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Some Properties for Certain Subclasses of Analytic Functions Associated with $k-$Integral Operators

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Abstract

In this paper, the k-integral operators for analytic functions defined in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ are introduced. Several new subclasses of analytic functions satisfying certain relations involving these operators are also introduced. Further, we establish the inclusion relation for these subclasses. Next, the integral preserving properties of a k-integral operator satisfied by these newly introduced subclasses are obtained. Some applications of the results are discussed. Concluding remarks are also given.

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1. Introduction

Geometric function theory is one of the most important branches of complex analysis which focus on the geometric properties of analytic functions. Geometric function theory was evolved around the turn of the 20^{th} century and developed deep connections with other fields of mathematics and physics like hyperbolic geometry, theory of partial differential equations, fluid dynamics etc. In this paper, we study certain classes based on some important geometric properties like starlikeness, convexity, close-to-convexity and quasi-convexity

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of univalent analytic functions and associated with certain k-integral operators.

Let A be the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

where f is analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Let P be the class of function $h(z)$ of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} h_n z^n \qquad (z \in U),$$

which are analytic and convex in U and satisfy the following condition:

$$Re(h(z)) > 0 \qquad (z \in U).$$

For the analytic functions f and g in U, we say that the function g is subordinate to f in U [10], and write

$$g(z) \prec f(z) \quad \text{or} \quad g \prec f\ ,$$

if there exists a Schwarz function w, which is analytic in U with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$g(z) = f(w(z)) \qquad (z \in U).$$

Al-Shaqsi and Darus [1] defined the subclasses $S^*(\mu;\phi),\ K(\mu;\phi),\ C(\mu,\eta;\phi,\psi)$ and $C^*(\mu,\eta;\phi,\psi)$ of the class A in terms of the subordination principle between certain analytic functions. These subclasses are as follows:

$$S^*(\mu;\phi) = \Bigg\{ f \in A : \frac{1}{1-\mu}\left(\frac{zf'(z)}{f(z)} - \mu\right) \prec \phi(z) \qquad (\phi \in P;\ 0 \le \mu < 1;\ z \in U) \Bigg\}, \tag{2}$$

$$K(\mu;\phi) = \Bigg\{ f \in A : \frac{1}{1-\mu}\left(\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \mu\right) \prec \phi(z) \qquad (\phi \in P;\ 0 \le \mu < 1;\ z \in U) \Bigg\}. \tag{3}$$

$$C(\mu,\eta;\phi,\psi) = \Bigg\{ f \in A : \exists\ g \in S^*(\mu;\phi) \text{ such that } \frac{1}{1-\eta}\left(\frac{zf'(z)}{g(z)} - \eta\right) \prec \psi(z) \quad (\phi,\ \psi \in P;\ 0 \le \mu,\ \eta < 1;\ z \in U) \Bigg\}. \tag{4}$$

$$C^*(\mu,\eta;\phi,\psi)=\Bigg\{f\in A:\exists\ g\in K(\mu;\phi) \text{ such that}$$
$$\frac{1}{1-\eta}\left(\left\{1+\frac{zf''(z)}{g'(z)}\right\}-\eta\right)\prec\psi(z)\,(\phi,\psi\in P;\ 0\leq\mu,\ \eta<1;\ z\in U)\Bigg\}. \tag{5}$$

The classes $S^*(\mu;\phi)$, $K(\mu;\phi)$, $C(\mu,\eta;\phi,\psi)$ and $C^*(\mu,\eta;\phi,\psi)$ consist of all analytic functions which are starlike of order μ, convex of order μ, close to convex of order η and quasi-convex of order η, respectively (see e.g. [3], [4], [7], [9], [13], [16]).

Now, we recall that the Jung-kim-Srivastava integral operator Q^μ_λ are defined as [7, p. 1788 (1.10)]:

$$\begin{aligned} Q^\mu_\lambda f(z)&=\binom{\lambda+\mu}{\lambda}\frac{\mu}{z^\lambda}\int_0^z\left(1-\frac{t}{z}\right)^{\mu-1}t^{\lambda-1}f(t)dt\\ &=z+\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda+\mu+n)}a_nz^n,\end{aligned} \tag{6}$$

where $(\lambda>-1;\ \mu>0;\ f\in A)$. The Bernardi integral operator $J_\mu f(z)$ are defined as [2]:

$$\begin{aligned} J_\mu f(z)&=\frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1}f(t)dt\\ &=z+\sum_{n=2}^{\infty}\left(\frac{\mu+1}{\mu+n}\right)a_nz^n \qquad (\ \mu>-1;\ f\in A),\end{aligned} \tag{7}$$

where $f(t)=t+\sum_{n=2}^{\infty}a_nt^n$.

Next, we recall the k-Gamma function, given by [5]:

$$\Gamma_k(z)=\int_0^\infty e^{\frac{-t^k}{k}}t^{z-1}dt\quad (Re(z)>0;\ k>0), \tag{8}$$

satisfying the following properties [5]:

$$\Gamma_k(k)=1$$

and

$$\Gamma_k(z+k)=z\Gamma_k(z). \tag{9}$$

Also, the Pochhammer k-symbol is defined as [5]:

$$(\gamma)_{n,k}=\begin{cases}\dfrac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}, & k\in\mathbb{R},\ \gamma\in\mathbb{C}/\{0\}\\ \gamma(\gamma+k)\dots(\gamma+(n-1)k), & n\in\mathbb{N},\ \gamma\in\mathbb{C}.\end{cases} \tag{10}$$

The k-Beta function is defined as [5]:

$$B_k(x,y)=\frac{1}{k}B\left(\frac{x}{k},\frac{y}{k}\right)=\frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt\qquad , \tag{11}$$

where $k>0$, $Re(x)>0$ and $Re(y)>0$ and it satisfies the following property [5]:

$$B_k(x,y)=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{12}$$

It is clear that, for $k=1$, equations (8), (10) and (11), give the Gamma function $\Gamma(z)$, Pochhammer symbol $(\gamma)_n$ and Beta function $B(x,y)$, respectively.

In the next section, we define certain k-integral operators and then, we introduce four subclasses $S^{\alpha}_{\beta,k}(\mu;\phi)$, $K^{\alpha}_{\beta,k}(\mu;\phi)$, $C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ and $QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ of the class A. These subclasses contain functions satisfying certain relations involving these newly introduced k-integral operators. Further, we deduce certain inclusion relations for the classes $S^{\alpha}_{\beta,k}(\mu;\phi)$, $K^{\alpha}_{\beta,k}(\mu;\phi)$, $C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ and $QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$.

2. Inclusion Results

First, we define the k-integral operator $M^{\alpha}_{\beta,k}:A\to A$ as:

$$M^{\alpha}_{\beta,k}f(z)=\frac{\Gamma_k(\alpha+\beta+k)}{kz^{\beta/k}\Gamma_k(\alpha)\Gamma_k(\beta+k)}\int_0^z t^{\frac{\beta}{k}-1}\left(1-\frac{t}{z}\right)^{\frac{\alpha}{k}-1}f(t)dt,$$
$$(\alpha>0;\ \beta>-1;\ k>0;\ f\in A), \tag{13}$$

which for $k=1$, $\beta=\lambda$ and $\alpha=\mu$, give the Jung-kim-Srivastava integral operator Q^{μ}_{λ}, given by equation (6).

Using equations (11) and (12), we obtain that the function $M^{\alpha}_{\beta,k}f$, defined in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$ has the following series representation:

$$M^{\alpha}_{\beta,k}f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma_k(\alpha+\beta+k)\Gamma_k(\beta+nk)}{\Gamma_k(\beta+k)\Gamma_k(\alpha+\beta+nk)}a_nz^n,$$
$$(\alpha>0;\ \beta>-1;\ k>0;\ f\in A) \tag{14}$$

which on using equations (9) and (10), gives

$$M^{\alpha}_{\beta,k}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{\alpha+\beta}{\beta}\right)\frac{(\beta)_{n,k}}{(\alpha+\beta)_{n,k}}a_nz^n,$$
$$(\alpha>0;\ \beta>-1;\ k>0;\ f\in A). \tag{15}$$

Using equations (9) and (14), we can verify that the function $M^{\alpha}_{\beta,k}f$ satisfies the following recurrence relation:

$$z\left(M^{\alpha+k}_{\beta,k}f(z)\right)'=\left(\frac{\alpha+\beta+k}{k}\right)M^{\alpha}_{\beta,k}f(z)-\left(\frac{\alpha+\beta}{k}\right)M^{\alpha+k}_{\beta,k}f(z). \tag{16}$$

Remark 2.1. *Substituting $k=1$, $\beta=\lambda$ and $\alpha=\mu$ in equation (16), we get the recurrence relation satisfied by the integral operator Q^{μ}_{λ} [7, p. 1790 (2.2)].*

Also, we define another k-integral operator $\eth_k^\sigma$ by putting $\alpha = k$ and $\beta = \sigma$ in equation (2.1), as:

$$\eth_k^\sigma f(z) = \frac{(\sigma + k)}{kz^{\frac{\sigma}{k}}} \int_0^z t^{\frac{\sigma}{k}-1} f(t)dt \quad (\sigma > -1;\ k > 0;\ f \in A;\ z \in U) \tag{17}$$

which has the following series representation:

$$\eth_k^\sigma f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\sigma + k}{\sigma + nk}\right) a_n z^n \quad (\sigma > -1;\ k > 0;\ f \in A;\ z \in U). \tag{18}$$

In view of equations (7), (17) and (18), it is clear that for $k = 1$ and $\sigma = \mu$, the k-integral operator $\eth_k^\sigma$ reduces to the Bernardi integral operator J_μ.

In view of the definitions of the subclasses $S^*(\mu;\phi)$, $K(\mu;\phi)$, $C(\mu,\eta;\phi,\psi)$ and $C^*(\mu,\eta;\phi,\psi)$ of A, deducing from the equations (2), (3), (4) and (5), respectively, we define the following classes of analytic functions satisfying certain relations involving the function $M_{\beta,k}^\alpha f$:

Definition 2.2. *The function $f \in A$ is said to be in the class $S_{\beta,k}^\alpha(\mu;\phi)$, if it satisfies the following differential subordination:*

$$\frac{1}{1-\mu}\left(\frac{z\left(M_{\beta,k}^\alpha f(z)\right)'}{M_{\beta,k}^\alpha f(z)} - \mu\right) \prec \phi(z), \tag{19}$$

where $\alpha > 0;\ \beta > -1;\ k > 0;\ \phi \in P;\ 0 \le \mu < 1;\ z \in U$.

Definition 2.3. *The function $f \in A$ is said to be in the class $K_{\beta,k}^\alpha(\mu;\phi)$, if it satisfies the following differential subordination:*

$$\frac{1}{1-\mu}\left(\left\{1 + \frac{z\left(M_{\beta,k}^\alpha f(z)\right)''}{\left(M_{\beta,k}^\alpha f(z)\right)'}\right\} - \mu\right) \prec \phi(z), \tag{20}$$

where $\alpha > 0;\ \beta > -1;\ k > 0;\ \phi \in P;\ 0 \le \mu < 1;\ z \in U$.

Definition 2.4. *The function $f \in A$ is said to be in the class $C_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$, if it satisfies the following differential subordination:*

$$\frac{1}{1-\eta}\left(\frac{z\left(M_{\beta,k}^\alpha f(z)\right)'}{M_{\beta,k}^\alpha g(z)} - \eta\right) \prec \psi(z), \tag{21}$$

where $g \in S_{\beta,k}^\alpha(\mu;\phi)$; $\alpha > 0;\ \beta > -1;\ k > 0;\ \phi,\psi \in P;\ 0 \le \mu,\eta < 1;\ z \in U$.

Definition 2.5. *The function $f \in A$ is said to be in the class $QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$, if it satisfies the following differential subordination:*

$$\frac{1}{1-\eta}\left(\left\{1 + \frac{z\left(M_{\beta,k}^\alpha f(z)\right)''}{\left(M_{\beta,k}^\alpha g(z)\right)'}\right\} - \eta\right) \prec \psi(z), \tag{22}$$

where $g \in K_{\beta,k}^\alpha(\mu;\phi)$; $\alpha > 0;\ \beta > -1;\ k > 0;\ \phi,\ \psi \in P;\ 0 \le \mu,\eta < 1;\ z \in U$.

Taking $\alpha = k$ and $\beta = \sigma$ in the Definitions 2.1-2.4, we define the classes $S_k^\sigma(\mu;\phi)$, $K_k^\sigma(\mu;\phi)$, $C_k^\sigma(\mu,\eta;\phi,\psi)$ and $QC_k^\sigma(\mu,\eta;\phi,\psi)$ of analytic functions, which satisfy certain relations involving the function $\eth_k^\sigma f(z)$, as :

$$S_k^\sigma(\mu;\phi) = \Bigg\{ f \in A : \frac{1}{1-\mu}\left(\frac{z\left(\eth_k^\sigma f(z)\right)'}{\eth_k^\sigma f(z)} - \mu\right) \prec \phi(z) \\ (\sigma > -1;\ k > 0;\ \phi \in P;\ 0 \le \mu < 1;\ z \in U) \Bigg\}. \tag{23}$$

$$K_k^\sigma(\mu;\phi) = \Bigg\{ f \in A : \frac{1}{1-\mu}\left(\left\{1 + \frac{z\left(\eth_k^\sigma f(z)\right)''}{\left(\eth_k^\sigma f(z)\right)'}\right\} - \mu\right) \prec \phi(z) \\ (\sigma > -1;\ k > 0;\ \phi \in P;\ 0 \le \mu < 1;\ z \in U) \Bigg\}. \tag{24}$$

$$C_k^\sigma(\mu,\eta;\phi,\psi) = \Bigg\{ f \in A : \exists\ g \in S_k^\sigma(\mu;\phi) \text{ such that} \\ \frac{1}{1-\eta}\left(\frac{z\left(\eth_k^\sigma f(z)\right)'}{\eth_k^\sigma g(z)} - \eta\right) \prec \psi(z) \\ (\sigma > -1;\ k > 0;\ \phi,\ \psi \in P;\ 0 \le \mu,\ \eta < 1;\ z \in U) \Bigg\}.$$

$$QC_k^\sigma(\mu,\eta;\phi,\psi) = \Bigg\{ f \in A : \exists\ g \in K_k^\sigma(\mu;\phi) \text{such that} \\ \frac{1}{1-\eta}\left(\left\{1 + \frac{z\left(\eth_k^\sigma f(z)\right)''}{\left(\eth_k^\sigma g(z)\right)'}\right\} - \eta\right) \prec \psi(z) \\ (\sigma > -1;\ k > 0;\ \phi,\ \psi \in P;\ 0 \le \mu,\ \eta < 1;\ z \in U) \Bigg\}.$$

It is clear from the Definitions 2.2-2.5 and equations (23)-(26) that the classes $S_{\beta,k}^\alpha(\mu;\phi)$, $K_{\beta,k}^\alpha(\mu;\phi)$, $C_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$, $QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$, $S_k^\sigma(\mu;\phi)$,
$K_k^\sigma(\mu;\phi)$, $C_k^\sigma(\mu,\eta;\phi,\psi)$ and $QC_k^\sigma(\mu,\eta;\phi,\psi)$ are the subclasses of the class A of analytic functions of the form (1) and defined in the open unit disc U.

Remark 2.6. *In view of the Definitions 2.2-2.5 and equations (2)-(5), we obtain the following relations:*

$$f \in S_{\beta,k}^\alpha(\mu;\phi) \ \Leftrightarrow M_{\beta,k}^\alpha f \in S^*(\mu;\phi), \tag{25}$$

$$f \in K_{\beta,k}^\alpha(\mu;\phi) \Leftrightarrow M_{\beta,k}^\alpha f \in K(\mu;\phi), \tag{26}$$

$$f \in C_{\beta,k}^\alpha(\mu,\eta;\phi,\psi) \Leftrightarrow M_{\beta,k}^\alpha f \in C(\mu,\eta;\phi,\psi) \tag{27}$$

and

$$f \in QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi) \Leftrightarrow M_{\beta,k}^\alpha f \in C^*(\mu,\eta;\phi,\psi). \tag{28}$$

Similarly, in view of equations (2)-(5) and equations (23)-(26), we obtain the following relations:

$$f \in S_k^{\sigma}(\mu;\phi) \Leftrightarrow \eth_k^{\sigma} f \in S^*(\mu;\phi), \tag{29}$$

$$f \in K_k^{\sigma}(\mu;\phi) \Leftrightarrow \eth_k^{\sigma} f \in K(\mu;\phi), \tag{30}$$

$$f \in C_k^{\sigma}(\mu,\eta;\phi,\psi) \Leftrightarrow \eth_k^{\sigma} f \in C(\mu,\eta;\phi,\psi) \tag{31}$$

and

$$f \in QC_k^{\sigma}(\mu,\eta;\phi,\psi) \Leftrightarrow \eth_k^{\sigma} f \in C^*(\mu,\eta;\phi,\psi). \tag{32}$$

Remark 2.7. *Since, it is easy to verify that* $z\left(M_{\beta,k}^{\alpha} f(z)\right)' = M_{\beta,k}^{\alpha}(zf'(z))$ *and hence* $z\left(\eth_k^{\sigma} f(z)\right)' = \eth_k^{\sigma}(zf'(z))$, *in view of Definitions 2.2-2.5 and equations (23)-(26), we deduce the following relations:*

$$f \in K_{\beta,k}^{\alpha}(\mu;\phi) \Leftrightarrow zf' \in S_{\beta,k}^{\alpha}(\mu;\phi), \tag{33}$$

$$f \in QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) \Leftrightarrow zf' \in C_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi), \tag{34}$$

$$f \in K_k^{\sigma}(\mu;\phi) \Leftrightarrow zf' \in S_k^{\sigma}(\mu;\phi) \tag{35}$$

and

$$f \in QC_k^{\sigma}(\mu,\eta;\phi,\psi) \Leftrightarrow zf' \in C_k^{\sigma}(\mu,\eta;\phi,\psi). \tag{36}$$

In particular, for the special choice of the function $\phi(z)$, we set

$$S_{\beta,k}^{\alpha}\left(\mu;\frac{1+Az}{1+Bz}\right) = S_{\beta,k}^{\alpha}(\mu;A,B) \qquad (-1 \le B < A \le 1),$$

$$K_{\beta,k}^{\alpha}\left(\mu;\frac{1+Az}{1+Bz}\right) = K_{\beta,k}^{\alpha}(\mu;A,B) \qquad (-1 \le B < A \le 1),$$

$$S_k^{\sigma}\left(\mu;\frac{1+Az}{1+Bz}\right) = S_k^{\sigma}(\mu;A,B) \qquad (-1 \le B < A \le 1)$$

and

$$K_k^{\sigma}\left(\mu;\frac{1+Az}{1+Bz}\right) = K_k^{\sigma}(\mu;A,B) \qquad (-1 \le B < A \le 1).$$

Now, we proceed to establish several inclusion properties of the subclasses $S_{\beta,k}^{\alpha}(\mu;\phi)$, $K_{\beta,k}^{\alpha}(\mu;\phi)$, $C_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi)$ and $QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi)$ associated with the function $M_{\beta,k}^{\alpha} f$, given by equation (14).

First, we need to mention the following lemma [6] to establish the inclusion property of the class $S_{\beta,k}^{\alpha}$:

Lemma 2.8. *Let* $u, v \in \mathbb{C}$ *and* $p(z)$ *be convex and univalent in* U *such that* $p(0) = 1$ *and* $Re(up(z) + v) > 0$. *If* $q(z)$ *is analytic in* U *with* $q(0) = 1$, *then the subordination*

$$q(z) + \frac{zq'(z)}{uq(z)+v} \prec p(z),$$

implies $q(z) \prec p(z) \qquad (z \in U)$.

Next, we establish the following inclusion relation for the class $S_{\beta,k}^{\alpha}(\mu;\phi)$:

Theorem 2.9. *Let* $0 \leq \mu < 1$ *and* $\phi \in P$*, then*

$$S^{\alpha}_{\beta,k}(\mu;\phi) \subset S^{\alpha+k}_{\beta,k}(\mu;\phi), \tag{37}$$

where

$$Re\,(\phi(z)) > max\left\{0, -\frac{\alpha+\beta+\mu k}{(1-\mu)k}\right\} \qquad (\alpha > 0;\ \beta > -1;\ k > 0). \tag{38}$$

Proof. Let $f \in S^{\alpha}_{\beta,k}(\mu;\phi)$, then from Definition 2.2, we have

$$\frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \mu\right) \prec \phi(z). \tag{39}$$

We assume that

$$u(z) := \frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha+k}_{\beta,k}f(z)\right)'}{M^{\alpha+k}_{\beta,k}f(z)} - \mu\right), \tag{40}$$

where u is analytic in U and $u(0) = 1$. Making use of equation (16) in the above equation, we get

$$(1-\mu)u(z) + \mu + \left(\frac{\alpha+\beta}{k}\right) = \frac{(\alpha+\beta+k)M^{\alpha}_{\beta,k}f(z)}{k\ M^{\alpha+k}_{\beta,k}f(z)}. \tag{41}$$

Taking Logarithm of equation (43), then differentiating with respect to z and multiplying the resultant equation with z, yields

$$\begin{aligned}\frac{zu'(z)}{(1-\mu)u(z) + \mu + \dfrac{\alpha+\beta}{k}} &= \frac{1}{(1-\mu)}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \mu\right) \\ &- \frac{1}{(1-\mu)}\left(\frac{z\left(M^{\alpha+k}_{\beta,k}f(z)\right)'}{M^{\alpha+k}_{\beta,k}f(z)} - \mu\right).\end{aligned} \tag{42}$$

Using equation (42) in equation (44), we get

$$\frac{zu'(z)}{(1-\mu)u(z) + \mu + \dfrac{\alpha+\beta}{k}} + u(z) = \frac{1}{(1-\mu)}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \mu\right). \tag{43}$$

From subordination (41), we have

$$\frac{zu'(z)}{(1-\mu)u(z) + \mu + \dfrac{\alpha+\beta}{k}} + u(z) \prec \phi(z). \tag{44}$$

Since $u(0) = 1$, therefore in view of Lemma 2.1, the subordination (46) implies

$$u(z) \prec \phi(z) \tag{45}$$

with the condition

$$Re\left((1-\mu)\phi(z) + \mu + \frac{\alpha+\beta}{k}\right) > 0. \tag{46}$$

Since $\alpha,\ \beta,\ \mu \in \mathbb{R}$, therefore the condition (48) is equivalent to the condition (30).

Using equation (42) in subordination (47), we get

$$\frac{1}{1-\mu}\left(\frac{z\left(M_{\beta,k}^{\alpha+k}f(z)\right)'}{M_{\beta,k}^{\alpha+k}f(z)}-\mu\right)\prec\phi(z), \tag{47}$$

which in view of the Definition 2.2, gives

$$f \in S_{\beta,k}^{\alpha+k}(\mu;\phi).$$

Hence, we establish the inclusion relation (39) subject to the condition (40).

□

In view of the Theorem 2.9, we get the following corollary by mathematical induction:

Corollary 2.10. *Let* $0 \le \mu < 1$ *and* $\phi \in P$ *such that condition (40) holds, then*

$$S_{\beta,k}^{\alpha}(\mu;\phi) \subset S_{\beta,k}^{\alpha+nk}(\mu;\phi) \qquad (n \ge 1;\ \alpha > 0;\ \beta > -1;\ k > 0).$$

Taking $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)$ $(-1 \le B < A \le 1;\ z \in U)$ in Corollary 2.10, we obtain the following corollary:

Corollary 2.11. *Let* $0 \le \mu < 1$*, then*

$$S_{\beta,k}^{\alpha}(\mu;A,B) \subset S_{\beta,k}^{\alpha+nk}(\mu;A,B)$$
$$(n \ge 1;\ -1 \le B < A \le 1;\ \alpha > 0;\ \beta > -1;\ k > 0),$$

where

$$\frac{1+Az}{1+Bz} > max\left\{0, -\frac{\alpha+\beta+\mu k}{k(1-\mu)}\right\}. \tag{48}$$

Again, taking $\alpha = k$ and $\beta = \sigma$ in Theorem 2.9 and Corollary 2.10, we obtain the following result for the k-integral operator $\eth_k^\sigma$:

Corollary 2.12. *Let* $0 \le \mu < 1$ *and* $\phi \in P$*, then*

$$S_{\sigma}^{k}(\mu;\phi) \subset S_{\sigma}^{nk}(\mu;\phi) \qquad (n > 1;\ \sigma > -1;\ k > 0),$$

where

$$Re\left(\phi(z)\right) > max\left\{0, -\frac{\sigma+(1+\mu)k}{(1-\mu)k}\right\}. \tag{49}$$

Also, taking $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)$ $(-1 \le B < A \le 1;\ z \in U)$ in Corollary 2.12, we obtain the following corollary:

Corollary 2.13. *Let* $0 \leq \mu < 1$, *then*

$$S_{\sigma}^{k}(\mu; A, B) \subset S_{\sigma}^{nk}(\mu; A, B) \;\; (n > 1;\; -1 \leq B < A \leq 1;\; \sigma > -1;\; k > 0)\,,$$

where

$$\frac{1 + Az}{1 + Bz} > max\left\{0, -\frac{\sigma + (1 + \mu)k}{(1 - \mu)k}\right\}. \tag{50}$$

Next, we establish the following inclusion relation for the subclass $K_{\beta,k}^{\alpha}$:

Theorem 2.14. *Let* $0 \leq \mu < 1$ *and* $\phi \in P$. *If the condition (40) holds, then*

$$K_{\beta,k}^{\alpha}(\mu; \phi) \subset K_{\beta,k}^{\alpha+k}(\mu; \phi) \qquad (\alpha > 0;\; \beta > -1;\; k > 0). \tag{51}$$

Proof. Making use of relation (35) and Theorem 2.9, we have

$$\begin{aligned} f \in K_{\beta,k}^{\alpha}(\mu; \phi) &\Leftrightarrow zf' \in S_{\beta,k}^{\alpha}(\mu; \phi) \\ &\Rightarrow zf' \in S_{\beta,k}^{\alpha+k}(\mu; \phi) \quad \text{with the condition (40)} \\ &\Leftrightarrow f \in K_{\beta,k}^{\alpha+k}(\mu; \phi). \end{aligned}$$

Hence, we get the inclusion property (53) .

□

In view of Theorem 2.14, we get the following corollary by mathematical induction:

Corollary 2.15. *Let* $0 \leq \mu < 1$, $\phi \in P$ *with condition (2.28) holds, then*

$$K_{\beta,k}^{\alpha}(\mu; \phi) \subset K_{\beta,k}^{\alpha+nk}(\mu; \phi) \qquad (n \geq 1;\; \alpha > 0;\; \beta > -1;\; k > 0).$$

Taking $\phi(z) = \left(\frac{1 + Az}{1 + Bz}\right)$ $(-1 \leq B < A \leq 1;\; z \in U)$ in Corollary 2.15, we obtain the following Corollary:

Corollary 2.16. *Let* $0 \leq \mu < 1$ *and the condition (40) holds, then*

$$K_{\beta,k}^{\alpha}(\mu; A, B) \subset K_{\beta,k}^{\alpha+nk}(\mu; A, B)$$
$$(n \geq 1;\; -1 \leq B < A \leq 1;\; \alpha > 0;\; \beta > -1;\; k > 0)\,.$$

Again, taking $\alpha = k$ and $\beta = \sigma$ in Corollary 2.15, we obtain the following corollary for the subclass $K_{\sigma}^{k}(\mu; \phi)$ associated with the k-integral operator $\eth_{k}^{\sigma}$:

Corollary 2.17. *Let* $0 \leq \mu < 1$, $\phi \in P$ *and the condition (51) holds, then*

$$K_{\sigma}^{k}(\mu; \phi) \subset K_{\sigma}^{nk}(\mu; \phi)$$
$$(n > 1;\; \sigma > -1;\; k > 0)\,.$$

Also, taking $\phi(z) = \left(\frac{1+Az}{1+Bz}\right)$ $(-1 \le B < A \le 1;\ z \in U)$ in Corollary 2.17, we obtain the following corollary:

Corollary 2.18. *Let* $0 \le \mu < 1$ *and the condition (52) holds, then*

$$K_{\sigma}^{k}(\mu; A, B) \subset K_{\sigma}^{nk}(\mu; A, B)$$
$$(n > 1;\ -1 \le B < A \le 1;\ \sigma > -1;\ k > 0)\,.$$

Further, we need to mention the following lemma [11] to establish the inclusion property for the subclass $C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$:

Lemma 2.19. *Let* $h(z)$ *be convex and univalent function in* U *and* $q(z)$ *be analytic in* U *with* $Re(q(z)) \ge 0$. *If* $\varphi(z)$ *is analytic in* U *with* $\varphi(0) = h(0)$, *then the subordination*

$$\varphi(z) + q(z) z \varphi'(z) \prec h(z) \qquad (z \in U),$$

implies $\varphi(z) \prec h(z)$.

Now, we establish the following inclusion relation for the subclass $C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$:

Theorem 2.20. *Let* $0 \le \mu, \eta < 1$ *and* $\phi, \psi \in P$. *If the condition (40) holds, then*

$$C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi) \subset C^{\alpha+k}_{\beta,k}(\mu, \eta; \phi, \psi) \qquad (\alpha > 0;\ \beta > -1;\ k > 0). \tag{52}$$

Proof. Let $f \in C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$, then from Definition 2.4, we have

$$\frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k} f(z)\right)'}{M^{\alpha}_{\beta,k} g(z)} - \eta\right) \prec \psi(z), \tag{53}$$

with $g \in S^{\alpha}_{\beta,k}(\mu; \phi)$. Hence, by Theorem 2.9, if condition (40) holds, then we have $g \in S^{\alpha+k}_{\beta,k}(\mu; \phi)$, which in view of Definition 2.2, gives

$$\frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha+k}_{\beta,k} g(z)\right)'}{M^{\alpha+k}_{\beta,k} g(z)} - \mu\right) \prec \phi(z), \tag{54}$$

where $0 \le \mu < 1$ and $\phi \in P$.

Let

$$p(z) = \frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha+k}_{\beta,k} g(z)\right)'}{M^{\alpha+k}_{\beta,k} g(z)} - \mu\right), \tag{55}$$

which gives

$$[p(z)(1-\mu) + \mu]\, M^{\alpha+k}_{\beta,k} g(z) = z\left(M^{\alpha+k}_{\beta,k} g(z)\right)'. \tag{56}$$

Next, we suppose that

$$w(z)=\frac{1}{1-\eta}\left(\frac{z\left(M_{\beta,k}^{\alpha+k}f(z)\right)'}{M_{\beta,k}^{\alpha+k}g(z)}-\eta\right), \tag{57}$$

or, equivalently

$$[(1-\eta)w(z)+\eta]M_{\beta,k}^{\alpha+k}g(z)=z\left(M_{\beta,k}^{\alpha+k}f(z)\right)', \tag{58}$$

where w is analytic in U with $w(0)=1$. Making use of recurrence relation (16) in the above equation, we get

$$\begin{aligned}[(1-\eta)w(z)+\eta]\,M_{\beta,k}^{\alpha+k}g(z) &= \left(\frac{\alpha+\beta+k}{k}\right)M_{\beta,k}^{\alpha}f(z)\\ &- \left(\frac{\alpha+\beta}{k}\right)M_{\beta,k}^{\alpha+k}f(z).\end{aligned} \tag{59}$$

Differentiating both the sides of equation (61) with respect to z and then multiplying the resultant equation with z, we obtain

$$\begin{aligned}&[(1-\eta)w(z)+\eta]\left(z\left(M_{\beta,k}^{\alpha+k}g(z)\right)'\right)\\ &+(1-\eta)zw'(z)M_{\beta,k}^{\alpha+k}g(z)=\left(\frac{\alpha+\beta+k}{k}\right)\left(z\left(M_{\beta,k}^{\alpha}f(z)\right)'\right)\\ &-\left(\frac{\alpha+\beta}{k}\right)\left(z\left(M_{\beta,k}^{\alpha+k}f(z)\right)'\right).\end{aligned} \tag{60}$$

Using equation (58) in the left hand side of equation (2.50), we get

$$\begin{aligned}&[(1-\eta)w(z)+\eta]\,[(1-\mu)p(z)+\mu]\,M_{\beta,k}^{\alpha+k}g(z)\\ &\qquad+(1-\eta)zw'(z)M_{\beta,k}^{\alpha+k}g(z)\\ &=\left(\frac{\alpha+\beta+k}{k}\right)\left(z\left(M_{\beta,k}^{\alpha}f(z)\right)'\right)\\ &-\left(\frac{\alpha+\beta}{k}\right)\left(z\left(M_{\beta,k}^{\alpha+k}f(z)\right)'\right).\end{aligned} \tag{61}$$

Using equation (60) in the right hand side of equation (63), we get

$$\begin{aligned}&[(1-\eta)w(z)+\eta]\,[(1-\mu)p(z)+\mu]\,M_{\beta,k}^{\alpha+k}g(z)+(1-\eta)zw'(z)M_{\beta,k}^{\alpha+k}g(z)\\ &=\left(\frac{\alpha+\beta+k}{k}\right)\left(z\left(M_{\beta,k}^{\alpha}f(z)\right)'\right)\\ &-\left(\frac{\alpha+\beta}{k}\right)[w(z)(1-\eta)+\eta]\,M_{\beta,k}^{\alpha+k}g(z).\end{aligned} \tag{62}$$

On simplifying equation (64), we get

$$\begin{aligned}&\frac{(1-\eta)zw'(z)}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}}+[w(z)(1-\eta)+\eta]\\ &=\frac{(\alpha+\beta+k)/k}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}}\left(\frac{z\left(M_{\beta,k}^{\alpha}f(z)\right)'}{M_{\beta,k}^{\alpha+k}g(z)}\right).\end{aligned} \tag{63}$$

Now, using recurrence relation (16) for the function g in equation (58), we get

$$\frac{(\alpha+\beta+k)/k}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}} = \frac{M^{\alpha+k}_{\beta,k}g(z)}{M^{\alpha}_{\beta,k}g(z)}. \tag{64}$$

Using equation (66) in the right hand side of equation (65), we get

$$\frac{(1-\eta)zw'(z)}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}} + [w(z)(1-\eta)+\eta] = \frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}g(z)} \tag{65}$$

or, equivalently

$$\frac{zw'(z)}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}} + w(z) = \frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}g(z)} - \eta\right). \tag{66}$$

Using subordination (55) in equation (68), we get

$$\frac{zw'(z)}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}} + w(z) \prec \psi(z). \tag{67}$$

Subordination (56) and equation (57), give

$$p(z) \prec \phi(z), \tag{68}$$

with the condition (40) which is equivalent to

$$Re\left((1-\mu)\phi(z)+\mu+\frac{\alpha+\beta}{k}\right) > 0.$$

Further, if we take $q(z) = \dfrac{1}{(1-\mu)p(z)+\mu+\dfrac{\alpha+\beta}{k}}$, then from subordination (60) and the above inequality $Re(q(z)) > 0$.

Thus, applying Lemma 2.19 to subordination (69), we get

$$w(z) = \frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha+k}_{\beta,k}f(z)\right)'}{M^{\alpha+k}_{\beta,k}g(z)} - \eta\right) \prec \psi(z). \tag{69}$$

Since $g \in S^{\alpha+k}_{\beta,k}(\mu;\phi)$, therefore in view of Definition 2.4, subordination (71), gives

$$f \in C^{\alpha+k}_{\beta,k}(\mu,\eta;\phi,\psi).$$

Hence, we establish the inclusion relation (54) subject to the condition (40). □

In view of Theorem 2.20, we get the following corollary by mathematical induction:

Corollary 2.21. *Let* $0 \leq \mu, \eta < 1$ *and* $\phi, \psi \in P$. *If the condition (40) holds, then*

$$C_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) \subset C_{\beta,k}^{\alpha+nk}(\mu,\eta;\phi,\psi) \qquad (n \geq 1;\ \alpha > 0;\ \beta > -1;\ k > 0).$$

Taking $\alpha = k$ and $\beta = \sigma$ in Corollary 2.21, we obtain the following corollary for the subclass $C_{\sigma}^{k}(\mu,\eta;\phi,\psi)$ associated with the k-integral operator $\eth_k^{\sigma}$:

Corollary 2.22. *Let* $0 \leq \mu < 1$ *and* $\phi, \psi \in P$. *If the condition (51) holds, then*

$$C_{\sigma}^{k}(\mu,\eta;\phi,\psi) \subset C_{\sigma}^{nk}(\mu,\eta;\phi,\psi) \qquad (n > 1;\ \sigma > -1;\ k > 0)\,.$$

Next, we establish the following inclusion relation for the subclass $QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi)$:

Theorem 2.23. *Let* $0 \leq \mu, \eta < 1$ *and* $\phi, \psi \in P$. *If the condition (40) holds, then*

$$QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) \subset QC_{\beta,k}^{\alpha+k}(\mu,\eta;\phi,\psi) \qquad (\alpha > 0;\ \beta > -1;\ k > 0). \tag{70}$$

Proof. Making use of relation (36) and Theorem 2.20, we get

$$\begin{aligned} f \in QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) &\Leftrightarrow zf' \in C_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) \\ &\Rightarrow zf' \in C_{\beta,k}^{\alpha+k}(\mu,\eta;\phi,\psi) \qquad \text{with the condition (40)} \\ &\Leftrightarrow f \in QC_{\beta,k}^{\alpha+k}(\mu,\eta;\phi,\psi). \end{aligned}$$

Hence, we establish the inclusion property (72).

□

In view of Theorem 2.23, we get the following corollary by mathematical induction:

Corollary 2.24. *Let* $0 \leq \mu, \eta < 1$ *and* $\phi, \psi \in P$. *If the condition (40) holds, then*

$$QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi) \subset QC_{\beta,k}^{\alpha+nk}(\mu,\eta;\phi,\psi) \qquad (n \geq 1;\ \alpha > 0;\ \beta > -1;\ k > 0).$$

Taking $\alpha = k$ and $\beta = \sigma$ in Corollary 2.24, we obtain the following corollary for the subclasses $QC_{\sigma}^{k}(\mu,\eta;\phi,\psi)$ associated with the k-integral operator $\eth_k^{\sigma}$:

Corollary 2.25. *Let* $0 \leq \mu, \eta < 1$ *and* $\phi,\ \psi \in P$. *If the condition (41) holds, then*

$$QC_{\sigma}^{k}(\mu,\eta;\phi,\psi) \subset QC_{\sigma}^{nk}(\mu,\eta;\phi,\psi) \qquad (n > 1;\ \sigma > -1;\ k > 0)\,.$$

In the next section, we establish the integral preserving property of the k-integral operator $\eth_k^{\alpha}$ and satisfied by the subclasses $S_{\beta,k}^{\alpha}(\mu;\phi)$, $K_{\beta,k}^{\alpha}(\mu;\phi)$,
$C_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi)$ and $QC_{\beta,k}^{\alpha}(\mu,\eta;\phi,\psi)$ of the class A.

3. Integral Preserving Properties

In this section, we obtain the conditions under which the preserving property of the k-integral operator $\eth_k^\alpha$ is satisfied by the classes $S^\alpha_{\beta,k}(\mu;\phi)$,
$K^\alpha_{\beta,k}(\mu;\phi), C^\alpha_{\beta,k}(\mu,\eta;\phi,\psi)$ and $QC^\alpha_{\beta,k}(\mu,\eta;\phi,\psi)$.

For this, we define the integral operator $J^\alpha_{\beta,\sigma,k}$ in terms of the function $M^\alpha_{\beta,k}f$, given by equation (14), as:

$$J^\alpha_{\beta,\sigma,k}f(z) := \eth_k^\sigma\left(M^\alpha_{\beta,k}f(z)\right) = \frac{(\sigma+k)}{kz^{\frac{\sigma}{k}}}\int_0^z t^{\frac{\sigma}{k}-1}M^\alpha_{\beta,k}f(t)dt \tag{71}$$

where $\alpha>0;\ \beta,\sigma>-1;\ k>0;\ f\in A$.

Remark 3.1. *Using equations (14) and (16), we get* $\eth_k^\sigma\left(M^\alpha_{\beta,k}f(z)\right) = M^\alpha_{\beta,k}\left(\eth_k^\sigma f(z)\right)$*; that is, from equation (73), we get* $J^\alpha_{\beta,\sigma,k}f(z) = M^\alpha_{\beta,k}\left(\eth_k^\sigma f(z)\right)$.

First, we establish the following integral preserving property satisfied by the class $S^\alpha_{\beta,k}(\mu;\phi)$:

Theorem 3.2. *Let* $f\in S^\alpha_{\beta,k}(\mu;\phi)$ *with* $\phi\in P$ *and*

$$Re(\phi(z)) > max\left\{0, -\frac{\sigma+\mu k}{(1-\mu)k}\right\} \qquad (\alpha>0;\ \beta,\sigma>-1;\ k>0), \tag{72}$$

then $\eth_k^\sigma f$*, defined by equation (17), belongs to the class* $S^\alpha_{\beta,k}(\mu;\phi)$.

Proof. Let $f\in S^\alpha_{\beta,k}(\mu;\phi)$, then from Definition 2.2, we have

$$\frac{1}{1-\mu}\left(\frac{z\left(M^\alpha_{\beta,k}f(z)\right)'}{M^\alpha_{\beta,k}f(z)} - \mu\right) \prec \phi(z). \tag{73}$$

We take

$$Y(z) = \frac{1}{1-\mu}\left(\frac{z\left(J^\alpha_{\beta,\sigma,k}f(z)\right)'}{J^\alpha_{\beta,\sigma,k}f(z)} - \mu\right), \tag{74}$$

where Y is analytic in U with $Y(0)=1$.

Multiplying equation (73) with $z^{\frac{\sigma}{k}}$ and then differentiating both sides of resultant equation with respect to z, we get

$$\frac{\sigma}{k} + \frac{z\left(J^\alpha_{\beta,\sigma,k}f(z)\right)'}{J^\alpha_{\beta,\sigma,k}f(z)} = \left(\frac{\sigma}{k}+1\right)\frac{M^\alpha_{\beta,k}f(z)}{J^\alpha_{\beta,\sigma,k}f(z)}. \tag{75}$$

Using equation (76) in the left hand side of equation (77), we get

$$(1-\mu)Y(z) + \mu + \frac{\sigma}{k} = \left(\frac{\sigma}{k}+1\right)\frac{M^\alpha_{\beta,k}f(z)}{J^\alpha_{\beta,\sigma,k}f(z)}. \tag{76}$$

Taking the logarithm of equation (78), then differentiating with respect to z and multiplying the resultant equation with z, yields

$$\frac{(1-\mu)zY'(z)}{(1-\mu)Y(z)+\mu+\dfrac{\sigma}{k}} = \frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}f(z)}, \tag{77}$$

or, equivalently

$$\begin{aligned}\frac{zY'(z)}{(1-\mu)Y(z)+\mu+\dfrac{\sigma}{k}} &= \frac{1}{(1-\mu)}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \mu\right)\\ &- \frac{1}{(1-\mu)}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}f(z)} - \mu\right),\end{aligned}$$

which on using equation (76), gives

$$\frac{zY'(z)}{(1-\mu)Y(z)+\mu+\dfrac{\sigma}{k}} + Y(z) = \frac{1}{(1-\mu)}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}f(z)} - \mu\right). \tag{78}$$

Again, using subordination (75) in equation (80), we get

$$\frac{zY'(z)}{(1-\mu)Y(z)+\mu+\dfrac{\sigma}{k}} + Y(z) \prec \phi(z). \tag{79}$$

In view of Lemma 2.8, subordination (81) implies

$$Y(z) \prec \phi(z), \tag{80}$$

with the condition $Re\left((1-\mu)\phi(z)+\mu+\dfrac{\sigma}{k}\right) > 0$, which is equivalent to the condition (74).

Using equation (76) in subordination (82), we get

$$\frac{1}{1-\mu}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}f(z)} - \mu\right) \prec \phi(z). \tag{81}$$

In view of Remark 3.1, subordination (83) has the following equivalent form:

$$\frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k}f(z)\right)\right)'}{M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k}f(z)\right)} - \mu\right) \prec \phi(z).$$

Therefore, in view of Definition 2.2, we get

$$\eth^{\sigma}_{k}f(z) \in S^{\alpha}_{\beta,k}(\mu;\phi).$$

□

Remark 3.3. *If $f \in S^{\alpha}_{\beta,k}(\mu;\phi)$, then in view of subordination (83) and equation (2), we get $J^{\alpha}_{\beta,\sigma,k}f(z) \in S^{*}(\mu;\phi)$, provided the condition (74) holds.*

In view of Theorem 3.2, we get the following corollary by mathematical induction:

Corollary 3.4. *If $f \in S^{\alpha}_{\beta,k}(\mu;\phi)$, and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n} f \in S^{\alpha}_{\beta,k}(\mu;\phi)$ $(n \geq 1;\ k > 0;\ 0 \leq \mu < 1;\ \sigma > -1;\ \phi \in P)$, where $(\eth^{\sigma}_{k})^{n} f := (\underbrace{\eth^{\sigma}_{k} \cdots \eth^{\sigma}_{k}}_{n\ times}) f$.*

For the choice $\phi(z) = \dfrac{1+Az}{1+Bz}$ $\quad(-1 \leq B < A \leq 1;\ z \in U)$, we have the following corollary:

Corollary 3.5. *Let $f \in S^{\alpha}_{\beta,k}(\mu;A,B)$ with*

$$\frac{1+Az}{1+Bz} > max\left\{0, -\frac{\sigma+\mu k}{(1-\mu)k}\right\}, \tag{82}$$

then $(\eth^{\sigma}_{k})^{n} f \in S^{\alpha}_{\beta,k}(\mu;A,B)$ $\quad(n \geq 1;\ 0 \leq \mu < 1,\ k > 0,\ \sigma > -1)$.

For the choices $\alpha = k$ and $\beta = \sigma$, we have the following corollary:

Corollary 3.6. *If $f \in S^{\sigma}_{k}(\mu;\phi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n} f \in S^{\sigma}_{k}(\mu;\phi)$ $\quad(n \geq 1;\ k > 0;\ 0 \leq \mu < 1;\ \sigma > -1;\ \phi \in P)$.*

Remark 3.7. *In view of relation (31), the above corollary can be restated as:*
"If $\eth^{\sigma}_{k} f \in S^{*}(\mu;\phi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n} f \in S^{*}(\mu;\phi)$ $\quad(n > 1;\ k > 0;\ 0 \leq \mu < 1;\ \sigma > -1;\ \phi \in P)$."

Now, we establish the following integral preserving property satisfied by the class $K^{\alpha}_{\beta,k}(\mu;\phi)$:

Theorem 3.8. *Let $f \in K^{\alpha}_{\beta,k}(\mu;\phi)$ with $\phi \in P$ and the condition (74) holds, then $\eth^{\sigma}_{k} f(z)$, defined by equation (17), belongs to the class $K^{\alpha}_{\beta,k}(\mu;\phi)$*
$(\alpha > 0;\ \beta, \sigma > -1,\ k > 0;\ z \in U)$.

***Proof*.** Making use of Remark 2.7, relation (35) and Theorem 3.2, we have

$$\begin{aligned} f \in K^{\alpha}_{\beta,k}(\mu;\phi) &\Leftrightarrow zf' \in S^{\alpha}_{\beta,k}(\mu;\phi) \\ &\Rightarrow \eth^{\sigma}_{k}(zf')(z) \in S^{\alpha}_{\beta,k}(\mu;\phi) \qquad \text{provided the condition (74) holds} \\ &\Leftrightarrow z\left(\eth^{\sigma}_{k} f(z)\right)' \in S^{\alpha}_{\beta,k}(\mu;\phi) \\ &\Leftrightarrow \eth^{\sigma}_{k} f(z) \in K^{\alpha}_{\beta,k}(\mu;\phi). \end{aligned}$$

□

In view of Theorem 3.8, we get the following corollary by mathematical induction:

Corollary 3.9. *If $f \in K^{\alpha}_{\beta,k}(\mu;\phi)$, and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n} f \in K^{\alpha}_{\beta,k}(\mu;\phi)$ $\quad(n \geq 1;\ k > 0;\ 0 \leq \mu < 1;\ \alpha > 0;\ \beta, \sigma > -1;\ \phi \in P)$, where $(\eth^{\sigma}_{k})^{n} f := (\underbrace{\eth^{\sigma}_{k} \cdots \eth^{\sigma}_{k}}_{n\ times}) f$.*

For the choice $\phi(z) = \dfrac{1+Az}{1+Bz} \quad (-1 \le B < A \le 1;\ z \in U)$, Corollary 3.9 gives the following corollary:

Corollary 3.10. *Let $f \in K^{\alpha}_{\beta,k}(\mu; A, B)$ and the condition (84) holds, then $(\eth^{\sigma}_{k})^{n} f \in K^{\alpha}_{\beta,k}(\mu; A, B) \quad (n \ge 1;\ k > 0;\ 0 \le \mu < 1;\ \alpha > 0;\ \beta, \sigma > -1;\ -1 \le B < A \le 1)$.*

For the choices $\alpha = k$ and $\beta = \sigma$, Corollary 3.9 give the following corollary:

Corollary 3.11. *If $f \in K^{\sigma}_{k}(\mu; \phi)$ and the condition (3.2) holds, then $(\eth^{\sigma}_{k})^{n} f \in K^{\sigma}_{k}(\mu; \phi) \quad (n \ge 1;\ \ 0 \le \mu < 1,\ k > 0,\ \sigma > -1,\ \phi \in P)$.*

Remark 3.12. *In view of the relation (32), the above corollary can be restated as:*
"If $\eth^{\sigma}_{k} f \in K(\mu; \phi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n} f \in K(\mu; \phi) \quad (n \ge 1;\ \ 0 \le \mu < 1,\ k > 0,\ \sigma > -1,\ \phi \in P)$."

Next, we establish the following integral preserving property satisfied by the class $C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$:

Theorem 3.13. *Let $f \in C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$ with $\phi, \psi \in P$ and condition (74) holds, then $\eth^{\sigma}_{k} f(z)$, defined by equation (17), belongs to the class*
$C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi) \quad (\alpha > 0;\ \beta, \sigma > -1,\ k > 0;\ z \in U)$.

Proof. Let $f \in C^{\alpha}_{\beta,k}(\mu, \eta; \phi, \psi)$, then by using Definition 2.4, we have

$$\frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k} f(z)\right)'}{M^{\alpha}_{\beta,k} g(z)} - \eta\right) \prec \psi(z), \tag{83}$$

with $g \in S^{\alpha}_{\beta,k}(\mu; \phi)$. Hence by Theorem 3.1 , we have $\eth^{\sigma}_{k} g(z) \in S^{\alpha}_{\beta,k}(\mu; \phi)$, which in view of Definition 2.2, gives

$$\frac{1}{1-\mu}\left(\frac{z\left(M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k} g(z)\right)\right)'}{M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k} g(z)\right)} - \mu\right) \prec \phi(z),$$

or, equivalently

$$\frac{1}{1-\mu}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k} g(z)\right)'}{J^{\alpha}_{\beta,\sigma,k} g(z)} - \mu\right) \prec \phi(z), \tag{84}$$

with the condition (74).

If we take

$$V(z) = \frac{1}{1-\mu}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k} g(z)\right)'}{J^{\alpha}_{\beta,\sigma,k} g(z)} - \mu\right), \tag{85}$$

then V is analytic in U with $V(0) = 1$.

Next, we take

$$N(z)=\frac{1}{1-\eta}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}-\eta\right), \tag{86}$$

where N is analytic in U with $N(0)=1$. By using equation (77) in equation (88), we get

$$\left[N(z)(1-\eta)+\eta\right]J^{\alpha}_{\beta,\sigma,k}g(z)+\left(\frac{\sigma}{k}\right)J^{\alpha}_{\beta,\sigma,k}f(z)=\left(\frac{\sigma}{k}+1\right)M^{\alpha}_{\beta,k}f(z). \tag{87}$$

By differentiating both sides of equation (89) with respect to z, then multiplying the resultant equation with $\left(\frac{z}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right)$, we obtain

$$\begin{aligned}&\left[N(z)(1-\eta)+\eta\right]\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}g(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right)+(1-\eta)zN'(z)+\\&\quad\frac{\sigma}{k}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right)=\left(\frac{\sigma}{k}+1\right)\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right).\end{aligned} \tag{88}$$

Using equations (87) and (88) in equation (90), we get

$$\begin{aligned}&\left[N(z)(1-\eta)+\eta\right]\left[V(z)(1-\mu)+\mu\right]+(1-\eta)zN'(z)+\\&\quad\frac{\sigma}{k}\left[N(z)(1-\eta)+\eta\right]=\left(\frac{\sigma}{k}+1\right)\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right).\end{aligned} \tag{89}$$

On simplifying equation (91), we obtain

$$\begin{aligned}&\frac{zN'(z)}{V(z)(1-\mu)+\mu+\frac{\sigma}{k}}+N(z)+\frac{\eta}{1-\eta}=\\&\frac{\left(\frac{\sigma}{k}+1\right)}{V(z)(1-\mu)+\mu+\frac{\sigma}{k}}\left\{\frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}\right)\right\}.\end{aligned} \tag{90}$$

Using equation (77) for the function g in equation (92), then using equation (87) in the resultant equation, we get

$$\frac{zN'(z)}{V(z)(1-\mu)+\mu+\frac{\sigma}{k}}+N(z)=\frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k}f(z)\right)'}{M^{\alpha}_{\beta,k}g(z)}-\eta\right),$$

which on again using subordination (85), gives

$$\frac{zN'(z)}{V(z)(1-\mu)+\mu+\frac{\sigma}{k}}+N(z)\prec\psi(z). \tag{91}$$

Now, in view of subordination (86) and equation (87), we have

$$V(z)\prec\phi(z), \tag{92}$$

with the condition (74), which is equivalent to the condition

$$Re\left(\phi(z)(1-\mu)+\mu+\frac{\sigma}{k})\right)>0.$$

Since, if we take $q(z)=\dfrac{1}{V(z)(1-\mu)+\mu+\dfrac{\sigma}{k}}$, then from subordination (3.22) and the above inequality $Re(q(z))>0$.

Therefore, in view of Lemma 2.19, subordination (93) implies $N(z)\prec\psi(z)$, which on using equation (88), gives

$$\frac{1}{1-\eta}\left(\frac{z\left(J^{\alpha}_{\beta,\sigma,k}f(z)\right)'}{J^{\alpha}_{\beta,\sigma,k}g(z)}-\eta\right)\prec\psi(z)$$

or, equivalently

$$\frac{1}{1-\eta}\left(\frac{z\left(M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k}f(z)\right)\right)'}{M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k}g(z)\right)}-\eta\right)\prec\psi(z), \tag{93}$$

In view of Definition 2.4, subordination (95) implies that
$\eth^{\sigma}_{k}f(z)\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$. □

In view of Theorem 3.13, we get the following corollary by mathematical induction:

Corollary 3.14. *If $f\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n}f\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ $(n\geq 1;\ \ 0\leq\mu<1;\ k>0,\ ;\ \alpha>0;\ \beta,\sigma>-1;\ \phi\in P)$, where $(\eth^{\sigma}_{k})^{n}f:=(\underbrace{\eth^{\sigma}_{k}\cdots\eth^{\sigma}_{k}}_{n\ times})f$.*

For the choices $\alpha=k$ and $\beta=\sigma$, Corollary 3.14 gives the following corollary:

Corollary 3.15. *If $f\in C^{\sigma}_{k}(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n}f\in C^{\sigma}_{k}(\mu,\eta;\phi,\psi)$ $(n\geq 1;\ \ 0\leq\mu<1,\ k>0,\ \sigma>-1,\ \phi\in P)$.*

Remark 3.16. *In view of the relation (93), the above corollary can be restated as:*
" If $\eth^{\sigma}_{k}f\in C(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth^{\sigma}_{k})^{n}$
$f\in C(\mu,\eta;\phi,\psi)$ $\quad(n\geq 1;\ \ 0\leq\mu<1,\ k>0,\ \sigma>-1,\ \phi\in P)$."

Finally, we establish the following integral preserving property satisfied by the class $QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$:

Theorem 3.17. *Let $f\in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ with $\phi,\psi\in P$ and the condition (74) holds, then $\eth^{\sigma}_{k}f(z)$ defined by equation (73) belongs to the class $QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)$ $\quad(\alpha>0;\ \beta,\sigma>-1,\ k>0;\ z\in U)$.*

Proof. Making use of Remark 2.7, relation (36) and Theorem 3.13, we have

$$\begin{aligned} f\in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) &\Leftrightarrow zf'\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)\\ &\Rightarrow \eth^{\sigma}_{k}(zf')(z)\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)\quad \text{provided the condition (74) holds}\\ &\Leftrightarrow z\left(\eth^{\sigma}_{k}f\right)'(z)\in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi)\end{aligned}$$

$$\Leftrightarrow \eth_k^\sigma f(z) \in QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi).$$

□

In view of Theorem 3.18, we get the following corollary by mathematical induction:

Corollary 3.18. *If $f \in QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth_k^\sigma)^n f \in QC_{\beta,k}^\alpha(\mu,\eta;\phi,\psi)$ $(n \geq 1;\ 0 \leq \mu < 1;\ k > 0;\ \alpha > 0;\ \beta,\sigma > -1;\ \phi \in P)$, where $(\eth_k^\sigma)^n f := (\underbrace{\eth_k^\sigma \cdots \eth_k^\sigma}_{n\ times})f$.*

For the choices $\alpha = k$ and $\beta = \sigma$, Corollary 3.19 give the following corollary:

Corollary 3.19. *If $f \in QC_k^\sigma(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth_k^\sigma)^n f \in QC_k^\sigma(\mu,\eta;\phi,\psi)$ $(n \geq 1;\ 0 \leq \mu < 1;\ k > 0;\ \sigma > -1;\ \phi \in P)$.*

Remark 3.20. *In view of the relation (36), the above corollary can be restated as:*
" If $\eth_k^\sigma f \in C^*(\mu,\eta;\phi,\psi)$ and the condition (74) holds, then $(\eth_k^\sigma)^n f \in C^*(\mu,\eta;\phi,\psi)$ $(n \geq 1;\ 0 \leq \mu < 1;\ k > 0;\ \sigma > -1;\ \phi \in P)$."

In the next section, we discuss some applications of the results, established in the previous sections.

4. Application

4.1. First, we consider the Hypergeometric function $F(a,b;c;z)$, defined as [15]:

$$F(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \qquad (a,b,c \in \mathbb{R},\ c > 0,\ z \in \mathbb{C}).$$

The function $F_1(z)$ is defined as [12]:

$$F_1(z) := zF(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} z^n \ (a,b,c \in \mathbb{R},\ c > 0,\ z \in \mathbb{C}).$$

Now, applying the integral operator $M_{\beta,k}^\alpha$, defined by equation (15), on the function $F_1(z)$, we get

$$M_{\beta,k}^\alpha F_1(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+\beta}{\beta}\right) \frac{(a)_{n-1}(b)_{n-1}(\beta)_{n,k}}{(c)_{n-1}(\alpha+\beta)_{n,k}(n-1)!} z^n$$
$$(\alpha > 0;\ \beta > -1;\ a,b,c \in \mathbb{R};\ c > 0;\ k > 0;\ z \in U).$$

For $\alpha = k$ and $\beta = \sigma$, the above equation gives $\eth_k^\sigma F_1(z)$, which is as follows:

$$\eth_k^\sigma F_1(z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}(\sigma+k)}{(c)_{n-1}(n-1)!(\sigma+nk)} z^n$$
$$(\sigma > -1;\ \ a,b,c \in \mathbb{R};\ k > 0;\ f \in A;\ z \in U).$$

1. Taking $f(z) = F_1(z)$ in Theorems 2.9, 2.14, 2.20, 2.23, Corollaries 2.10, 2.15, 2.21, 2.24 and then combining the results, we get the following assertions for the function $F_1(z)$:

If condition (40) holds, then

$$F_1 \in S^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow F_1 \in S^{\alpha+nk}_{\beta,k}(\mu;\phi).$$

$$F_1 \in K^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow F_1 \in K^{\alpha+nk}_{\beta,k}(\mu;\phi).$$

$$F_1 \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow F_1 \in C^{\alpha+nk}_{\beta,k}(\mu,\eta;\phi,\psi).$$

$$F_1 \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow F_1 \in QC^{\alpha+nk}_{\beta,k}(\mu,\eta;\phi,\psi),$$

for $n \geq 1$, respectively.

2. Taking $f(z) = F_1(z)$ in Theorems 3.2,3.8,3.13, 3.18, Corollaries 3.4, 3.9, 3.14, 3.19 and then combining the results, we get the following assertions for the function $F_1(z)$:

If condition (74) holds, then

$$F_1 \in S^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow (\eth^{\sigma}_{k})^n F_1 \in S^{\alpha}_{\beta,k}(\mu;\phi).$$

$$F_1 \in K^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow (\eth^{\sigma}_{k})^n F_1 \in K^{\alpha}_{\beta,k}(\mu;\phi).$$

$$F_1 \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow (\eth^{\sigma}_{k})^n F_1 \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi).$$

$$F_1 \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow (\eth^{\sigma}_{k})^n F_1 \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi),$$

for $n \geq 1$, respectively.

4.2. Next, we consider the function $u_{p,b,c}(z)$, defined as [14]:

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(q)_n n!} z^n \qquad (q = p + \frac{b+1}{2} > 0,\ p,b,c \in \mathbb{R},\ z \in U).$$

The function $\omega(z)$ is defined as [14]:

$$\omega(z) := z u_{p,b,c}(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(q)_{n-1}(n-1)!} z^n$$
$$(q = p + \frac{b+1}{2} > 0, p, b, c \in \mathbb{R}, z \in U).$$

Now, applying the integral operator $M^{\alpha}_{\beta,k}$, defined by equation (15), on the function $\omega(z)$, we get

$$M^{\alpha}_{\beta,k}\omega(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+\beta}{\beta}\right) \frac{(-c/4)^{n-1}(\beta)_{n,k}}{(q)_{n-1}(\alpha+\beta)_{n,k}(n-1)!} z^n$$
$$(\alpha > 0;\ \beta > -1;\ c \in \mathbb{R};\ q > 0;\ k > 0;\ f \in A).$$

For $\alpha = k$ and $\beta = \sigma$, above equation gives $\eth^{\sigma}_{k}\omega(z)$, which is as follows:

$$\eth^{\sigma}_{k}\omega(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}(\sigma+k)}{(q)_{n-1}(n-1)!(\sigma+nk)} z^n$$
$$(\sigma > -1;\ \ b, c, p, q \in \mathbb{R};\ k > 0;\ f \in A;\ z \in U).$$

1. Taking $f(z)=\omega(z)$ in Theorems 2.9, 2.14, 2.20, 2.23, Corollaries 2.10, 2.15, 2.21, 2.24 and then combining the results, we get the following assertions for the function $\omega(z)$:

If condition (40) holds, then

$$\omega \in S^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow \omega \in S^{\alpha+nk}_{\beta,k}(\mu;\phi).$$
$$\omega \in K^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow \omega \in K^{\alpha+nk}_{\beta,k}(\mu;\phi).$$
$$\omega \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow \omega \in C^{\alpha+nk}_{\beta,k}(\mu,\eta;\phi,\psi).$$
$$\omega \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow \omega \in QC^{\alpha+nk}_{\beta,k}(\mu,\eta;\phi,\psi),$$

for $n \geq 1$, respectively.

2. Taking $f(z)=\omega(z)$ in Theorems 3.2, 3.8, 3.13 and 3.18, Corollaries 3.4, 3.9, 3.14 and 3.19 and then combining the results, we get the following assertions for the function $\omega(z)$:

If condition (74) holds, then

$$\omega \in S^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow (\eth^{\sigma}_{k})^n\omega \in S^{\alpha}_{\beta,k}(\mu;\phi).$$
$$\omega \in K^{\alpha}_{\beta,k}(\mu;\phi) \Rightarrow (\eth^{\sigma}_{k})^n\omega \in K^{\alpha}_{\beta,k}(\mu;\phi).$$
$$\omega \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow (\eth^{\sigma}_{k})^n\omega \in C^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi).$$
$$\omega \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi) \Rightarrow (\eth^{\sigma}_{k})^n\omega \in QC^{\alpha}_{\beta,k}(\mu,\eta;\phi,\psi),$$

for $n \geq 1$, respectively.

In the next section, some concluding remarks are given:

5. Concluding Remarks

In this section, we obtain the integral preserving properties of the k-integral operator $M^{\alpha}_{\beta,k}$, defined by equation (13). For this, we define the integral operator $I^{\alpha}_{\beta,\sigma,k}$ in terms of the function $\eth^{\sigma}_{k}$, given by equation (17), as:

$$\begin{aligned} I^{\alpha}_{\beta,\sigma,k}f(z) &:= M^{\alpha}_{\beta,k}\left(\eth^{\sigma}_{k}f(z)\right) \\ &= \frac{\Gamma_k(\alpha+\beta+k)}{kz^{\beta/k}\Gamma_k(\alpha)\Gamma_k(\beta+k)}\int_0^z t^{\frac{\beta}{k}-1}\left(1-\frac{t}{z}\right)^{\frac{\alpha}{k}-1}\eth^{\sigma}_{k}f(t)dt \\ &\quad (\alpha>0;\ \beta,\sigma>-1;\ k>0;\ f\in A), \end{aligned} \tag{94}$$

which in view of Remark 3.1, gives $I^{\alpha}_{\beta,\sigma,k}=J^{\alpha}_{\beta,\sigma,k}$.

In view of equations (13)-(16) and equation (96), we obtain the following integral preserving properties of $\eth^{\sigma}_{k}$ by using the same steps involved in the proofs of Theorems 3.2, 3.8, 3.13 and 3.18 :

Theorem 5.1. *Let $f \in S^{\sigma}_{k}(\mu;\phi)$ with $\phi \in P$ and*

$$Re(\phi(z)) > max\left\{0, -\frac{\beta+\mu k}{(1-\mu)k}\right\} \qquad (\alpha>0;\ \beta,\sigma>-1;\ k>0), \tag{95}$$

then $M^{\alpha}_{\beta,k}f$, defined by equation (14), belongs to the class $S^{\sigma}_{k}(\mu;\phi)$ and hence $(M^{\alpha}_{\beta,k})^n f \in S^{\sigma}_{k}(\mu;\phi)$ $\quad (n \geq 1;\ \ 0 \leq \mu < 1,\ k>0,\ \sigma>-1,\ \phi\in P)$.

Theorem 5.2. *Let $f \in K_k^\sigma(\mu;\phi)$ with $\phi \in P$, satisfying the condition (97), then $M_{\beta,k}^\alpha f$, defined by equation (14) belongs to the class $K_k^\sigma(\mu;\phi)$ and hence $(M_{\beta,k}^\alpha)^n f \in K_k^\sigma(\mu;\phi)$ $\quad (n \geq 1;\ \ 0 \leq \mu < 1,\ k > 0,\ \sigma > -1,\ \phi \in P)$.*

Theorem 5.3. *Let $f \in C_k^\sigma(\mu,\eta;\phi,\psi)$ with $\phi \in P$, satisfying the condition (97), then $M_{\beta,k}^\alpha f$, defined by equation (14) belongs to the class $C_k^\sigma(\mu,\eta;\phi,\psi)$ and hence $(M_{\beta,k}^\alpha)^n f \in C_k^\sigma(\mu,\eta;\phi,\psi)$ $\quad (n \geq 1;\ \ 0 \leq \mu < 1,\ k > 0,\ \sigma > -1,\ \phi \in P)$.*

Theorem 5.4. *Let $f \in QC_k^\sigma(\mu,\eta;\phi,\psi)$ with $\phi \in P$, satisfying the condition (97), then $M_{\beta,k}^\alpha f$, defined by equation (14) belongs to the class $QC_k^\sigma(\mu,\eta;\phi,\psi)$ and hence $(M_{\beta,k}^\alpha)^n f \in QC_k^\sigma(\mu,\eta;\phi,\psi)$ $\quad (n \geq 1;\ \ 0 \leq \mu < 1,\ k > 0,\ \sigma > -1,\ \phi \in P)$.*

Since $I_{\beta,\sigma,k}^\alpha = J_{\beta,\sigma,k}^\alpha$, therefore, in view of equations (31)-(44), Theorems 5.1-5.4 give Remarks 3.7, 3.12, 3.16 and 3.21.

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