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Araştırma Makalesi - Research Article

Exact Solutions of the Benney–Luke Equation via (1/G')-Expansion Method

(1/G')-Açılım Yöntemi ile Benney-Luke Denkleminin Tam Çözümleri

Hülya Durur[1*], Asıf Yokuş[2]

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ABSTRACT

In this study, the $(1/G')$-expansion method was implemented to solve the Benney–Luke (BL) equation. Exact solutions of the BL equation were obtained via this method. The solutions obtained from the BL equation were in hyperbolic form. 3D, 2D and contour graphs of obtained solutions are presented. Results show that the $(1/G')$-expansion method provides an efficient and straightforward mathematical instrument for finding solutions of nonlinear evolution equations (NLEEs).

Keywords- $(1/G')$ -Expansion Method, Benney–Luke Equation, Exact Solution, Traveling Wave Solution

ÖZ

Bu çalışmada, Benney-Luke (BL) denklemini çözmek için $(1/G')$ -açılım yöntemi uygulanmıştır. BL denkleminin tam çözümleri bu yöntem ile elde edilmektedir. BL denkleminden elde edilen çözümler hiperbolik formdadır. Elde edilen çözümlerin 3 boyutlu, 2 boyutlu ve kontur grafikleri sunulmaktadır. Sonuçlar, $(1/G')$ -açılım yönteminin doğrusal olmayan evrim denklemlerinin çözümlerini bulmak için etkili ve basit bir matematiksel enstrüman olduğu gösterilmiştir.

Anahtar Kelimeler- $(1/G')$ -Açılım Yöntemi, Benney–Luke Denklemi, Tam Çözüm, Yürüyen Dalga Çözümü

[1]*Sorumlu yazar iletişim: hulyadurur@ardahan.edu.tr (http:// orcid.org/0000-0002-9297-6873)
Bilgisayar Mühendisliği Bölümü, Mühendislik Fakültesi, Ardahan Üniversitesi, Ardahan, 75000, Türkiye
[2]İletişim: asfyokus@yahoo.com (https://orcid.org/0000-0002-1460-8573)
Matematik Bölümü, Fen Fakültesi, Fırat Üniversitesi, Elazığ, 23100, Türkiye

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I. INTRODUCTION

NLEEs are usually used to describe the nonlinear phenomena of waves in plasma physics, quantum mechanics, solid-state physics, and variety branches of engineering. There are many methods for obtaining exact solutions of NLEEs have been employed successfully, such as Homotopy analysis and Homotopy-Pade methods [1], (G'/G)-expansion method [2], Variational iteration algorithm-I [3], $(G'/G,1/G)$-expansion method [4], sumudu transform method [5], $(1/G')$-expansion method [6-9] the Clarkson–Kruskal direct method [10], the auto-Bäcklund transformation method [11], decomposition method [12], homogeneous balance method [13], the first integral method [14], residual power series method [15], collocation method [16], modified Kudryashov method [17], sine-Gordon expansion method [18,19], the improved Bernoulli sub-equation function method, [20] and so on [21-23,30-43].

Consider the BL equation of the form [24],

$$u_{tt} - u_{xx} + \alpha u_{xxxx} - \beta u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \tag{1}$$

the relation between the constants in Eq. (1) is $\alpha - \beta = \sigma - 1/3$ and where α, β are positive numbers. Also, where σ is named the bond number, it is an officially valid approach to capture the effects of surface tension and gravitational force and to define bi-directional water wave propagation in the presence of surface tension [25].

The BL equation appears in a variety study, such as exact solutions of BL equation were found by using enhanced (G'/G)-expansion method [26] analytic solutions of BL equation were obtained with the help of improved (G'/G)-expansion method [27], the exact solution was obtained using homogeneous balance method for BL equation [28], the shock wave solution of the BL equation was obtained using ansatz method [29], exact solutions of the BL equation were found by using modified simple equation method [24].

In this study, we consider obtaining exact solutions for the BL equation using $(1/G')$-expansion method.

II. MATERIAL AND METHOD

A. Description of the Method

Consider a form of NLPDEs,

$$P\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots\right) = 0. \tag{2}$$

Let $u = u(x,t) = u(\xi)$, $\xi = x - vt$, $v \neq 0$, here v is the velocity of the wave and constant. We can transform it the following nODE for $u(\xi)$:

$$q(u, u', u'', \ldots) = 0. \tag{3}$$

The solution of Eq. (3) is supposed that with the form

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i \left(\frac{1}{G'}\right)^i, \tag{4}$$

where a_i, $i = (1, \ldots, n)$ are scalars, $G = G(\xi)$ ensures following second-order IODE

$$G'' + \lambda G' + \mu = 0, \tag{5}$$

here μ and λ are constants to be determined,

$$\frac{1}{G'(\xi)} = \frac{1}{-\frac{\mu}{\lambda} + B\cosh[\xi\lambda] - B\sinh[\xi\lambda]}. \tag{6}$$

The wanted derivatives of Eq. (4) were calculated and written into Eq. (3), obtaining a polynomial with $(1/G')$. Equating the coefficients of this polynomial to zero, an algebraic system of equations was created. The equation was solved via the package program and the default Eq. (3) was put in its place in the solution function. Eventually, the solutions of Eq. (2) were found.

III. SOLUTIONS OF THE BL EQUATION

The traveling wave transmutation allows us to convert Eq. (1) into an ODE for $u = u(\xi)$,

$$\left(v^2 - 1\right)u'' + \left(\alpha - \beta v^2\right)u^{iv} - 3vu'u'' = 0, \tag{7}$$

In Eq. (7), once the integration is taken according to ξ and the integration constants are equal to zero, we attain

$$\left(v^2 - 1\right)u' + \left(\alpha - \beta v^2\right)u''' - \frac{3}{2}v\left(u'\right)^2 = 0. \tag{8}$$

In Eq. (8), we find $n = 1$ from balancing term and in Eq. (4), the following situation is attain

$$u(\xi) = a_0 + a_1\left(\frac{1}{G'}\right), \quad a_1 \neq 0. \tag{9}$$

By writing Eq. (9) into Eq. (8) and equating the coefficients of Eq. (1) to zero, systems of equations can be found in the form

$$\begin{aligned}
&\left(\frac{1}{G'[\xi]}\right)^1 : -\lambda^2 a_1 + v^2\lambda^2 a_1 + \alpha\lambda^4 a_1 - v^2\beta\lambda^4 a_1 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^2 : -3\lambda\mu a_1 + 3v^2\lambda\mu a_1 + 15\alpha\lambda^3\mu a_1 - 15v^2\beta\lambda^3\mu a_1 - 3v\lambda^3 a_1^2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^3 : -2\mu^2 a_1 + 2v^2\mu^2 a_1 + 50\alpha\lambda^2\mu^2 a_1 - 50v^2\beta\lambda^2\mu^2 a_1 - 12v\lambda^2\mu a_1^2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^4 : 60\alpha\lambda\mu^3 a_1 - 60v^2\beta\lambda\mu^3 a_1 - 15v\lambda\mu^2 a_1^2 = 0,\\
&\left(\frac{1}{G'[\xi]}\right)^5 : 24\alpha\mu^4 a_1 - 24v^2\beta\mu^4 a_1 - 6v\mu^3 a_1^2 = 0.
\end{aligned} \tag{10}$$

We can offer the following solutions using the system of Eq. (10) computer technology.

Case1.

$$a_1 = -\frac{4\left(-\mu + v^2\mu\right)}{v\lambda^2}, \quad \alpha = \frac{1 - v^2 + v^2\beta\lambda^2}{\lambda^2}, \tag{11}$$

by writing the values in Eq. (11) in Eq. (9), we get hyperbolic type solution of Eq. (1)

$$u_1(x,t) = -\frac{4\left(-\mu + v^2\mu\right)}{v\lambda^2\left(-\frac{\mu}{\lambda} + B\cosh\left[\left(-tv + x\right)\lambda\right] - B\sinh\left[\left(-tv + x\right)\lambda\right]\right)} + a_0. \tag{12}$$

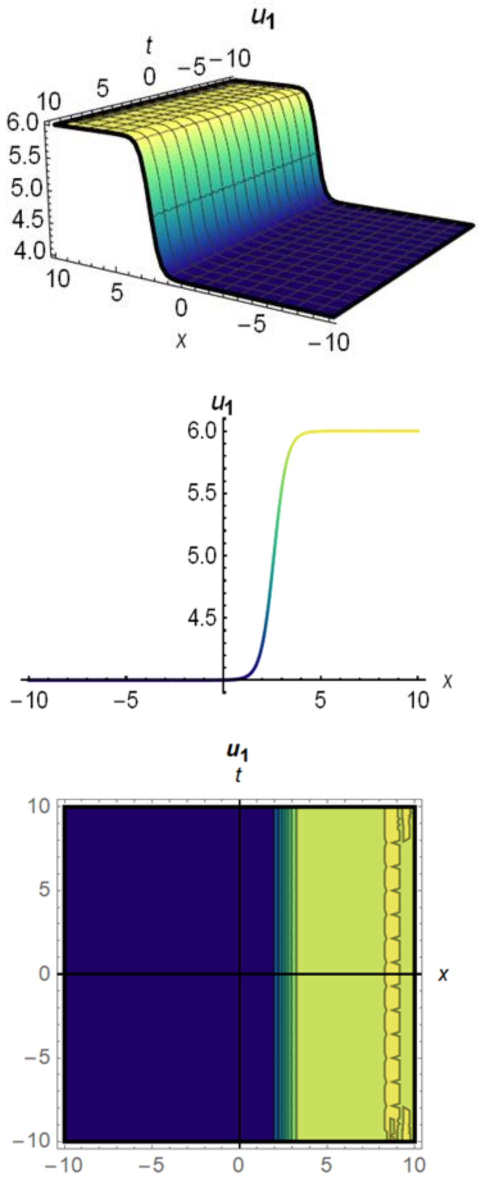

Figure 1. 3D, 2D and contour graphs for $B = 0.6,\ \mu = -0.3,\ v = 2,\ \lambda = 3,\ a_0 = 4,\ t = 1$ values of Eq. (12).

Case2.

$$a_1 = -\frac{4\left(-\alpha\mu + v^2\alpha\mu\right)}{v}, \quad \beta = \alpha, \quad \lambda = -\frac{1}{\sqrt{\alpha}}, \tag{13}$$

by writing the values in Eq. (13) in Eq. (9), we get hyperbolic type solution of Eq. (1)

$$u_2\left(x,t\right) = -\frac{4\left(-\alpha\mu + v^2\alpha\mu\right)}{v\left(\sqrt{\alpha}\,\mu + B\cosh\left[\frac{-tv+x}{\sqrt{\alpha}}\right] + B\sinh\left[\frac{-tv+x}{\sqrt{\alpha}}\right]\right)} + a_0. \tag{14}$$

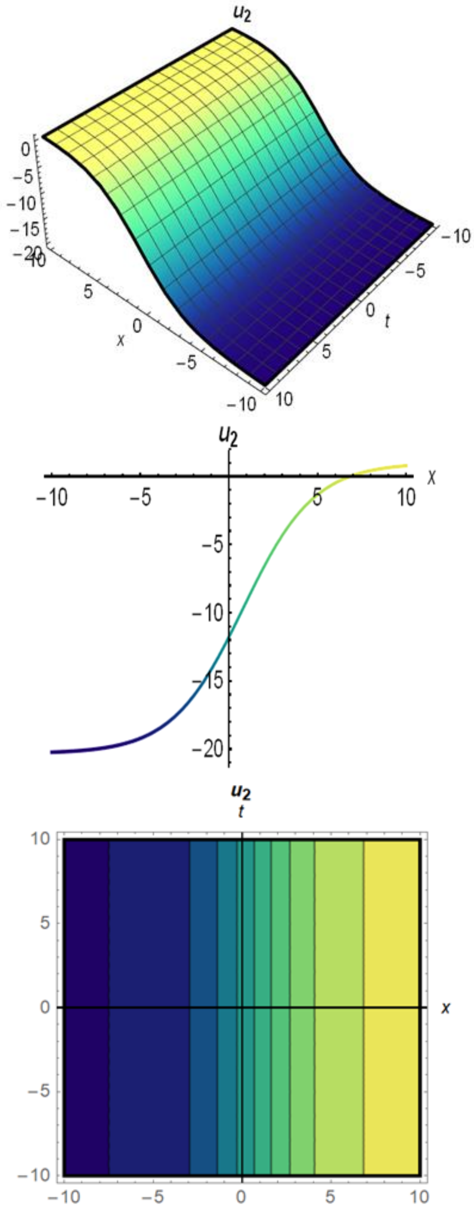

Figure 2. 3D, 2D and contour graphs for $B = 0.6, \mu = 0.1, t = 1, \lambda = 2, \alpha = 4, v = 3, a_0 = 1$ values of Eq. (14).

Case3.

$$a_1 = \frac{4}{-1+\alpha\lambda^2}\left(\frac{\alpha\sqrt{-1+\alpha\lambda^2}\,\mu}{\sqrt{-1+\beta\lambda^2}} - \frac{\beta\sqrt{-1+\alpha\lambda^2}\,\mu}{\sqrt{-1+\beta\lambda^2}}\right), \quad v = -\frac{\sqrt{-1+\alpha\lambda^2}}{\sqrt{-1+\beta\lambda^2}}, \tag{15}$$

by writing the values in Eq. (15) in Eq. (9), we get hyperbolic type solution of Eq. (1)

$$u_3(x,t) = \frac{4\left(\frac{\alpha\sqrt{-1+\alpha\lambda^2}\,\mu}{\sqrt{-1+\beta\lambda^2}} - \frac{\beta\sqrt{-1+\alpha\lambda^2}\,\mu}{\sqrt{-1+\beta\lambda^2}}\right)}{\left(-1+\alpha\lambda^2\right)\left(-\frac{\mu}{\lambda} + B\cosh\left[\lambda\left(x+\frac{t\sqrt{-1+\alpha\lambda^2}}{\sqrt{-1+\beta\lambda^2}}\right)\right] - B\sinh\left[\lambda\left(x+\frac{t\sqrt{-1+\alpha\lambda^2}}{\sqrt{-1+\beta\lambda^2}}\right)\right]\right)} + a_0. \tag{16}$$

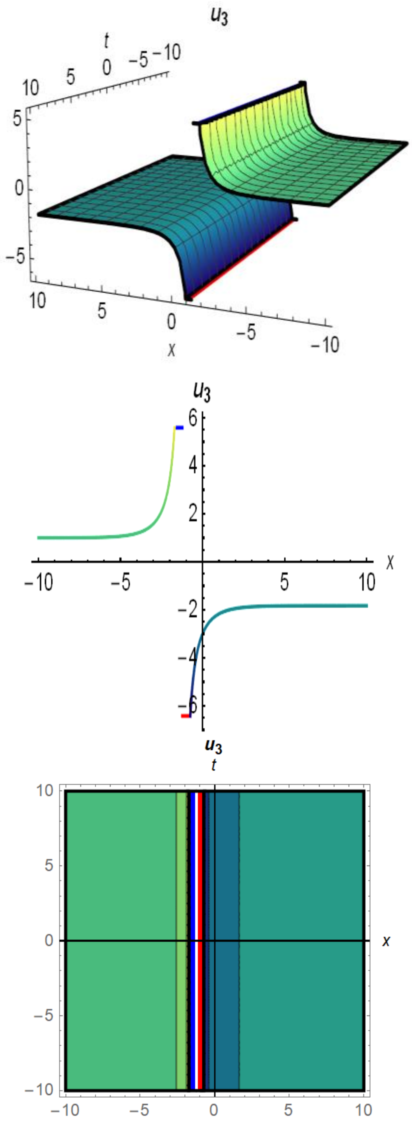

Figure 3. 3D, 2D and contour graphs for $B = 0.6,\ \mu = 0.5,\ t = 1,\ \lambda = 1,\ \alpha = 3,\ v = 2,\ a_0 = 1,\ \beta = 2$ values of Eq. (16).

The graphs presented in Figures 1-2-3 are hyperbolic type traveling wave solution and represent the standing wave at any time. It has been observed that the Eqs. (12-14-16) used in drawing these figures provide the BL equation.

IV. CONCLUSIONS

In this article, we achieved hyperbolic type exact solutions for the BL equation with the aim of $(1/G')$-expansion method. Literature review of some methods used to obtain analytical solutions for NPDEs in mathematics was conducted. Also, studies for certain equations in the literature were mentioned. In this study, the methodology of the $(1/G')$-expansion method, which we discussed in this study, was presented with its main lines. Then unlike the solutions presented in the literature, the solutions in Eq. (6) format were successfully obtained with this method. 3D, 2D and contour graphs of the solutions attained were drawn. A computer package program was utilized in the construction of these solutions. The BL equation, which plays a significant role in mathematical physics, was tested by the effectiveness and reliability of the method.

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