GÜFBED/*GUSTIJ* (2021) 11 (2): 417-423
DOI: 10.17714/gumusfenbil.830575 Araştırma Makalesi / *Research Article*

The eigenvalues of circulant matrices with generalized tetranacci numbers

Genelleştirilmiş tetranacci sayıları ile tanımlı circulant matrislerin özdeğerleri

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• Geliş tarihi / *Received*: 24.11.2020 • Düzeltilerek geliş tarihi / *Received in revised form*: 15.02.2021 • Kabul tarihi / *Accepted*: 28.02.2021

Abstract
Let the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ be the generalized tetranacci sequence. Define the $n \times n$ circulant matrix $\mathrm{C}(\mathcal{T})$ by $\mathrm{c}_{\mathrm{ij}} = \begin{cases} \mathcal{T}_{\mathrm{j}-i} & , \mathrm{j} \geq i \\ \mathcal{T}_{n+\mathrm{j}-i} & , \mathrm{j} < i \end{cases}$ for $i, \mathrm{j} = 1, 2, \dots, n$. In this paper, the eigenvalue of $C(\mathcal{T})$ is studied. By using this value, the determinant value of this matrix is delivered.

Keywords: Determinant, Eigenvalue, Tetranacci numbers

Öz
$(\mathcal{T}_n)_{n\in\mathbb{N}}$ *genelleştirilmiş tetranacci dizisi ve* $C(\mathcal{T})$, $n \times n$ *tipinde* $i, j = 1, 2, \dots, n$ *için*
$c_{ij} = \begin{cases} \mathcal{T}_{j-i} & , j \geq i \\ \mathcal{T}_{n+j-i} & , j < i \end{cases}$
biçimde tanımlı circulant matris olsun. Bu çalışmada, $C(\mathcal{T})$ *'nin özdeğerleri çalışılmıştır. Bu değer kullanılarak, circulant matrisin determinant değeri hesaplanmıştır.*

Anahtar kelimeler*: Determinant, Özdeğer, Tetranacci sayıları*

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1. Introduction

The calculation of the eigenvalues and eigenvectors of a system is high-level subject in mathematics and engineering, where it is mentioned in such many applications as analysis and small oscillations of vibrating systems. Also eigenvalues are often introduced in the context of matrix theory. Developments of eigenvalues were initiated by Cauchy in (Cauchy, 1829). He proved that the eigenvalues of a symmetric matrix are real. This was extended by Hermite in (Hermite, 1855) to what are now called Hermitian matrices. After these progresses, many mathematicians worked in these problems for improving theory of eigenvalue.
An $n-$step Fibonacci sequence $\left(F_k^{(n)}\right)_{k=1}^{\infty}$ is defined by letting $F_k^{(n)}=0$ for $k\leq 0$, $F_1^{(n)}=F_2^{(n)}=1$, and other terms according to the following linear recurrence relation $F_k^{(n)}=\sum_{i=1}^{n}F_{k-i}^{(n)}$ for $k>2$. Tetranacci numbers are the $n=4$ case of the Fibonacci $n-$step numbers. Firstly the tetranacci numbers which also called Quadranacci were described in (Feinberg, 1963).
Waddill generalized the tetranacci series in his work in (Waddill, 1992). Then, some new properties and results for tetranacci numbers were obtained in (Kirkpatrick, 1977; Spickerman,1982; Spickerman and Joyner, 1984; Zaveri and Patel, 2015).
There is no hesitation that circulant and $r-$circulant matrices have a wide range of applications in some differential equations, communication linear forecast, coding theory and so on. The $r-$circulant matrix $C_r=[d_{ij}]$, which is $j-i\equiv k\ (mod\ n)$, is defined as form

$$d_{ij}=\begin{cases} d_{j-i} & , j\geq i \\ r.d_{n+j-i}, & j<i \end{cases} \text{ for } i,j=1,2,\dots,n.$$

Particularly, for $r=1$, The matrix $C=[c_{ij}]$ of type nxn, is called the circulant matrix and generic element is shown as

$$c_{ij}=\begin{cases} c_{j-i} & , j\geq i \\ c_{n+j-i} & , j<i \end{cases}.$$

Circulant matrices via special numbers have widely applications in several studies for example Solak's paper (Solak, 2005; Bahsi and Solak, 2014). For instance, Kocer et al. (Kocer et al, 2007) have studied the norms of circulant matrices which terms are Horadam numbers. In (Shen and Cen, 2010), Shen and Cen have obtained the bounds for the norms of $r-$circulant matrices Bahsi in (Bahşi, 2015) has computed norms of circulant matrices with the generalized Fibonacci and Lucas numbers. In (Tuglu and Kızılateş, 2015a; Kızılateş and Tuglu, 2016; Kızılateş and Tuglu, 2018), Tuglu and Kızılateş have given some matrix norms of circulant, $r-$circulant and geometric circulant matrices with the special Fibonacci numbers. Also, Bahşi calculated the matrix norms of circulant matrices with Tribonacci sequence (Bahşi, 2015). Then, Özkoç and Ardıyok calculated the spectral and Euclidean norms of the circulant and negacyclic matrices via tetranacci sequence (Özkoç and Ardıyok, 2016). Taşçı and Acar studied Gaussian tetranacci numbers with their initial values being Gaussian integer (Tascı and Acar, 2017). Yesil Baran et al. calculated some matrix norms for the circulant matrices consisting of elements of the generalized tetranacci number sequence (Yesil Baran and Yetiş, 2019). Also Tuglu et al. obtained the norms of some special matrices with Fibonacci numbers (Tuglu and Kızılateş, 2015b). In addition to this, Kızılateş et al. showed that some properties of Harmonic Fibonacci numbers and Quadra Lucas-Jacobsthal numbers (Kızılateş,2017; Tuglu et al., 2015).
In the light of these informations, the target of this study is to present eigen values and determinants of circulant matrix which terms are generalized tetranacci sequence with the help of fourth recurrence relation. Now, we give some preliminaries about concept of circulant matrix and tetranacci sequence.

2. Preliminaries

Firstly, because of that results of this study, we need to introduce some concepts which include special sequences.
Tetranacci sequence which is shown $(M_n)_{n\in\mathbb{N}}$ is defined by the recurrence relation

$$M_n=M_{n-1}+M_{n-2}+M_{n-3}+M_{n-4}\quad (n\geq 4) \tag{1}$$

where initial conditions for $M_0=M_1=0$, $M_2=M_3=1$ .The elements of this sequence are called Tetranacci numbers (Waddill, 1992). Binet formula for this sequence is

$$M_n=\frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(Zaveri and Patel, 2015). Here $\alpha, \beta, \gamma, \delta$ are the roots of the characteristic equation of (1).
The sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is defined as the generalized tetranacci sequence with recurrence relation as

$$\mathcal{T}_n = p\mathcal{T}_{n-1} + q\mathcal{T}_{n-2} + r\mathcal{T}_{n-3} + s\mathcal{T}_{n-4} \quad (n \geq 4) \tag{2}$$

where initial conditions for $\mathcal{T}_0 = a$, $\mathcal{T}_1 = b$, $\mathcal{T}_2 = c$, $\mathcal{T}_3 = d$ and $1 - p - q - r - s \neq 0$ (Yesil Baran and Yetis,2019). Let α, β, γ and δ are the roots of characteristic equation of (2) . Binet formula for $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is obtained as

$$\mathcal{T}_n = \frac{A\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{B\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{C\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{D\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{3}$$

where

$$D = d - cp - bq - ar$$
$$C = (\gamma - \delta)[c - bp - aq] + D$$
$$B = (b - ap)[(\gamma - \beta)(\delta - \beta)] - \frac{C(\beta-\delta)+D(\gamma-\beta)}{(\gamma-\delta)}$$
$$A = \frac{a(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)+B(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} + \frac{-C(\alpha-\beta)(\alpha-\delta)(\beta-\delta)+D(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)}{(\beta-\gamma)(\beta-\delta)(\gamma-\delta)}.$$

Vieta's formula is about to the coefficients of a polynomial to sums and products of its roots.
For $a \neq 0$, Vieta's formula for the quartic

$$ax^4 + bx^3 + cx^2 + dx + e = 0 = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

gives in four variables as

$$x_1 + x_2 + x_3 + x_4 = \frac{-b}{a}$$
$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = \frac{c}{a}$$
$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = \frac{-d}{a}$$
$$x_1x_2x_3x_4 = \frac{e}{a}.$$

Lemma 2.1. Let $\mathcal{A} = C(a_0, a_1, \cdots, a_{n-1})$ be an $n \times n$ circulant matrix. Then we have

$$\lambda_j(\mathcal{A}) = \sum_{k=0}^{n-1} a_k\, w^{-jk}$$

where $w = e^{\frac{2\pi i}{n}}$, $i = \sqrt{-1}$, $j = 0,1, \cdots, n-1$ **(**Davis,1979**).**

Lemma 2.2. Let the $w = e^{\frac{2\pi i}{n}}$ satisfy the $n - th$ primitive root of unity, where $i = \sqrt{-1}$ and $a, b, c, d, g \epsilon \mathbb{C}$, following equation holds

$$\prod_{k=1}^{n}\left(a - bw^{-k} + cw^{-2k} - dw^{-3k}\right) = a^n - d^n + (2^{-n} - 2^{1-2n})b^n + 2^{1-n}\left(\frac{c-2ad}{b}\right)^n + 2^n\left(\frac{ad}{b}\right)^n$$
$$\prod_{k=1}^{n}\left(a - bw^{-k} + cw^{-2k} - dw^{-3k} + gw^{-4k}\right) = a^n + g^n + 2^{2-2n}(b^n + d^n) + 2^{1-3n}\left(\frac{4ac+b}{a}\right)^n + $$
$$+2^{2-4n}\left(\frac{b}{a}\right)^n.$$

Proof: The proof of lemma 2.2. was shown in **(**Davis,1979**).**

3. Main section

In this section, we formulate eigenvalues and determinants of circulant matrix with fourth recurrence relation. Firstly the nxn circulant matrix which terms are generalized tetranacci numbers is defined by

$$C(\mathcal{T})=\begin{bmatrix}\mathcal{T}_0 & \mathcal{T}_1 & \mathcal{T}_2 & \cdots & \mathcal{T}_{n-1}\\ \mathcal{T}_{n-1} & \mathcal{T}_0 & \mathcal{T}_1 & \cdots & \mathcal{T}_{n-2}\\ \mathcal{T}_{n-2} & \mathcal{T}_{n-1} & \mathcal{T}_0 & \cdots & \mathcal{T}_{n-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \mathcal{T}_1 & \mathcal{T}_2 & \mathcal{T}_3 & \cdots & \mathcal{T}_0\end{bmatrix}.$$

The following theorem gives us to the eigenvalues of $C(\mathcal{T})$.

Theorem 3.1. Let $C(\mathcal{T})$ be circulant matrix. The eigenvalues of matrix A are

$$\lambda_j(C(\mathcal{T}))=\frac{\begin{bmatrix}-s\left\{\mathcal{T}_{n-1}+\frac{1}{\alpha}A^*+\frac{1}{\beta}B^*+\frac{1}{\gamma}C^*+\frac{1}{\delta}D^*\right\}w^{-3j}+\{-q\mathcal{T}_n+p\mathcal{T}_{n+1}-\mathcal{T}_{n+2}+L\}w^{-2j}\\ +\{p\mathcal{T}_n-\mathcal{T}_{n+1}-[(p-\alpha)A^*+(p-\beta)B^*+(p-\gamma)C^*+(p-\delta)D^*]\}w^{-j}+\mathcal{T}_n-\{A^*+B^*+C^*+D^*\}\end{bmatrix}}{-sw^{-4j}-rw^{-3j}-qw^{-2j}+pw^{-j}+1}$$

where

$$L=A^*(\beta\gamma+\beta\delta+\gamma\delta)+B^*(\alpha\gamma+\alpha\delta+\gamma\delta)+C^*(\alpha\beta+\alpha\delta+\beta\delta)+D^*(\alpha\beta+\alpha\gamma+\beta\gamma)\ .$$

and

$$A^*=\frac{A}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\ ,B^*=\frac{B}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)},C^*=\frac{C}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)},D^*=\frac{D}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\ .$$

Proof: From Lemma 2.1. and (3), we have

$$\begin{aligned}\lambda_j(C(\mathcal{T}))&=\sum_{k=0}^{n-1}\mathcal{T}_k\,w^{-jk}\\ &=\sum_{k=0}^{n-1}\left(A^*\alpha^k+B\beta^k+C^*\gamma^k+D^*\delta^k\right)w^{-jk}\\ &=A^*\frac{\left(\alpha w^{-j}\right)^n-1}{\alpha w^{-j}-1}+B^*\frac{\left(\beta w^{-j}\right)^n-1}{\beta w^{-j}-1}+C^*\frac{\left(\gamma w^{-j}\right)^n-1}{\gamma w^{-j}-1}+D^*\frac{\left(\delta w^{-j}\right)^n-1}{\delta w^{-j}-1}.\end{aligned}$$

For the $\left(\alpha w^{-j}\right)^n=\alpha^n$, $\left(\beta w^{-j}\right)^n=\beta^n$, $\left(\gamma w^{-j}\right)^n=\gamma^n$, $\left(\delta w^{-j}\right)^n=\delta^n$, the RHS of equation equals to

$$\lambda_j(C(\mathcal{T}))=\frac{\begin{matrix}A^*(\alpha^n-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1)\\ +B^*(\beta^n-1)(\alpha w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1)\\ +C^*(\gamma^n-1)(\alpha w^{-j}-1)(\beta w^{-j}-1)(\delta w^{-j}-1)\\ +D^*(\delta^n-1)(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)\end{matrix}}{(\alpha w^{-j}-1)(\beta w^{-j}-1)(\gamma w^{-j}-1)(\delta w^{-j}-1)}\ .$$

After regulations of numerator, we obtain

$$\begin{Bmatrix}\alpha\beta\gamma\delta\{A^*\alpha^{n-1}+B^*\beta^{n-1}+C^*\gamma^{n-1}+D^*\delta^{n-1}\}\\ -\{A^*\beta\gamma\delta+B^*\alpha\gamma\delta+C^*\alpha\beta\delta+D^*\alpha\beta\gamma\}\end{Bmatrix}w^{-3j}$$

$$+\begin{Bmatrix}A^*\alpha^n(\beta\gamma+\beta\delta+\gamma\delta)+B^*\beta^n(\alpha\gamma+\alpha\delta+\gamma\delta)\\ +C^*\gamma^n(\alpha\beta+\alpha\delta+\beta\delta)+D^*\delta^n(\alpha\beta+\alpha\gamma+\beta\gamma)-L\end{Bmatrix}w^{-2j}$$

$$+\begin{Bmatrix}A^*\alpha^n(\beta+\gamma+\delta)+B^*\beta^n(\alpha+\gamma+\delta)+C^*\gamma^n(\alpha+\beta+\delta)\\ +D^*\delta^n(\alpha+\beta+\gamma)-A^*(\beta+\gamma+\delta)-B^*(\alpha+\gamma+\delta)-C^*(\alpha+\beta+\delta)-D^*(\alpha+\beta+\gamma)\end{Bmatrix}w^{-j}$$

$$+A^*\alpha^n+B^*\beta^n+C^*\gamma^n+D^*\beta+\gamma+\delta\delta^n-(A^*+B^*+C^*+D^*).$$

Using Vieta formulas and (3), if we replace

$$\alpha\beta\gamma\delta=-s,$$

$$A^*\alpha^{n-1} + B^*\beta^{n-1} + C^*\gamma^{n-1} + D^*\delta^{n-1} = \mathcal{T}_{n-1},$$
$$\beta + \gamma + \delta = p - \alpha,$$
$$\alpha + \gamma + \delta = p - \beta,$$
$$\alpha + \beta + \delta = p - \gamma,$$
$$\alpha + \beta + \gamma = p - \delta,$$

then we get

$$\begin{bmatrix} -s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}w^{-3j} + \{-q\mathcal{T}_n + p\mathcal{T}_{n+1} - \mathcal{T}_{n+2} + L\}w^{-2j} \\ +\{p\mathcal{T}_n - \mathcal{T}_{n+1} - [(p-\alpha)A^* + (p-\beta)B^* + (p-\gamma)C^* + (p-\delta)D^*]\}w^{-j} + \mathcal{T}_n - \{A^* + B^* + C^* + D^*\} \end{bmatrix}.$$

After regulations of denominator, we obtain

$$\left[\alpha\beta\gamma\delta w^{-4j} - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)w^{-3j} + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)w^{-2j} - (\alpha + \beta + \gamma + \delta)w^{-j} + 1\right].$$

Using Vieta formulas, if we change

$$\alpha\beta\gamma\delta = -s,\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = r,$$
$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -q,$$
$$\alpha + \beta + \gamma + \delta = -p,$$

we reach

$$\left[-sw^{-4j} - rw^{-3j} - qw^{-2j} - (\alpha + \beta + \gamma + \delta) + pw^{-j} + 1\right].$$

Finally we obtain the eigenvalues of matrix $C(\mathcal{T})$ are

$$\lambda_j(C(\mathcal{T})) = \frac{\begin{bmatrix} -s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}w^{-3j} + \{-q\mathcal{T}_n + p\mathcal{T}_{n+1} - \mathcal{T}_{n+2} + L\}w^{-2j} \\ +\{p\mathcal{T}_n - \mathcal{T}_{n+1} - [(p-\alpha)A^* + (p-\beta)B^* + (p-\gamma)C^* + (p-\delta)D^*]\}w^{-j} + \mathcal{T}_n - \{A^* + B^* + C^* + D^*\} \end{bmatrix}}{-sw^{-4j} - rw^{-3j} - qw^{-2j} + pw^{-j} + 1}$$

Theorem 3.2. Let $C(\mathcal{T})$ be circulant matrix. The determinant value of matrix $C(\mathcal{T})$ is

$$\det(A) = \frac{\begin{bmatrix} (\mathcal{T}_n - \{A^*+B^*+C^*+D^*\})^n + \left(-s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}^n\right) \\ -\left(2^{-n} - 2^{1-2n}\right)\{p\mathcal{T}_n - \mathcal{T}_{n+1} - [(p-\alpha)A^* + (p-\beta)B^* + (p-\gamma)C^* + (p-\delta)D^*]\}^n \\ +2^{1-n}\left(\{-q\mathcal{T}_n + p\mathcal{T}_{n+1} - \mathcal{T}_{n+2} + L\} - 2(\mathcal{T}_n - \{A^*+B^*+C^*+D^*\})s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}\right)^n \\ +2^n\left(\frac{\mathcal{T}_n - \{A^*+B^*+C^*+D^*\}s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}}{-\{p\mathcal{T}_n - \mathcal{T}_{n+1} - [(p-\alpha)A^* + (p-\beta)B^* + (p-\gamma)C^* + (p-\delta)D^*]\}}\right)^n \end{bmatrix}}{1 + (-s)^n + 2^{2-2n}((-p)^n + r^n) + 2^{1-3n}(-4q-p)^n + 2^{2-4n}(-p)^n}.$$

Proof: From Theorem 3.1., we have

$$\det(C(\mathcal{T})) = \prod_{j=0}^{n-1} \lambda_j(C(\mathcal{T}))$$
$$= \frac{\begin{bmatrix} -s\left\{\mathcal{T}_{n-1} + \frac{1}{\alpha}A^* + \frac{1}{\beta}B^* + \frac{1}{\gamma}C^* + \frac{1}{\delta}D^*\right\}w^{-3j} + \{-q\mathcal{T}_n + p\mathcal{T}_{n+1} - \mathcal{T}_{n+2} + L\}w^{-2j} \\ +\{p\mathcal{T}_n - \mathcal{T}_{n+1} - [(p-\alpha)A^* + (p-\beta)B^* + (p-\gamma)C^* + (p-\delta)D^*]\}w^{-j} + \mathcal{T}_n - \{A^*+B^*+C^*+D^*\} \end{bmatrix}}{-sw^{-4j} - rw^{-3j} - qw^{-2j} + pw^{-j} + 1}.$$

By considering Lemma 2.2., we obtain

$$\det(C(\mathcal{T}))=\frac{\left[\begin{array}{c}(\mathcal{T}_n-\{A^*+B^*+C^*+D^*\})^n+\left(-s\left\{\mathcal{T}_{n-1}+\frac{1}{\alpha}A^*+\frac{1}{\beta}B^*+\frac{1}{\gamma}C^*+\frac{1}{\delta}D^*\right\}^n\right)\\ -\left(2^{-n}-2^{1-2n}\right)\{p\mathcal{T}_n-\mathcal{T}_{n+1}-[(p-\alpha)A^*+(p-\beta)B^*+(p-\gamma)C^*+(p-\delta)D^*]\}^n\\ +2^{1-n}\left(\{-q\mathcal{T}_n+p\mathcal{T}_{n+1}-\mathcal{T}_{n+2}+L\}-2(\mathcal{T}_n-\{A^*+B^*+C^*+D^*\})s\left\{\mathcal{T}_{n-1}+\frac{1}{\alpha}A^*+\frac{1}{\beta}B^*+\frac{1}{\gamma}C^*+\frac{1}{\delta}D^*\right\}\right)^n\\ +2^n\left(\frac{\mathcal{T}_n-\{A^*+B^*+C^*+D^*\}s\left\{\mathcal{T}_{n-1}+\frac{1}{\alpha}A^*+\frac{1}{\beta}B^*+\frac{1}{\gamma}C^*+\frac{1}{\delta}D^*\right\}}{-\{p\mathcal{T}_n-\mathcal{T}_{n+1}-[(p-\alpha)A^*+(p-\beta)B^*+(p-\gamma)C^*+(p-\delta)D^*]\}}\right)^n\end{array}\right]}{1+(-s)^n+2^{2-2n}((-p)^n+r^n)+2^{1-3n}(-4q-p)^n+2^{2-4n}(-p)^n}.$$

Therefore the proof is completed.

4. Conclusion

In this paper, we investigate the eigenvalue of $C(\mathcal{T})$ which is defined by tetranacci numbers. Tetranacci series of numbers each term is added to the next term by adding the four terms before and they ara a continuing sequence of numbers. So tetranacci series of numbers is generalization of Fibonacci numbers. So all theorems and conclusions, which are found for tetranacci numbers, can be applied to Fibonacci numbers. Consequently, These theorems are generalization of the eigenvalues and determinants of the circulant matrix which is defined by Fibonacci numbers.

Acknowledgment

This work was supported by Amasya University Research Fund for financial support through Project number FMB-BAP 19-0419. The author thanks the Amasya University Scientific Research Projects Unit for financial support.

References

Bahşi, M. (2015). On the norms of circulant matrices with the generalized Fibonacci and Lucas numbers. Turkic World Mathematical Society Journal of Pure and Applied Mathematics, 6(1), 84-92.

Bahsi, M. and Solak, S. (2014). On the norms of r-circulant matrices with the hyper-Fibonacci and Lucas numbers. Journal of Mathematical Inequalities, 8(4), 693-705.

Cauchy, A.L. (1829). Sur 1'bquation a l'aide de laquelle on determine les inegalities sqculaires des mouvements des planbtes, mineralogy and petrology. Exercices de Mathématiques, 4 = Oeuvres, (2)9, 174-95.

Davis, P. J. (1979). Circulant matrices. John Wiley and Sons: New York.

Feinberg, M. (1963). Fibonacci-Tribonacci. The Fibonacci Quarterly, 1(1), 71-74.

Hermite, C. (1855). Remarque sur un theoreme de M. Cauchy. Comptes Rendus de L'Académie des Sciences., 41 = Oeuvres 1, 459- 481.

Kızılateş, C. (2017). On the quadra Lucas-Jacobsthal numbers. Karaelmas Science and Engineering Journal, 7(2), 619-621.

Kızılateş, C. and Tuglu, N. (2016). On the bounds for the spectral norms of geometric circulant matrices. Journal of Inequalities and Applications, 2016(1), 1-15.

Kızılateş, C. and Tuglu, N. (2018). On the norms of geometric and symmetric geometric circulant matrices with the Tribonacci Number. Gazi University Journal of Science, 31(2), 555-567.

Kirkpatrick, T.B. (1977). Fibonacci sequences and additive triangles of higher order and degree. Fibonacci Quarterly, 15 (4), 319–322.

Kocer, E. G., Mansour, T. and Tuglu, N. (2007). Norms of circulant and semicirculant matrices with horadam's Numbers. Ars Combinatoria, 85, 353-359.

Özkoç, A. and Ardıyok, E. (2016). Circulant and negacyclic matrices via tetranacci numbers. Honam Mathematical Journal, 38(4), 725-738. https://doi.org/10.5831/HMJ.2016.38.4.725.

Shen, S. Q. and Cen, J. M. (2010). On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers. International Journal of Contemporary Mathematical Sciences, 5(12), 569-578.

Solak, S. (2005). On the norms of circulant matrices with the Fibonacci and Lucas numbers. Applied Mathematics and Computation, 160(1), 125-132. https://doi.org/10.1016/j.amc.2003.08.126.

Spickerman, W.R. (1982). Binet's formula for the tribonacci sequence. The Fibonacci Quarterly, 20 (2), 118-120.

Spickerman, W.R. and Joyner, R.N. (1984). Binet's formula for the recursive sequence of order k. The Fibonacci Quarterly, 22 (4), 327–331.

Tascı, D. and Acar, H. (2017). Gaussian tetranacci numbers. Communications in Mathematics ans Applications, 8(3), 379-386.

Tuglu, N. and Kızılateş, C. (2015a). On the norms of circulant and r-circulant matrices with the hyperharmonic Fibonacci numbers. Journal of Inequalities and Applications, 253(2015).

Tuglu, N. and Kızılateş, C. (2015b). On the norms of some special matrices with the harmonic Fibonacci numbers. Gazi University Journal of Science, 28(3), 497-501.

Tuglu, N., Kızılateş, C. and Kesim, S. (2015). On the harmonic and hyperharmonic Fibonacci numbers. Advances in Difference Equations, 2015(1), 1-12.

Waddill, M. E. (1992). The tetranacci sequence and generalizations. The Fibonacci Quarterly, 30(1), 9-20.

Yesil Baran, F. and Yetiş, T. (2019). On The norms of circulant matrices via generalized tetranacci numbers. Bilecik Seyh Edebali University Journal of Science, 6(2), 444-454. https://doi.org/10.35193/bseufbd.662239.

Zaveri M. N. and Patel, J. K. (2015). Binet's formula for the tetranacci sequence. International Journal of Science and Research, 78-96.