Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

Fixed Points for Generalized Type Contractions in Partially Ordered Metric Spaces

Seher Sultan YEŞİLKAYA[1,*]

[1]*Kahramanmaraş Sütçü İmam University, Department of Mathematics, 46040, Kahramanmaraş, Türkiye*
sultanseher20@gmail.com, ORCID: 0000-0002-1748-2398

Received: 03.11.2020 **Accepted:** 10.09.2021 **Published:** 31.12.2021

Abstract

In this article, we define ordered weak θ-contractive and ordered Ćirić type weak θ-contractive mappings in partially ordered metric spaces. We also introduce some fixed point theorems for such mappings. These theorems extend the main theorems of many comparable results from the current literature. Finally, an example is showed to support the new theorems.

Keywords: Fixed point theorem; Partially ordered metric spaces; Regular mapping.

Kısmi Sıralı Metrik Uzaylarda Genelleştirilmiş Tip Daralmalar için Sabit Noktalar

Öz

Bu makalede, kısmi sıralı metrik uzaylarda sıralı zayıf θ-daralma ve sıralı Ćirić tipi zayıf θ-daralma dönüşümleri tanımlanmıştır. Ayrıca, bu tür dönüşümler için bazı sabit nokta teoremleri tanıtılmıştır. Bu teoremler, mevcut literatürden birçok karşılaştırılabilir sonucun ana teoremlerini genişletir. Son olarak, yeni teoremleri destekleyen bir örnek gösterilmiştir.

Anahtar Kelimeler: Sabit nokta teoremi; Kısmi sıralı metrik uzaylar; Düzenli dönüşüm.

* Corresponding Author
DOI: 10.37094/adyujsci.820255

1. Introduction and Preliminaries

Fixed point theory in metric spaces begins with the Banach contraction mapping, introduced in 1922 [1]. The existence of fixed points in partially ordered metric spaces has been investigated in [2]. Fixed points of operators in partially ordered metric spaces are very significant and have been studied by several authors [3-9].

We said that X is regular if the ordered metric spaces $(X, \preceq, d)$ supplies the following cases: If $\{x_r\}$ is an increasing sequence in X with respect to $\preceq$ such that $x_r \to v \in X$, then $x_r \preceq v$ for all $r \in \mathbb{N}$.

Banach contraction principle has been weakened and generalized by many researchers. For example, the notions of θ-contraction [10], F-contraction [11], Ćirić contraction [12], weak generalized contraction [4], have been established, and several generalizations of this principle are obtained.

Jleli and Samet denote by Θ the set of functions $\theta: (0, \infty) \to (1, \infty)$ satisfying the following conditions:

(Θ_1) θ is non-decreasing;

(Θ_2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0^+$;

(Θ_3) there exist $r \in (0,1)$ and $l \in (0, \infty]$ such that $\lim_{t\to 0^+} \frac{\theta(t)-1}{t^r} = l$.

Define by Ψ the set of functions $\Psi: [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is non-decreasing;

(ii) for each $k > 1$, $\lim_{n\to\infty} \psi^n(k) = 0$;

(iii) $\psi(0) = 0$, and for each $k > 1$, $\psi(k) < k$.

In this study, we present new contractive mappings in partially ordered metric spaces, inspired by the papers of Samira et al. [13], Ćirić [12], Jleli and Samet [10].

2. Main Results

Now, we introduce some fixed point theorems for ordered weak θ-contractive and ordered Ćirić type weak θ-contractive mappings in partially ordered metric space. We begin this section with the definition of ordered weak θ-contractive.

Definition 1. Let $(X,\preceq,d)$ be a partially ordered metric space. and $K: X\to X$ be a self mapping. Let

$$Q=\{(x,y)\in X\times X: x\preceq y,\ d(Kx,Ky)>0\} \tag{1}$$

and $\theta\in\Theta$ and $\psi\in\Psi$. We say that K is an ordered weak θ-contractive if there exists $\beta\in(0,1)$ such that

$$\theta(d(Kx,Ky))\leq[\theta(\psi(d(x,y)))]^{\beta}, \tag{2}$$

for all $(x,y)\in Q$.

Theorem 1. Let $(X,\preceq,d)$ be a partially ordered complete metric space and $K: X\to X$ is an ordered weakly θ-contractive. Suppose that $\theta\in\Theta$, $\psi\in\Psi$ and K is non-decreasing and there exists $x_0\in X$ such that $x_0\preceq Kx_0$. Therefore, K has a fixed point in X provided that at least one of the following conditions holds

(i) K is continuous,

(ii) X is regular.

Proof. Starting from an arbitrary point $x_0\in X$. We consider the constructive sequence $\{x_r\}\subset X$ which is defined by $x_r=Kx_{r-1}=K^r x_0$, for all $r\in\mathbb{N}$. Suppose that there exists $r_0\in\mathbb{N}$ such that $x_{r_0}=x_{r_0+1}$, then $x_{r_0}=x_{r_0+1}=Kx_{r_0}$ and so the proof is completed.

Now assume that for all $r\in\mathbb{N}$, $x_{r+1}\neq x_r$. Since $x_0\preceq Kx_0$ and K is non-decreasing, we get

$$x_0\preceq x_1\preceq x_3\preceq\cdots\preceq x_r\preceq\cdots.$$

From $x_r\preceq x_{r+1}$ and $d(Kx_{r-1},Kx_r)>0$ for all $r\in\mathbb{N}$, we have $(x_r,x_{r+1})\in Q$. So, using Eqn. (2) we obtain

$$\theta(d(x_r,x_{r+1}))=\theta(d(Kx_{r-1},Kx_r))\leq[\theta\left(\psi\left(d(x_{r-1},x_r)\right)\right)]^{\beta}, \tag{3}$$

for all $r\in\mathbb{N}$. Since condition (Θ_1), we obtain

$$\begin{aligned}\theta(d(K^r x_0,K^{r+1}x_0))&\leq[\theta(\psi(d(K^{r-1}x_0,K^r x_0)))]^{\beta}\\&\leq[\theta(\psi(d(K^{r-2}x_0,K^{r-1}x_0)))]^{\beta^2}\\&\ \ \vdots\\&\leq[\theta(\psi(d(x_0,Kx_0)))]^{\beta^r}.\end{aligned} \tag{4}$$

Letting $r\to\infty$ in the above inequality, we get

$$\lim_{r\to\infty}\theta(d(x_r,x_{r+1}))=1. \tag{5}$$

From (Θ_2) this implies that

$$\lim_{r\to\infty}d(x_r,x_{r+1})=0^+.$$

Using (Θ_3), there exists $w\in(0,1)$ and $V\in(0,\infty]$ such that

$$\lim_{r\to\infty}\frac{\theta(d(x_r,x_{r+1}))-1}{(d(x_r,x_{r+1}))^w}=V. \tag{6}$$

Suppose that $V<\infty$. In this case, let $S=\frac{V}{2}>0$. From the definition of the limit, there exists $r_0\in\mathbb{N}$ such that

$$\left|\frac{\theta(d(x_r,x_{r+1}))-1}{(d(x_r,x_{r+1}))^w}-V\right|\leq S,\qquad \text{for all } r\geq r_0.$$

Then we get

$$\frac{\theta(d(x_r,x_{r+1}))-1}{(d(x_r,x_{r+1}))^w}\geq V-S=S,\qquad \text{for all } r\geq r_0.$$

Then for all $r\geq r_0$, we obtain

$$r(d(x_r,x_{r+1}))^w\leq Hr[\theta(d(x_r,x_{r+1}))-1],$$

where $H=\frac{1}{S}$. Assume that $V=\infty$. Let $S>0$ be an arbitrary positive number. Thus there exists $r_0\in\mathbb{N}$ such that

$$\frac{\theta(d(x_r,x_{r+1}))-1}{(d(x_r,x_{r+1}))^w}\geq S,$$

for all $r\geq r_0$. This implies that for all $r\geq r_0$,

$$r(d(x_r,x_{r+1}))^w\leq Hr[\theta(d(x_r,x_{r+1}))-1],$$

where $H=\frac{1}{S}$. Therefore, in two cases, there exists $H>0$ and $r_0\in\mathbb{N}$ such that, for all $r\geq r_0$,

$$r(d(x_r,x_{r+1}))^w\leq Hr[\theta(d(x_r,x_{r+1}))-1].$$

By using Eqn. (4), we get

$$r(d(x_r,x_{r+1}))^w\leq Hr([\theta(\psi(d(x_0,x_1)))]^{\beta^r}-1), \tag{7}$$

for all $r\geq r_0$. Letting $r\to\infty$ in Eqn. (7), we obtain

$$\lim_{r\to\infty}r(d(x_r,x_{r+1}))^w=0.$$

Therefore, there exists $r_1\in\mathbb{N}$ such that

$$d(x_r, x_{r+1}) \leq \frac{1}{r^{\frac{1}{w}}}, \text{ for all } r \geq r_1. \tag{8}$$

Next, we prove that $\{x_r\}$ is a Cauchy sequence in K. There exists sequences $r, p \in \mathbb{N}$ such that $p > r \geq r_1$. Then from Eqn. (8), we obtain

$$\begin{aligned} d(x_r, x_p) &\leq d(x_r, x_{r+1}) + d(x_{r+1}, x_{r+2}) + \cdots + d(x_{p-1}, x_p) \\ &\leq \sum_{i=r}^{p-1} \frac{1}{i^{\frac{1}{w}}} \leq \sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{w}}}. \end{aligned}$$

By the convergence of the series $\sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{w}}}$, in the limit $r \to \infty$, we get $d(x_r, x_p) \to 0$. Then $\{x_r\}$ is a Cauchy sequence in (X, d). Because (X, d) is a complete metric space, there exists $v \in X$ such that

$$\lim_{r \to \infty} x_r = v. \tag{9}$$

If K is continuous, then we have

$$v = \lim_{r \to \infty} x_{r+1} = \lim_{r \to \infty} K x_r = K \lim_{r \to \infty} x_r = Kv.$$

Thus $v = Kv$, and v is a fixed point of K.

We assume that X is regular, so $x_r \preceq v$ for all $r \in \mathbb{N}$. Two cases arise here:

Case 1. If there exists $u \in \mathbb{N}$ for $x_u = v$, so

$$Kv = Kx_u = x_{u+1} \preceq v.$$

In addition, we have $x_u \preceq x_{u+1}$. So, $v \preceq Kv$ and thus $v = Kv$.

Case 2. Given that $x_r \neq v$ for all $r \in \mathbb{N}$ and $d(v, Kv) > 0$. From $\lim_{r \to \infty} x_r = v$, there exists $r_1 \in \mathbb{N}$ such that $d(x_{r+1}, Kv) > 0$ and $d(x_r, v) < \frac{d(v,Kv)}{2}$ for every $r \geq r_1$. In addition, $(x_r, v) \in Q$. Thus, from (Θ_1) and ψ non-decreasing we obtain

$$\begin{aligned} \theta(d(Kx_r, Kv)) &\leq [\theta(\psi(d(x_r, v)))]^{\beta} \\ &\leq \theta(\psi(d(x_r, v))) \\ &\leq \theta\left(\psi\left(\frac{d(v, Kv)}{2}\right)\right). \end{aligned}$$

This implies that

$$d(x_{r+1}, Kv) \leq \psi(\frac{d(v, Kv)}{2}) < \frac{d(v, Kv)}{2}.$$

Taking limit as $r \to \infty$, we obtain

$$d(v, Kv) < \frac{d(v, Kv)}{2},$$

a contraction. Hence, we get $d(v, Kv) = 0$, that is, $v = Kv$. This concludes the proof.

Definition 2. Let $(X, \preceq, d)$ be a partially ordered metric space and $K: X \to X$ be a self mapping. Let

$$Q = \{(x, y) \in X \times X: x \preceq y,\ d(Kx, Ky) > 0\}. \tag{10}$$

Further, $\theta \in \Theta$ and $\psi \in \Psi$. We say that K is an ordered Ćirić type weak θ-contractive if there exists $\beta \in (0,1)$ such that

$$\theta(d(Kx, Ky)) \leq [\theta(\psi(P(x, y)))]^{\beta}, \tag{11}$$

for all $(x, y) \in Q$, where

$$P(x, y) = \max\{d(x, y), d(x, Kx), d(y, Ky), \tfrac{1}{2}[d(x, Ky) + d(y, Kx)]\}. \tag{12}$$

Theorem 2. Let $(X, \preceq, d)$ be a partially ordered complete metric space and $K: X \to X$ is an ordered Ćirić type weak θ-contractive. Assume that $\theta \in \Theta$, $\psi \in \Psi$ and K is non-decreasing and there exists $x_0 \in X$ such that $x_0 \preceq Kx_0$. Therefore, K has a fixed point in X provided that at least one of the following conditions holds

(i) K is continuous,

(ii) X is regular.

Proof. Given an arbitrary point $x_0 \in X$ we consider the constructive sequence $\{x_r\} \subset X$ which is defined by $x_r = Kx_{r-1} = K^r x_0$, for all $r \in \mathbb{N}$. Assume there exists $r_0 \in \mathbb{N}$ such that $x_{r_0} = x_{r_0+1}$, then $x_{r_0} = x_{r_0+1} = Kx_{r_0}$ and so the proof is completed.

Now assume that for all $r \in \mathbb{N}$, $x_{r+1} \neq x_r$. As $x_0 \preceq Kx_0$ and K is non-decreasing, we get

$$x_0 \preceq x_1 \preceq x_3 \preceq \cdots \preceq x_r \preceq \cdots.$$

From $x_r \preceq x_{r+1}$ and $d(Kx_{r-1}, Kx_r) > 0$, for all $r \in \mathbb{N}$, we have $(x_r, x_{r+1}) \in Q$. So, from Eqn. (11) we obtain

$$\begin{aligned}
\theta(d(x_r, x_{r+1})) = \theta(d(Kx_{r-1}, Kx_r)) &\leq [\theta(\psi(\max\{d(x_{r-1}, x_r), d(x_{r-1}, Kx_{r-1}), d(x_r, Kx_r),\\
&\quad \tfrac{1}{2}[d(x_{r-1}, Kx_r) + d(x_r, Kx_{r-1})]\}))]^{\beta}\\
&= [\theta\big(\psi(\max\{d(x_{r-1}, x_r), d(x_r, x_{r+1})\})\big)]^{\beta},
\end{aligned}$$

for all $r \in \mathbb{N}$. Then suppose that

$$d(x_{r-1}, x_r) < d(x_r, x_{r+1}).$$

We get

$$\theta(d(x_r, x_{r+1})) \leq [\theta(\psi(d(x_r, x_{r+1})))]^{\beta}$$

a contradiction. Then we obtain

$$d(x_r, x_{r+1}) \leq d(x_{r-1}, x_r).$$

Hence we have

$$\theta(d(Kx_{r-1}, Kx_r)) \leq [\theta(\psi(d(x_{r-1}, x_r)))]^{\beta}.$$

From condition (Θ_1), we get

$$\begin{aligned}\theta(d(K^r x_0, K^{r+1} x_0)) &\leq [\theta(\psi(d(K^{r-1} x_0, K^r x_0)))]^{\beta} \\ &\leq [\theta(\psi(d(K^{r-2} x_0, K^{r-1} x_0)))]^{\beta^2} \\ &\quad \vdots \\ &\leq [\theta(\psi(d(x_0, Kx_0)))]^{\beta^r}. \end{aligned} \tag{13}$$

Letting $r \to \infty$ in the above inequality, we get

$$\lim_{r\to\infty} \theta(d(x_r, x_{r+1})) = 1, \tag{14}$$

From (Θ_2) this implies that

$$\lim_{r\to\infty} d(x_r, x_{r+1}) = 0^+.$$

Using (Θ_3), there exist $w \in (0,1)$ and $V \in (0, \infty]$ such that

$$\lim_{r\to\infty} \frac{\theta(d(x_r,x_{r+1}))-1}{(d(x_r,x_{r+1}))^w} = V. \tag{15}$$

Suppose that $V < \infty$. In this case, let $S = \frac{V}{2} > 0$. From the definition of the limit, there exists $r_0 \in \mathbb{N}$ such that

$$\left|\frac{\theta(d(x_r, x_{r+1})) - 1}{(d(x_r, x_{r+1}))^w} - V\right| \leq S, \qquad \text{for all } r \geq r_0.$$

Thereupon, we get

$$\frac{\theta(d(x_r, x_{r+1})) - 1}{(d(x_r, x_{r+1}))^w} \geq V - S = S, \qquad \text{for all } r \geq r_0.$$

So, for all $r \geq r_0$, we obtain

$$r(d(x_r, x_{r+1}))^w \leq Hr[\theta(d(x_r, x_{r+1})) - 1],$$

where $H = \frac{1}{S}$. Assume that $V = \infty$. Let $S > 0$ be an arbitrary positive number. Thus there exists $r_0 \in \mathbb{N}$ such that

$$\frac{\theta(d(x_r, x_{r+1})) - 1}{(d(x_r, x_{r+1}))^w} \geq S,$$

for all $r \geq r_0$. This implies that for all $r \geq r_0$,

$$r(d(x_r, x_{r+1}))^w \leq Hr[\theta(d(x_r, x_{r+1})) - 1],$$

where $H = \frac{1}{S}$. Therefore, in two cases, there exists $H > 0$ and $r_0 \in \mathbb{N}$ such that, for all $r \geq r_0$,

$$r(d(x_r, x_{r+1}))^w \leq Hr[\theta(d(x_r, x_{r+1})) - 1].$$

Using (13), we have

$$r(d(x_r, x_{r+1}))^w \leq Hr([\theta(\psi(d(x_0, x_1)))]^{\beta^r} - 1), \tag{16}$$

for all $r \geq r_0$. Letting $r \to \infty$ in Eqn. (16), we obtain

$$\lim_{r\to\infty} r(d(x_r, x_{r+1}))^w = 0.$$

Therefore, there exists $r_1 \in \mathbb{N}$ such that

$$d(x_r, x_{r+1}) \leq \frac{1}{r^{\frac{1}{w}}}, \text{ for all } r \geq r_1. \tag{17}$$

Next we prove that $\{x_r\}$ is a Cauchy sequence in K. There exist sequences $r, p \in \mathbb{N}$ such that $p > r \geq r_1$. Then from Eqn. (17), we obtain

$$d(x_r, x_p) \leq d(x_r, x_{r+1}) + d(x_{r+1}, x_{r+2}) + \cdots + d(x_{p-1}, x_p)$$
$$\leq \sum_{i=r}^{p-1} \frac{1}{i^{\frac{1}{w}}} \leq \sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{w}}}.$$

By the convergence of the series $\sum_{i=r}^{\infty} \frac{1}{i^{\frac{1}{w}}}$, in the limit $r \to \infty$, we get $d(x_r, x_p) \to 0$. Then $\{x_r\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, then there exists $v \in X$ such that

$$\lim_{r\to\infty} x_r = v. \tag{18}$$

If K is continuous, then we have

$$v = \lim_{r\to\infty} x_{r+1} = \lim_{r\to\infty} Kx_r = K \lim_{r\to\infty} x_r = Kv.$$

Thus $v = Kv$, and, v is a fixed point of K. We given that X is regular, so $x_r \preceq v$ for all $r \in \mathbb{N}$. Two conditions arise here:

Case 1. If there exists $u \in \mathbb{N}$ for that $x_u = v$, so

$$Kv = Kx_u = x_{u+1} \preceq v.$$

In addition, we have $x_u \preceq x_{u+1}$. So, $v \preceq Kv$ and thus $v = Kv$.

Case 2. Given that $x_r \neq v$ for all $r \in \mathbb{N}$ and $d(v, Kv) > 0$. From $\lim_{r\to\infty} x_r = v$, there exists $r_1 \in \mathbb{N}$ such that $d(x_{r+1}, Kv) > 0$ and $d(x_r, v) < \frac{d(v,Kv)}{2}$ for every $r \geq r_1$. In addition, $(x_r, v) \in Q$. Thus, from (Θ_1) and ψ non-decreasing we obtain

$$\begin{aligned}\theta(d(Kx_r, Kv)) &\leq [\theta(\psi(P(x_r, v)))]^{\beta} \\ &\leq \theta(\psi(\max\{d(x_r, v), d(x_r, Kx_r), d(v, Kv), \\ &\quad \frac{1}{2}[d(x_r, Kv) + d(v, Kx_r)]\})) \\ &\leq \theta\left(\psi\big(d(Kv, v)\big)\right).\end{aligned}$$

This implies that

$$d(x_{r+1}, Kv) \leq \psi(d(Kv, v)) < d(Kv, v).$$

Taking limit as $r \to \infty$, we obtain

$$d(v, Kv) \leq d(Kv, v),$$

a contraction. Hence, we get $d(v, Kv) = 0$, that is, $v = Kv$ this concludes the proof.

Example 1. Let $X = [0,1] \cup \{2,3\}$ and $d(x, y) = |x - y|$, for all $x, y \in X$. Define an order relation $\preceq$ on X, where $\preceq$ is usual order. $(X, \preceq, d)$ is complete and define a mapping $K: X \to X$ by

$$K(x) = \begin{cases} \frac{x}{6}, & x \in [0,1] \\ x, & x \in \{2,3\} \end{cases}.$$

Then, K is non-decreasing. We claim that K is an ordered weakly θ-contractive with $\theta(p) = e^{pe^p}$, $\psi(s) = \frac{s}{2}$ and $\beta = e^{-\frac{x}{3}}$ .Therefore, Theorem 1 and Theorem 2 are satisfied.

3. Conclusion

Agarwal et. al., [4] defined weak generalized contractions in partially ordered metric spaces. Ćirić [12] denoted Ćirić contraction and Jleli and Samet [10] defined θ-contraction in metric spaces. We introduce new contraction mappings by combining the ideas of Agarwal et al., Ćirić, Jleli and Samet. In addition, we present an example to show that the new theorems are applicable.

Acknowledgements

The author would like to thank the referees for their careful reading of the manuscript and valuable suggestions.

References

[1] Banach, S., *Sur les opérations dans les ensembles abstraits et leur application aux équations integrals*, Fundamenta Mathematicae, 3(1), 133-181, 1922.

[2] Ran, A.C., Reurings, M.C., *A fixed point theorem in partially ordered sets and some applications to matrix equations,* Proceedings of the American Mathematical Society, 1435-1443, 2004.

[3] Abbas, M., Nazir, T., Radenović, S., *Common fixed points of four maps in partially ordered metric spaces*, Applied Mathematics Letters, 24(9), 1520-1526, 2011.

[4] Agarwal, R.P., El-Gebeily, M.A., O'Regan, D., *Generalized contractions in partially ordered metric spaces,* Applicable Analysis, 87(1), 109-116, 2008.

[5] Yeşilkaya, S.S., *Some fixed point theorems in ordered metric spaces having t-property*, Tbilisi Mathematical Journal, 8-2021, 219–226, 2021.

[6] Kumam, P., Rouzkard, F., Imdad, M., Gopal, D., *Fixed point theorems on ordered metric spaces through a rational contraction,* Abstract and Applied Analysis, 2013, 2013.

[7] Durmaz, G., Mınak, G., Altun, I., *Fixed points of ordered F-contractions*, Hacettepe Journal of Mathematics and Statistics, 45(1), 15-21, 2016.

[8] Minak, G., Altun, I., *Ordered θ-contractions and some fixed point results,* Nonlinear Functional Analysis, 41, 2017.

[9] Yeşilkaya, S.S., *Presic Type Operators for a Pair Mappings,* Turkish Journal of Mathematics and Computer Science, 13(1), 204-210, 2021.

[10] Jleli, M., Samet, B., *A new generalization of the Banach contraction principle,* Journal of Inequalities and Applications, 2014(1), 1-8, 2014.

[11] Wardowski, D., *Fixed points of a new type of contractive mappings in complete metric Spaces,* Fixed Point Theory and Applications, 2012(1), 1-6, 2012.

[12] Ciric, L.B., *Generalized contractions and fixed-point theorems,* Publications de l'Institut Mathématique, 12(26), 19-26, 1971.

[13] Razavi, S.S., Parvaneh Masiha, H., *Generalized f-contractions in partially ordered metric spaces,* Sahand Communications in Mathematical Analysis, 16(1), 93-104, 2019.