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Global Existence and General Decay of Solutions for Quasilinear System with Degenerate Damping Terms

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Abstract: In this work, we investigate a quasilinear system of two viscoelastic equations with degenarete damping, dispersion and source terms under Dirichlet boundary conditions. Under suitable conditions on the relaxation function h_i $(i = 1, 2)$ and initial data, we establish global existence and general decay results. This work generalizes and improves earlier results in the literature.

Keywords: General decay, Viscoelastic equations, Degenerate damping, Quasilinear equations.

1 Introduction

In this work, we considere the following quasilinear system of two viscoelastic equations with degenerate damping, dispersion and source terms:

$$\begin{cases} |u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + \left(|u|^k + |v|^l\right) |u_t|^{j-1} u_t = f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ |v_t|^{\eta} v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + \left(|v|^\theta + |u|^\varrho\right) |v_t|^{s-1} v_t = f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain with a sufficiently smooth boundary in R^n $(n \geq 1)$, $j, s \geq 1$, $\eta > 0$, $k, l, \theta, \varrho \geq 0$; $h_i(.) : R^+ \to R^+$ $(i = 1, 2)$ are positive relaxation functions which will be specified later. $\left(|(.)|^a + |(.)|^b\right) |(.)_t|^{\tau-1} (.)_t$ and $-\Delta (.)_{tt}$ are the degenerate damping term and the dispersion term, respectively.

By taking

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|u|^{\kappa} u |v|^{\kappa+2}, \\ f_2(u,v) &= a|u+v|^{2(\kappa+1)}(u+v) + b|v|^{\kappa} v |u|^{\kappa+2}, \end{aligned}$$

in which $a > 0$, $b > 0$, and

$$1 < \kappa < +\infty \quad \text{if } n = 1, 2 \text{ and } 1 < \kappa \leq \frac{3-n}{n-2} \text{ if } n \geq 3. \tag{2}$$

It is easy to show that

$$uf_1(u,v) + vf_2(u,v) = 2(\kappa+2) F(u,v), \ \forall (u,v) \in R^2, \tag{3}$$

where

$$F(u,v) = \frac{1}{2(\kappa+2)} \left[a|u+v|^{2(\kappa+2)} + 2b|uv|^{\kappa+2} \right]. \tag{4}$$

To motivate our problem (1) can trace back to the initial boundary value problem for the single viscoelastic equation of the form

$$|u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2} u_t = |u|^{p-2} u \tag{5}$$

which was studied by Wu [1]. The author established a general uniform decay result under some appropriate assumptions on the relaxation function h and the initial data. Then in [2], the author investigated same problem and obtained general decay result for $j = 2$.

For a coupled system, He [3] looked into the following problem

$$\begin{cases} |u_t|^{\eta} u_{tt} - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_{tt} + |u_t|^{j-2} u_t = f_1(u,v), \\ |v_t|^{\eta} v_{tt} - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_{tt} + |v_t|^{s-2} v_t = f_2(u,v), \end{cases} \tag{6}$$

where $\eta > 0$, $j, s \geq 2$. The author studied general decay results and a blow-up result. Then, in [4], the author investigated same problem without damping term and established a general decay result of solutions.

The rest paper is arranged as follows: In Section 2, as preliminaries, we give necessary assumptions and lemmas that will be used later. In section 3, we prove the global existence of solution. In last section, we studied the general decay of solutions.

2 Preliminaries

In this section, we will present some assumptions, notations, and lemmas that will be used later for our main results. Throughout this paper, we denote the standart $L^2(\Omega)$ norm by $\|.\| = \|.\|_{L^2(\Omega)}$ and $L^p(\Omega)$ norm $\|.\|_p = \|.\|_{L^p(\Omega)}$.

To state and prove our result, we need some assumptions:

$(\mathbf{A1})$ Regarding $h_i : [0,\infty) \to (0,\infty)$, $(i = 1,2)$ are C^1 functions and satisfy

$$h_i(\alpha) > 0, \;\; h_i'(\alpha) \leq 0, \quad 1 - \int_0^\infty h_i(\alpha)d\alpha = l_i > 0, \; \alpha \geq 0$$

and non-increasing differentiable positive C^1 functions ς_1 and ς_2 such that

$$h_i'(t) \leq -\varsigma_i(t) h_i^{\rho_i}(t), \; t \geq 0, \; 1 \leq \rho_i < \frac{3}{2} \text{ for } i = 1,2.$$

$(\mathbf{A2})$ For the nonlinearity, we assume that

$$\begin{cases} 1 \leq j, s \;\text{ if } n = 1,2, \\ 1 \leq j, s \leq \frac{n+2}{n-2} \;\text{ if } n \geq 3. \end{cases}$$

$(\mathbf{A3})$ Assume that η satisfies

$$\begin{cases} 0 < \eta \;\text{ if } n = 1,2, \\ 0 < \eta \leq \frac{2}{n-2} \text{ if } n \geq 3. \end{cases}$$

In addition, we present some notations:

$$(h_i^s \diamond \nabla w)(t) = \int_0^t h_i^s(t-s)\left\|\nabla w(t) - \nabla w(s)\right\|^2 ds,$$

$$l = \min\{l_1, l_2\}.$$

Remark 1. *(A1) is need to guarantee the hyperbolicity of the system* (1)*. Conditions $\rho_i < \frac{3}{2}$, $(i = 1,2)$ are imposed so that $\int_0^\infty h_i(s)\,ds < \infty$, $(i = 1,2)$.*

Lemma 2. *(Sobolev-Poincare inequality) [7]. Let q be a number with $2 \leq q < \infty$ $(n = 1,2)$ or $2 \leq q \leq 2n/(n-2)$ $(n \geq 3)$, then there is a constant $C_* = C_*(\Omega,\, q)$ such that*

$$\|u\|_q \leq C_* \|\nabla u\| \textit{ for } u \in H_0^1(\Omega).$$

Lemma 3. *[8] Suppose that* (4) *holds. Then there exist $\rho > 0$ such that for the solution (u,v)*

$$\|u+v\|_{2(\kappa+2)}^{2(\kappa+2)} + 2\|uv\|_{\kappa+2}^{\kappa+2} \leq \rho(l_1\|\nabla u\|^2 + l_2\|\nabla v\|^2)^{\kappa+2}. \tag{7}$$

Now, we state the local existence theorem that can be established by combining arguments of [1]-[6].

Theorem 4. *Assume that (A1), (A2), (A3) and* (2) *hold. Let $u_0, v_0 \in H_0^1(\Omega)$ and $u_1, v_1 \in L^2(\Omega)$ are given. Then, for some $T_m > 0$, problem* (1) *has a weak solution in the following class:*

$$u, v \in C\left([0,T_m); H_0^1(\Omega)\right),$$

$$u_t, v_t \in C\left([0,T_m); L^2(\Omega)\right).$$

We define the energy function as follows

$$E(t) = \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) + \frac{1}{2}\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t) + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\right]$$
$$+\frac{1}{2}\left[(1-\int_0^t h_1(s)ds)\|\nabla u(t)\|^2 + (1-\int_0^t h_2(s)ds)\|\nabla v(t)\|^2\right] - \int_\Omega F(u,v)\,dx. \tag{8}$$

Also, we define

$$I(t) = \|\nabla u_t\|^2 + \|\nabla v_t\|^2 + (1-\int_0^t h_1(s)ds)\|\nabla u(t)\|^2 + (1-\int_0^t h_2(s)ds)\|\nabla v(t)\|^2$$
$$+(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t) - 2(\kappa+2)\int_\Omega F(u,v)\,dx \tag{9}$$

and

$$J(t) = \frac{1}{2}\left[(1-\int_0^t h_1(s)ds)\|\nabla u(t)\|^2 + (1-\int_0^t h_2(s)ds)\|\nabla v(t)\|^2\right]$$
$$+\frac{1}{2}\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t) + \frac{1}{2}(\|\nabla u_t\|^2 + \|\nabla v_t\|^2)\right]$$
$$-\int_\Omega F(u,v)\,dx. \tag{10}$$

By computation, we get

$$\begin{aligned}\frac{d}{dt}E(t) &\leq \frac{1}{2}\left[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)\right] \\ &\quad -\frac{1}{2}\left(h_1(t)\|\nabla u\|^2 + h_2(t)\|\nabla v\|^2\right) \\ &\quad -\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{j+1}\,dx - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{s+1}\,dx \\ &\leq 0. \end{aligned} \tag{11}$$

3 Global Existence

This section is devoted to prove the global existence of solution (1).

Lemma 5. *[5]. Let $(u_0, v_0) \in H_0^1(\Omega)$, $(u_1, v_1) \in L^2(\Omega)$. Suppose that $(A1)-(A3)$ hold. If*

$$I(0) > 0 \text{ and } \beta = \rho\left(\frac{2(\kappa+2)}{\kappa+1}E(0)\right)^{\kappa+1} < 1, \tag{12}$$

then

$$I(t) > 0,\ \forall t > 0.$$

Theorem 6. *Suppose that the conditions of Lemma 5 hold, then the solution (1) is bounded and global in time.*

Proof: It suffices to show that

$$\|(u,v)\|_H := \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2$$

is bounded independently of t. For this pupose, we apply (8), (10) and (11) to get

$$\begin{aligned} E(0) &\geq E(t) = J(t) + \frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right) \\ &\geq \frac{\kappa+1}{2(\kappa+2)}\left(l_1\|\nabla u(t)\|^2 + l_2\|\nabla v(t)\|^2 + \|\nabla u_t\|^2 + \|\nabla v_t\|^2\right. \\ &\quad \left. +(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)\right) \\ &\quad +\frac{1}{\eta+2}\left(\|u_t\|_{\eta+2}^{\eta+2} + \|v_t\|_{\eta+2}^{\eta+2}\right). \end{aligned} \tag{13}$$

Thus,

$$\|(u,v)\|_H \leq CE(0),$$

where positive constant C, which depends only on κ, l_1, l_2. □

4 General Decay of Solutions

This section is devoted to prove the decay of solution (1). Set

$$\Gamma(t) := ME(t) + \varepsilon\Phi(t) + F(t), \tag{14}$$

where M and ε are some positive constants to be specified later and

$$\begin{aligned}\Phi(t) &= \delta_1(t)\left[\frac{1}{\eta+1}\int_\Omega |u_t|^\eta u_t u dx + \int_\Omega \nabla u_t \nabla u dx\right] \\ &\quad + \delta_2(t)\left[\frac{1}{\eta+1}\int_\Omega |v_t|^\eta v_t v dx + \int_\Omega \nabla v_t \nabla v dx\right],\end{aligned} \tag{15}$$

$$\begin{aligned}F(t) &= \delta_1(t)\left[\int_\Omega \left(\Delta u_t - \frac{|u_t|^\eta u_t}{\eta+1}\right)\int_0^t h_1(t-s)(u(t)-u(s))dsdx\right] \\ &\quad + \delta_2(t)\left[\int_\Omega \left(\Delta v_t - \frac{|v_t|^\eta v_t}{\eta+1}\right)\int_0^t h_2(t-s)(v(t)-v(s))dsdx\right].\end{aligned} \tag{16}$$

Lemma 7. *For ε small enough while M large enough, the relation*

$$\alpha_1\Gamma(t) \le E(t) \le \alpha_2\Gamma(t), \quad \forall t \ge 0. \tag{17}$$

holds for two positive constants α_1 and α_2.

Proof: As references [9]-[5], it is easy to see that $\Gamma(t)$ and $E(t)$ are equivalent in the sense that α_1 and α_2 are positive constants, depending on ε and M. □

Lemma 8. *[1] Assume that (12) holds. Let (u, v) be the solution of problem (1). Then, for $\sigma \ge 0$, we get*

$$\begin{cases} \int_\Omega \left(\int_0^t h_1(t-s)(u(t)-u(s))ds\right)^{\sigma+2} dx \le (1-l_1)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_1(\kappa+1)}\right)^{\frac{\sigma}{2}} (h_1 \diamond \nabla u)(t) \\ \int_\Omega \left(\int_0^t h_2(t-s)(v(t)-v(s))ds\right)^{\sigma+2} dx \le (1-l_2)^{\sigma+1} c_*^{\sigma+2}\left(\frac{2(\kappa+2)E(0)}{l_2(\kappa+1)}\right)^{\frac{\sigma}{2}} (h_2 \diamond \nabla v)(t) \end{cases}. \tag{18}$$

Lemma 9. *Let $u_0, v_0 \in H_0^1(\Omega)$, $u_1, v_1 \in L^2(\Omega)$ be given and satisfying (12). Assume that $(A1)-(A3)$ hold. Then, for any t_0, the functional $\Gamma(t)$ verifies, along solution of (1),*

$$\Gamma'(t) \le -\xi_1 E(t) + \xi_2\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)\right] \tag{19}$$

for some $\xi_i > 0,\ (i = 1, 2)$.

Proof: As references [9]-[5]-[1], it is easy to obtain desired result. We omit it. □

Now, we are ready to state our stability result.

Theorem 10. *Assume that (4), $(A1)-(A3)$ hold and that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(v_0, v_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and satisfy $E(0) < E_1$ and*

$$\left(l_1\|\nabla u_0\|^2 + l_2\|\nabla v_0\|^2\right)^{\frac{1}{2}} < \alpha_*. \tag{20}$$

Then for each , there exist two positive constants K and k such that the energy of (1) satisfies

$$E(t) \le K e^{-k\int_{t_0}^t \delta(s)ds}, \quad t \ge t_0 \tag{21}$$

where $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$.

Proof: Multiplying (19) by $\delta(t)$, we have

$$\delta(t)\Gamma'(t) \leq -\xi_1\delta(t)E(t) + \xi_2\delta(t)\left[(h_1 \diamond \nabla u)(t) + (h_2 \diamond \nabla v)(t)\right].$$

Since $(A2)$ and $\delta(t) := \min\{\delta_1(t), \delta_2(t)\}$ and using the fact that $-\left[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)\right] \leq -2E'(t)$ by (11), we get

$$\begin{aligned} \delta(t)\Gamma'(t) &\leq -\xi_1\delta(t)E(t) - \xi_2\delta(t)\left[(h_1' \diamond \nabla u)(t) + (h_2' \diamond \nabla v)(t)\right] \\ &\leq -\xi_1\delta(t)E(t) - 2\xi_2 E'(t),\ \forall t \geq t_0. \end{aligned} \tag{22}$$

That is

$$G'(t) \leq -c_*\delta(t)E(t) \leq -k\delta(t)G(t),\ \forall t \geq t_0, \tag{23}$$

where $G(t) = \delta(t)\Gamma(t) + CE(t)$ is equivalent to $E(t)$ due to (17) and k is a positive constant. A simple integration of (23) leads to

$$G(t) \leq G(t_0)e^{-k\int_{t_0}^{t}\delta(s)ds},\ \forall t \geq t_0 \tag{24}$$

This completes the proof. □

5 Conclusion

As far as we know, there is not any global existence and general decay results in the literature known for quasilinear viscoelastic equations with degenerate damping terms. Our work extends the works for some quasilinear viscoelastic equations treated in the literature to the quasilinear viscoelastic equation with degenerate damping terms.

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