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On Idempotency of Linear Combinations of Two 2×2 Idempotent Matrices

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ABSTRACT

Let A and B be 2×2 non-zero complex matrices. Let P be a linear combination of A and B in the form of $P = c_1A + c_2B$ where c_1, c_2 are nonzero scalar numbers. An idempotent matrix is a matrix which, when multiplied by itself, yields itself. In this study, we established the entries of idempotent matrix B according to a given A idempotent matrix such that P is also be an idempotent matrix. In addition, the result was obtained that this determined P matrix is a singular matrix.

İki 2×2 İdempotent Matrisin Lineer Kombinasyonunun İdempotentliği Üzerine

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ÖZET

A ve B, 2×2 tipinde sıfır olmayan kompleks matrisler olsun. c_1, c_2 sıfırdan farklı skaler sayılar olmak üzere P, A ile B nin $P = c_1A + c_2B$ formunda olan bir lineer kombinasyonu olsun. Bir idempotent matris, kendisiyle çarpıldığında kendisini veren bir matristir. Bu çalışmada, verilen A idempotent matrisine göre, B idempotent matrisinin bileşenleri, P matrisi de idempotent olacak şekilde belirlenmiştir. Ayrıca belirlenen bu P matrisinin singüler matris olduğu sonucu elde edilmiştir.

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1. Introduction

It is assumed throughout that $c_1, c_2 \in \mathbb{C}$ are nonzero complex numbers and A, B are two nonzero idempotent complex matrices. Let P be their linear combination of the form

$$P = c_1A + c_2B$$

For nonzero idempotent matrices a complete solution to the problem of characterizing all situations, where the linear combination of A and B preserves the idempotency property established by Baksalary and Baksalary [1]. They proved that if $AB = BA$ then there is no idempotent P except for the classical cases characterized by Halmos [2] and also they proved that if $AB \neq BA$ then P is idempotent if and only if $c_1 \in \mathcal{F}/\{0,1\}$, $c_2 = 1 - c_1$, $(A - B)^2 = 0$ (where $\mathcal{F}$ is a field). P_1, P_2 and P_3 being any three different nonzero mutually commutative idempotent matrices, and c_1,c_2 and c_3 being nonzero scalars, the problem of characterizing some situations, where a linear combination of the form $P=c_1P_1+c_2P_2$ or $P=c_1P_1+c_2P_2+c_3P_3$, is also an idempotent matrix considered by Özdemir and Özban [3] They established a complete solution to the problem of characterizing all situations, where the operation of combining linearly idempotent matrices preserves the idempotency property.

Differently from these studies in this paper we construct the entries of idempotent matrix B according to a given idempotent matrix A such that a linear combination of A and B of the form

$$P = c_1A + c_2B$$

is also be an idempotent matrix (where $A, B \in \mathbb{C}_{2\times 2}$ and $c_1, c_2 \in \mathbb{C}$ are nonzero complex numbers), and we obtained the result that B is the singular matrix

2. Results

By the Theorem 2.1.(a) [1], the problem considered in this paper becomes trivial when A is identity matrix or P is identity matrix. Consequently the case where A is identity and the case where P is identity excluded from the calculations in this note.

Theorem. For two $A = [a_{ij}], B = [b_{ij}] \in \mathbb{C}_{2\times 2}$ nonzero idempotent matrices, let P be their linear combinations of the form

$$P = c_1A + c_2B \qquad (1.1)$$

with nonzero scalars c_1 and c_2.

i) If $a_{11} = 1$ and $a_{12} = 0$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} = 1 - a_{21}b_{12}$, $b_{12}b_{21} = b_{12}a_{21}b_{11}$ and $b_{22} = a_{21}b_{12}$.

ii) If $a_{11} = 1$ and $a_{21} = 0$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} = 1 - a_{12}b_{21}$, $b_{21} = -a_{12}b_{11}$ and $b_{22} = a_{12}b_{21}$.

iii) If $a_{11} = 0$ and $a_{12} = 0$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} = a_{21}b_{12}$, $b_{21} = a_{21}(1 - b_{12}a_{21})$ and $b_{22} = 1 - a_{21}b_{12}$.

iv) If $a_{11} = 0$ and $a_{21} = 0$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} = a_{12}b_{21}$, $b_{21} = a_{21}b_{11}$ and $b_{22} = 1 - a_{12}b_{21}$.

v) If $a_{11} = \frac{1}{2}$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} - b_{11}^2 = = 2a_{12}b_{21}(1 - 2a_{12}b_{21})$, $b_{12} = 2a_{12}(1 - 2a_{12}b_{21})$ and $b_{22} = 1 - b_{11}$.

vi) If $a_{11} \notin \{0, 1, \frac{1}{2}\}$ then P is idempotent if and only if $c_2 = 1 - c_1$, $b_{11} = \frac{a_{12}b_{21} + b_{12}a_{21} - a_{11}}{1 - 2a_{11}}$, $b_{12}b_{21} = b_{11} - b_{11}^2$ and $b_{22} = 1 - b_{11}$.

Proof. Since $A = [a_{ij}]$, $B = [b_{ij}] \in \mathbb{C}_{2\times 2}$ idempotent matrices, A satisfies

$$a_{11}^2 + a_{12}a_{21} = a_{11}, \qquad (1.2)$$

$$a_{22} = 1 - a_{11} \, . \qquad (1.3)$$

and B satisfies

$$b_{11}^2 + b_{12}b_{21} = b_{11}, \qquad (1.4)$$

$$b_{22} = 1 - b_{11}. \qquad (1.5)$$

Direct calculations show that P of form (1.1) is idempotent if and only if

$$(c_1a_{11} + c_2b_{11})^2 + (c_1a_{21} + c_2b_{21})(c_1a_{12} + +c_2b_{12}) = (c_1a_{11} + c_2b_{11}) \quad (1.6)$$

and

$$1 - (c_1a_{22} + c_2b_{22}) = c_1a_{11} + c_2b_{11}. \qquad (1.7)$$

The equality (1.7) simplify to

$$c_2 = 1 - c_1 \qquad (1.8)$$

when substituting (1.3) and (1.5).

And the equality (1.6) simplifies to

$$a_{11} - 2a_{11}b_{11} + b_{11} = a_{12}b_{21} + b_{12}a_{21} \qquad (1.9)$$

when substituting (1.2), (1.4) and (1.8).

Therefore;

i) If $a_{11} = 1$ and $a_{12} = 0$ then from (1.9) we get $b_{11} = 1 - a_{21}b_{12}$. From Equation (1.4) and (1.5), it follows that $b_{12}b_{21} = = b_{12}a_{21}b_{11}$ and $b_{22} = a_{21}b_{12}$.

So B must be one of the forms below to ensure P's idempotency,

a) $\begin{bmatrix} 1 & 0 \\ b_{21} & 0 \end{bmatrix}$ $(for\ b_{12} = 0)$,

b) $\begin{bmatrix} 1 - a_{21}b_{12} & b_{12} \\ a_{21}(1 - a_{21}b_{12}) & a_{21}b_{12} \end{bmatrix}$ $(for\ b_{12} \neq 0)$.

ii) If $a_{11} = 1$ and $a_{21} = 0$ then from (1.9) we get $b_{11} = 1 - a_{12}b_{21}$ .From (1.4) and (1.5), it follows that $b_{12}b_{21} = a_{12}b_{11}b_{21}$ and $b_{22} = = a_{12}b_{21}$.

So B must be one of the forms below to ensure P's idempotency,

a) $\begin{bmatrix} 1 & b_{12} \\ 0 & 0 \end{bmatrix}$ $(for\ b_{21} = 0)$,

b) $\begin{bmatrix} 1 - a_{12}b_{21} & a_{12}(1 - a_{12}b_{21}) \\ b_{21} & a_{12}b_{21} \end{bmatrix}$ $(for\ b_{21} \neq 0)$.

iii) If $a_{11} = 0$ and $a_{12} = 0$ then from (1.9) we get $b_{11} = a_{21}b_{12}$. From (1.4) and (1.5), it follows that $b_{12}b_{21} = b_{12}a_{21}(1 - b_{12}a_{21})$ and $b_{22} = 1 - a_{21}b_{12}$.

So B must be one of the forms below to ensure P's idempotency,

a) $\begin{bmatrix} 0 & b_{12} \\ b_{21} & 1 \end{bmatrix}$ with $b_{21}b_{12} = 0$ ($for\ b_{11} = 0$),

b) $\begin{bmatrix} a_{21}b_{12} & b_{12} \\ a_{21}(1 - b_{12}a_{21}) & 1 - a_{21}b_{12} \end{bmatrix}$
($for\ b_{11} \neq 0$).

iv) If $a_{11} = 0$ and $a_{21} = 0$ then from (1.9) we get $b_{11} = a_{12}b_{21}$, from (1.4) and (1.5) , it follows that $b_{12}b_{21} = b_{21}a_{12}(1 - b_{21}a_{12})$ and $b_{22} = 1 - a_{12}b_{21}$.
So B must be one of the forms below to ensure P's idempotency,

a) $\begin{bmatrix} 0 & b_{12} \\ b_{21} & 1 \end{bmatrix}$ with $b_{21}b_{12} = 0$ ($for\ b_{11} = 0$),

b) $\begin{bmatrix} a_{12}b_{21} & a_{12}(1 - b_{21}a_{12}) \\ b_{21} & 1 - a_{12}b_{21} \end{bmatrix}$ ($for\ b_{11} \neq 0$).

v) If $a_{11} = \frac{1}{2}$ then from (1.2) we get $a_{12}a_{21} = \frac{1}{4}$. From (1.6) and (1.8), it follows that

$$\left(\frac{c_1}{2} + (1 - c_1)b_{11}\right)^2 + (c_1a_{12} + (1 - c_1)b_{12}).$$
$$.(c_1a_{21} + (1 - c_1)b_{21}) = \frac{c_1}{2} + (1 - c_1)b_{11}.$$

Substituting (1.4) to this equation simplifies the latter to the equality

$$a_{12}b_{21} + a_{21}b_{12} = \frac{1}{2}$$

and

$$a_{12}b_{21} + \frac{b_{12}}{4a_{12}} = \frac{1}{2}$$

Therefore $b_{12} = 2a_{12}(1 - 2a_{12}b_{21})$. This leads the equations (1.4) and (1.5) to
$b_{11} - b_{11}^2 = 2a_{12}b_{21}(1 - 2a_{12}b_{21})$ and
$b_{22} = 1 - b_{11}$.

So B is of the form

$$\begin{bmatrix} 1 - 2a_{12}b_{21} & 2a_{12}(1 - 2b_{21}a_{12}) \\ b_{21} & 2a_{12}b_{21} \end{bmatrix}.$$

vi) If $a_{11} \notin \left\{0, 1, \frac{1}{2}\right\}$ then from (1.6) we get

$$b_{11} = \frac{a_{12}b_{21} + b_{12}a_{21} - a_{11}}{1 - 2a_{11}}$$

Therefore, B must be one of the forms below to ensure P's idempotency.

a) If $b_{11} = 0$ then B is of the form

$$\begin{bmatrix} 0 & \frac{a_{11}}{a_{21}} \\ 0 & 1 \end{bmatrix}.$$

b) If $b_{11} \neq 0$ then B is of the form

$$\begin{bmatrix} \frac{a_{12}b_{21} + b_{12}a_{21} - a_{11}}{1 - 2a_{11}} & b_{12} \\ b_{21} & 1 - \frac{a_{12}b_{21} + b_{12}a_{21} - a_{11}}{1 - 2a_{11}} \end{bmatrix}$$

with $b_{12}b_{21} = b_{11} - b_{11}^2$.

Corollary. The matrix P is a singular matrix.

Statement of Conflict of Interest
Authors have declared no conflict of interest.

Author's Contributions
The contribution of the authors is equal.

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