frac{d^2s}{dt^2}\sin\frac{s}{17},-\frac{8}{289}\left(\frac{ds}{dt}\right)^2\sin\frac{s}{17}+\frac{8}{17}\frac{d^2s}{dt^2}\cos\frac{s}{17},\frac{15}{17}\frac{d^2s}{dt^2}\right).$$

Since δ is a unit speed curve, it is obvious that

$$\frac{ds}{dt}=1$$
$$\frac{d^2s}{dt^2}=0.$$

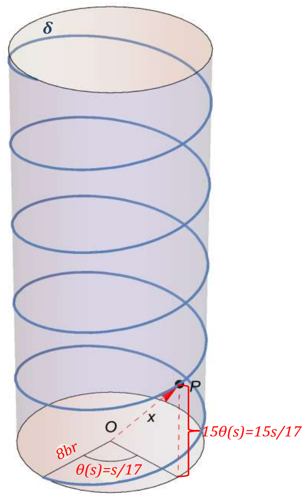

Figure 3.2: An illustration for the helix curve given in Example 3.4

On the other hand, the following equalities hold:

$$\begin{aligned} \mathbf{T}(s) &= \left(-\frac{8}{17}\sin\frac{s}{17}, \frac{8}{17}\cos\frac{s}{17}, \frac{15}{17}\right) \\ \mathbf{N}(s) &= \left(-\cos\frac{s}{17}, -\sin\frac{s}{17}, 0\right) \\ \mathbf{B}(s) &= \left(\frac{15}{17}\sin\frac{s}{17}, -\frac{15}{17}\cos\frac{s}{17}, \frac{8}{17}\right). \end{aligned} \tag{3.22}$$

From here, we obtain

$$\sqrt{{k_1}^2+{k_2}^2} = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\left(-\frac{8}{289}\cos\frac{s}{17}, -\frac{8}{289}\sin\frac{s}{17}, 0\right)\right\| = \frac{8}{289}.$$

By means of (2.4) *and* (3.22)*, the second and third Bishop bases are given by*

$$\mathbf{N_1} = \left(-\cos\varphi\cos\frac{s}{17} - \frac{15}{17}\sin\varphi\sin\frac{s}{17}, -\cos\varphi\sin\frac{s}{17} + \frac{15}{17}\sin\varphi\cos\frac{s}{17}, -\frac{8}{17}\sin\varphi\right) \tag{3.23}$$

and

$$\mathbf{N_2} = \left(-\sin\varphi\cos\frac{s}{17} + \frac{15}{17}\cos\varphi\sin\frac{s}{17}, -\sin\varphi\sin\frac{s}{17} - \frac{15}{17}\cos\varphi\cos\frac{s}{17}, \frac{8}{17}\cos\varphi\right). \tag{3.24}$$

Then we can write

$$\begin{aligned} a &= \langle \mathbf{x}, \mathbf{T}\rangle = \frac{225}{289}s \\ b &= \langle \mathbf{x}, -\cos\varphi\mathbf{N_1} - \sin\varphi\mathbf{N_2}\rangle = 8. \end{aligned}$$

So, we get

$$\begin{aligned} h &= 8 \\ r &= \sqrt{\frac{50625}{83521}s^2+64}. \end{aligned}$$

Substituting the obtained values of b, r, h, $\sqrt{{k_1}^2+{k_2}^2}$, $\frac{ds}{dt}$ *and* $\langle \mathbf{x}, \mathbf{x}\rangle$ *into* (3.15) *and* (3.18) *gives us the followings:*

$$S_t = \frac{225}{83521}s \tag{3.25}$$

and

$$S_r = -\frac{1}{289}\sqrt{\frac{50625}{83521}s^2+64}. \tag{3.26}$$

Finally, we must note that one can easily find the same solutions by means of the other options that are given in (3.9), (3.16) *and* (3.17).

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