Adıyaman University Journal of Science
https://dergipark.org.tr/en/pub/adyujsci
DergiPark AKADEMİK
ISSN 2147-1630
e-ISSN 2146-586X

A New Approach to Fibonacci Tessarines with Fibonacci and Lucas Number Components

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Received: 11.01.2021 **Accepted:** 10.09.2021 **Available online:** 31.12.2021

Abstract

In this paper, by using identities related to the tessarines, Fibonacci numbers and Lucas numbers we define Fibonacci tessarines and Lucas tessarines. We obtain Binet formulae, D'ocagnes identity and Cassini identity for these tessarines. We also give the identities of Fibonacci negatessarines and Lucas negatessarines and define new vector which are called Fibonacci tessarine vector.

Keywords: Tessarines; Fibonacci numbers; Fibonacci tessarine vector.

Fibonacci ve Lucas Sayı Bileşenleri ile Fibonacci Tessarinelere Yeni Bir Yaklaşım

Öz

Bu makalede, tessarineler, Fibonacci ve Lucas sayılarıyla ilgili özdeşlikleri kullanarak Fibonacci tessarineler ve Lucas tessarineleri tanımladık. Bu tessarineler için Binet formüllerini, D'ocagnes özdeşliğini ve Cassini özdeşliğini elde ettik. Ayrıca, negatif Fibonacci tessarineler ve negatif Lucas tessarinelerin özdeşliklerini verdik ve Fibonacci tessarine vektörü olarak yeni bir vektör tanımladık.

* Corresponding Author DOI: 10.37094/adyujsci.852037

Anahtar Kelimeler: Tessarineler; Fibonacci sayıları; Fibonacci tessarine vektör.

1. Introduction

A tessarine can be defined as follows [1-3];

$$\mathcal{T} = t_1 + t_2\, i + t_3\, j + t_4\, k,$$

where t_{1-4} are real numbers and $+1, i, j, k$ are governed by relations

$$i^2 = -1,\ j^2 = +1,\ ij = ji = k,\ jk = i,\ ki = -j\,.$$

Let $\mathcal{T}$ and $\mathcal{T}'$ be tessarines. The addition, subtraction and multiplication of these numbers are presented as follows

$$\mathcal{T} \mp \mathcal{T}' = (\,t_1 \mp t_1') + (\,t_2 \mp t_2')i + (t_3 \mp t_3')\,j + (\,t_4 \mp t_4')k$$

and

$$\begin{aligned}\mathcal{T} \times \mathcal{T}' &= (t_1 + t_2\, i + t_3\, j + t_4\, k) \times (t_1{}' + t_2{}'i + t_3{}'j + t_4{}'k)\\ &= (t_1t_1' - t_2\, t_2' + t_3t_3' - t_4t_4'\,) + (\,t_1t_2{}' + t_2t_1{}' + t_3\, t_4{}' + t_4t_3{}'\,)i\\ &\quad +(\,t_1t_3{}' + t_3t_1{}' - t_2\, t_4{}' - t_4t_2{}')j + (\,t_1\, t_4{}' + t_4t_1{}' + t_3t_2{}' + t_2t_3{}')k\,.\end{aligned}$$

The conjugates of a tessarine are described by $\mathcal{T}^i, \mathcal{T}^j$ and $\mathcal{T}^k$. In that case, there are different conjugations as follows:

$$\mathcal{T}^i = t_1 - t_2\, i + t_3\, j - t_4\, k,$$

$$\mathcal{T}^j = t_1 + t_2\, i - t_3\, j - t_4\, k,$$

$$\mathcal{T}^k = t_1 - t_2\, i - t_3\, j + t_4\, k.$$

Then, the following equalities are written

$$\mathcal{T} \times \mathcal{T}^i = t_1^2 + t_2^2 + t_3^2 + t_4^2 + 2j(t_1t_3 + t_2t_4),$$

$$\mathcal{T} \times \mathcal{T}^j = t_1^2 - t_2^2 - t_3^2 + t_4^2 + 2i(t_1t_2 - t_3t_4),$$

$$\mathcal{T} \times \mathcal{T}^k = t_1^2 + t_2^2 - t_3^2 - t_4^2 + 2k(t_1t_4 - t_2t_3).$$

Fibonacci numbers and Lucas numbers are defined in many studies. The relations between these numbers are given and computed in [4-8]. Horadam defined the generalized Fibonacci sequences [9].

In the present paper, we introduce and study the Fibonacci tessarines and Lucas tessarines by using some properties of Fibonacci and Lucas numbers, and we obtain some identities for them. The Fibonacci numbers f_n are defined for all integers *n* by the second order recurrence relation

$$f_{n+2} = f_{n+1} + f_n$$

and initial conditions $f_1 = f_2 = 1$. The Lucas numbers l_n are defined for all integers *n* by the same second order recurrence relation as

$$l_{n+2} = l_{n+1} + l_n,$$

but initial conditions $l_1 = 1,\ l_2 = 3$. The Binet formulae for the Fibonacci and Lucas numbers are as follows [10]

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } l_n = \alpha^n + \beta^n,$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

These facts are well-known and can be found in most basic references, e.g. [11, 12]. In this paper, we need some of them as follows,

$$f_{n+2}f_{n-1} = f_{n+1}^2 - f_n^2\ , \tag{1}$$

$$f_{n-1}f_{n+1} - f_n^2 = (-1)^n, \tag{2}$$

$$f_{2n+1} = f_n^2 + f_{n+1}^2, \tag{3}$$

$$l_n = f_{n-1} + f_{n+1}, \tag{4}$$

$$l_n = f_{n+2} - f_{n-2}, \tag{5}$$

$$l_{-n} = (-1)^n l_n, \tag{6}$$

$$5f_n f_m = l_{n+m} - (-1)^m l_{n-m}, \tag{7}$$

$$f_m f_{n+1} - f_{m+1} f_n = (-1)^n f_{m-n}, \tag{8}$$

$$f_{-n} = (-1)^{n+1} f_n, \tag{9}$$

$$l_{n-1} + l_{n+1} = 5f_n. \tag{10}$$

2. Fibonacci Tessarines and Lucas Tessarines

Definition 1. The n^{th} Fibonacci tessarine $\mathcal{T}_{\mathrm{n}}$ and the n^{th} Lucas tessarine $\mathcal{T}_{\mathrm{n}}'$ are defined by, respectively,

$$\mathcal{T}_{\mathrm{n}} = f_n + f_{n+1}i + f_{n+2}\, j + f_{n+3}k \tag{11}$$

and

$$\mathcal{T}_{\mathrm{n}}' = l_n + l_{n+1}i + l_{n+2}\, j + l_{n+3}k, \tag{12}$$

where f_n is the n^{th} Fibonacci number and l_n is the n^{th} Lucas number. Also i, j and k are arbitrary units which satisfy the relations;

$$i^2 = -1,\ j^2 = +1, k^2 = -1 \ \text{ and } \ ij = ji = k\,. \tag{13}$$

Starting from $n = 0$, the Fibonacci tessarines and Lucas tessarines can be written respectively as;

$$\mathcal{T}_0 = 1j + 2k;\ \mathcal{T}_1 = 1 + 1i + 2j + 3k;\ \mathcal{T}_2 = 1 + 2i + 3j + 5k, \ldots$$

$$\mathcal{T}_0' = 2 + 1i + 3j + 4k;\ \ \mathcal{T}_1' = 1 + 3i + 4j + 7k;\ \mathcal{T}_2' = 3 + 4i + 7j + 11k, \ldots$$

Now, let $\mathcal{T}_{\mathrm{n}} = f_n + f_{n+1}i + f_{n+2}\, j + f_{n+3}k$ and $\mathcal{T}_{\mathrm{m}} = f_m + f_{m+1}i + f_{m+2}\, j + f_{m+3}k$ be Fibonacci tessarines. Then we have

$$\mathcal{T}_{\mathrm{n}} \mp \mathcal{T}_{\mathrm{m}} = (f_n + f_m) + (f_{n+1} + f_{m+1})i + (f_{n+2} + f_{m+2})j + (f_{n+3} + f_{m+3})k$$

and

$$\begin{aligned}\mathcal{T}_{\mathrm{n}} \times \mathcal{T}_{\mathrm{m}} = {} & (f_n f_m - f_{n+1} f_{m+1} + f_{n+2} f_{m+2} - f_{n+3} f_{m+3}) \\ & +(f_n f_{m+1} + f_{n+1} f_m + f_{n+2} f_{m+3} + f_{n+3} f_{m+2})i \\ & +(f_n f_{m+2} + f_{n+2} f_m - f_{n+1} f_{m+3} - f_{n+3} f_{m+1})j \\ & +(f_n f_{m+3} + f_{n+3} f_m + f_{n+2} f_{m+1} + f_{n+1} f_{m+2})k.\end{aligned}$$

Definition 2. Let $\mathcal{T}_{\mathrm{n}}$ be a Fibonacci tessarine. For $n \geq 1$, there are three different conjugations with respect to i, j and k;

$$\mathcal{T}_n^i = f_n - f_{n+1}i + f_{n+2}\, j - f_{n+3}k,$$

$$\mathcal{T}_n^j = f_n + f_{n+1}i - f_{n+2}\, j - f_{n+3}k,$$

$$\mathcal{T}_n^k = f_n - f_{n+1}i - f_{n+2}\, j + f_{n+3}k.$$

Now by using definition 2 and the Eqn. (11), we can obtain

$$\mathcal{T}_n \times \mathcal{T}_n^i = (3+2j)f_{2n+3},$$

$$\mathcal{T}_n \times \mathcal{T}_n^j = (1-2i)f_{2n+3},$$

$$\mathcal{T}_n \times \mathcal{T}_n^k = -l_{2n+3} + (-1)^{n+1}k.$$

3. Some Identities on Fibonacci Tessarines and Lucas Tessarines

3.1. Identities

Let $n \geq 1$ be an integer. Then, we can give the following relations between Fibonacci tessarines

$$\mathcal{T}_n - \mathcal{T}_{n+1}i + \mathcal{T}_{n+2}\,j - \mathcal{T}_{n+3}k = (3+2j)l_{n+3},$$

$$\mathcal{T}_n + \mathcal{T}_{n+1}i - \mathcal{T}_{n+2}\,j - \mathcal{T}_{n+3}k = (1-2i)l_{n+3},$$

$$\mathcal{T}_n - \mathcal{T}_{n+1}i - \mathcal{T}_{n+2}\,j \mp \mathcal{T}_{n+3}k = -5f_{n+3}.$$

Proof. Now, we will give proof of identity $\mathcal{T}_n - \mathcal{T}_{n+1}i + \mathcal{T}_{n+2}j - \mathcal{T}_{n+3}k$. We have

$$\begin{aligned}\mathcal{T}_n - \mathcal{T}_{n+1}i + \mathcal{T}_{n+2}j - \mathcal{T}_{n+3}k &= (f_n + f_{n+1}i + f_{n+2}\,j + f_{n+3}k)\\ &\quad -(f_{n+1} + f_{n+2}i + f_{n+3}\,j + f_{n+4}k)i\\ &\quad +(f_{n+2} + f_{n+3}i + f_{n+4}\,j + f_{n+5}k)j\\ &\quad -(f_{n+} + f_{n+4}i + f_{n+5}\,j + f_{n+6}k)k\\ &= (f_n + f_{n+2}) + (f_{n+4} + f_{n+6}) + 2j(f_{n+2} + f_{n+4}).\end{aligned}$$

Here by using the Eqn. (4) and doing necessary calculations, we obtain

$$\begin{aligned}\mathcal{T}_n - \mathcal{T}_{n+1}i + \mathcal{T}_{n+2}j - \mathcal{T}_{n+3}k &= 2l_{n+3} + l_{n+3} + 2jl_{n+3} = 3l_{n+3} + 2jl_{n+3}\\ &= (3+2j)l_{n+3}.\end{aligned}$$

For all n, we compute the following identities with similar method.

$$\mathcal{T}_n + \mathcal{T}_{n+1}i - \mathcal{T}_{n+2}\,j - \mathcal{T}_{n+3}k = (1-2i)l_{n+3},$$

$$\mathcal{T}_n - \mathcal{T}_{n+1}i - \mathcal{T}_{n+2}\,j + \mathcal{T}_{n+3}k = -5f_{n+3}.$$

3.2. Identity

Let $\mathcal{T}_n^{\mathrm{k}}$ be the conjugation of Fibonacci tessarine with respect to the imaginary unit k. Then, for $n \geq 1$, we have

$$\mathcal{T}_n\mathcal{T}_n^{\mathrm{k}} + \mathcal{T}_{n-1}\mathcal{T}_{n-1}^{\mathrm{k}} = -5f_{2n+2}.$$

Proof. From the Eqn. (2) and Eqn. (3), we have

$$\begin{aligned}\mathcal{T}_n\mathcal{T}_n^{k} + \mathcal{T}_{n-1}\mathcal{T}_{n-1}^{k} &= f_n^2 + f_{n+1}^2 - \left(f_{n+2}^2 + f_{n+3}^2\right) + 2k(f_{n+3}f_n - f_{n+1}f_{n+2}) \\ &\quad + f_{n-1}^2 + f_n^2 - (f_{n+1}^2 + f_{n+2}^2) + 2k(f_{n+2}f_{n-1} - f_nf_{n+1}) \\ &= f_{2n+1} - f_{2n+5} + 2k\big((f_{n+2} + f_{n+1})f_n - f_{n+1}(f_{n+1} + f_n)\big) \\ &\quad + f_{2n-1} - f_{2n+3} + 2k\big((f_{n+1} + f_n)f_{n-1} - f_n(f_n + f_{n-1})\big) \\ &= -(f_{2n+5} - f_{2n+1}) + 2k(f_{n+2}f_n - f_{n+1}^2) - (f_{2n+3} - f_{2n-1}) \\ &\quad + 2k(f_{n+1}f_{n-1} - f_n^2) \\ &= -5f_{2n+2}.\end{aligned}$$

3.3. Identities (Binet Formulae)

Let $\mathcal{T}_n$ and $\mathcal{T}_n'$ be Fibonacci tessarine and Lucas tessarine, respectively. For $n \geq 1$, the Binet formulae for these numbers are given as follows;

$$\mathcal{T}_n = \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

and

$$\mathcal{T}_n' = \bar{\alpha}\alpha^{\mathrm{n}} + \bar{\beta}\beta\beta^{\mathrm{n}},$$

where

$$\bar{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3 \text{ and } \bar{\beta} = 1 + i\beta + j\beta^2 + k\beta^3.$$

Proof. By taking $\bar{\alpha} = 1 + i\alpha + j\alpha^2 + k\alpha^3$ and $\bar{\beta} = 1 + i\beta + j\beta^2 + k\beta^3$ and using the Binet formulae for Fibonacci and Lucas numbers, we obtain

$$\mathcal{T}_n = f_n + f_{n+1}i + f_{n+2}j + f_{n+3}k$$

$$= \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta} i + \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta} j + \frac{\alpha^{n+3} - \beta^{n+3}}{\alpha - \beta} k$$

$$= \frac{(\alpha^n(1 + i\alpha + j\alpha^2 + k\alpha^3) - \beta^n(1 + i\beta + j\beta^2 + k\beta^3)}{\alpha - \beta}$$

$$= \frac{\bar{\alpha}\alpha^n - \bar{\beta}\beta^n}{\alpha - \beta}$$

and

$$\mathcal{T}'_n = l_n + l_{n+1}i + l_{n+2}\, j + l_{n+3}k$$

$$= \alpha^n + \beta^n + (\alpha^{n+1} + \beta^{n+1})i + (\alpha^{n+2} + \beta^{n+2})j + (\alpha^{n+3} + \beta^{n+3})k$$

$$= \alpha(1 + i\alpha + j\alpha^2 + k\alpha^3) + \beta(1 + i\beta + j\beta^2 + k\beta^3)$$

$$= \bar{\alpha}\alpha^n + \bar{\beta}\beta^n.$$

3.4. Identity (D'ocagnes identity)

Let $m, n \geq 0$ be integers. Then, the D'ocagnes identity for Fibonacci tessarine is given by

$$\mathcal{T}_m\mathcal{T}_{n+1} - \mathcal{T}_{m+1}\mathcal{T}_n = (-1)^{m+1}\,(\mathcal{T}_{m-n} - 5f_{m-n} - f_{m-n+1}i).$$

Proof. If we use Eqn. (5), Eqn. (8) and Eqn. (9), we have

$$\mathcal{T}_m\mathcal{T}_{n+1} - \mathcal{T}_{m+1}\mathcal{T}_n = (f_m + f_{m+1}i + f_{m+2}j + f_{m+3}k)(f_{n+1} + f_{n+2}i + f_{n+3}j + f_{n+4}k)$$

$$-(f_{m+1} + f_{m+2}i + f_{m+3}j + f_{m+4}k)(f_n + f_{n+1}i + f_{n+2}j + f_{n+3}k)$$

$$= -4(-1)^{m+1}\, f_{m-n} + (-1)^{m+1}f_{m-n+2}j + (-1)^{m+1}\, f_{m-n+3}k$$

$$= (-1)^{m+1}[(\, f_{m-n} + f_{m-n+1}i + f_{m-n+2}j + f_{m-n+3}k)$$
$$-(5\, f_{m-n} + f_{m-n+1}i)]$$

$$= (-1)^{m+1}\,(\mathcal{T}_{m-n} - 5f_{m-n} - f_{m-n+1}i).$$

3.5. Identity (Cassini's identity)

Let $n \geq 0$ be an integer. The Cassini's identity is given as;

$$\mathcal{T}_{n+1}\mathcal{T}_{n-1} - \mathcal{T}_n^2 = (-1)^n(2\mathcal{T}_1 + 2 + 2j - 3k).$$

Proof. We have

$$\mathcal{T}_{n+1}\mathcal{T}_{n-1} - \mathcal{T}_n^2$$
$$= (f_{n+1} + f_{n+2}i + f_{n+3}j + f_{n+4}k)(f_{n-1} + f_n i + f_{n+1}j + f_{n+2}k)$$

$$-(f_n + f_{n+1}i + f_{n+2}j + f_{n+3}k)^2$$
$$= f_n^2 - f_{n-1}f_{n+1} - (f_{n+1}^2 + f_n f_{n+2})$$
$$-(f_{n+2}^2 - f_{n+1}f_{n+3}) + (f_{n+2}f_{n+4} - f_{n+3}^2)$$
$$+i(f_{n-1}f_{n+2} - f_n f_{n+1} + f_{n+1}f_{n+4} - f_{n+2}f_{n+3})$$
$$+j(f_{n+1}^2 - f_n f_{n+2} - f_{n+2}^2 + f_{n-1}f_{n+3} - f_{n+1}f_{n+3} + f_{n+1}f_{n+3} - f_n f_{n+2})$$
$$+j(f_{n+1}^2 - f_n f_{n+2} - f_{n+2}^2 + f_{n-1}f_{n+3} - f_{n+1}f_{n+3} + f_{n+1}f_{n+3} - f_n f_{n+2})$$
$$+k(f_{n-1}f_{n+4} - f_n f_{n+3})$$

which implies

$$\mathcal{T}_{n+1}\mathcal{T}_{n-1} - \mathcal{T}_n^2$$
$$= 4(-1)^n + i((-1)^{n+1}f_{-2} + (-1)^{n+3}f_{-2})) + j(2(-1)^n$$
$$+(-1)^{n+2}f_{-3} + (-1)^n f_3) + k(-1)^{n+3}f_{-4}$$
$$= 4(-1)^n + 2i(-1)^n f_2 + j(2(-1)^n + (-1)^n f_3 + (-1)^n f_3) + k(-1)^n f_4$$
$$= (-1)^n(4 + 2i + 6j + 3k).$$

Also by adding and subtracting the term $3k$, we complete the proof.

3.6. Identities (Fibonacci negatessarine and Lucas negatessarine)

Let $\mathcal{T}_n$ and $\mathcal{T}_n'$ be Fibonacci tessarine and Lucas tessarine, respectively. The identities of Fibonacci negatessarine and Lucas negatessarine are defined as;

$$\mathcal{T}_{-n} = (-1)^{n+1}\mathcal{T}_n + (-1)^n l_n(i + j + 2k)$$

and

$$\mathcal{T}_{-n}' = (-1)^n\mathcal{T}_n' + (-1)^{n+1} + 5f_n(i + j + 2k).$$

Proof. Now, we will give proof of identity $\mathcal{T}_{-n}$. We have

$$\mathcal{T}_{-n} = f_{-n} + f_{-n+1}i + f_{-n+2}j + f_{-n+3}k$$
$$= f_{-n} + f_{-(n-1)}i + f_{-(n-2)}j + f_{-(n-3)}k$$
$$= (-1)^{n+1}f_n + (-1)^n f_{n-1}i + (-1)^{n+1}f_{n-2}j + (-1)^n f_{n-3}k$$
$$= (-1)^{n+1}(f_n + f_{n+1}i + f_{n+2}j + f_{n+3}k) - (-1)^{n+1}f_{n+1}i - (-1)^{n+1}f_{n+2}j$$
$$-(-1)^{n+1}f_{n+3}k + (-1)^n f_{n-1}i + (-1)^{n+1}f_{n-2}j + (-1)^n f_{n-3}k$$

In this identity, taking into account the Eqn. (1), Eqn. (5) and Eqn. (9), we obtain

$$\mathcal{T}_{-n} = (-1)^{n+1}\mathcal{T}_{-n} + (-1)^n(f_{n+1} + f_{n-1})i + (-1)^n(f_{n+2} - f_{n-2})j$$

$$+(-1)^n(f_{n+3}+f_{n-3})k$$
$$=(-1)^{n+1}\mathcal{T}_n+(-1)^n l_n(i+j+2k).$$

Now, if we use the Eqn. (6), Eqn. (7) and Eqn. (10), we have

$$\mathcal{T}'_{-n}=l_{-n}+l_{-n+1}i+l_{-n+2}j+l_{-n+3}k$$
$$=l_{-n}+l_{-(n-1)}i+l_{-(n-2)}j+l_{-(n-3)}k$$
$$=(-1)^n l_n+(-1)^{n-1}l_{n-1}i+(-1)^n l_{n-2}j+(-1)^{n-1}l_{n-3}k$$
$$=(-1)^n(l_n+l_{n+1}i+l_{n+2}j+l_{n+3}k)-(-1)^n l_{n+1}i-(-1)^n l_{n+2}j$$
$$-(-1)^n l_{n+3}k+(-1)^{n-1}l_{n-1}i+(-1)^n l_{n-2}j+(-1)^{n-1}l_{n-3}k$$
$$=(-1)^n\mathcal{T}'_n+(-1)^{n+1}((l_{n+1}+l_{n-1})i+(l_{-n+2}-l_{n-2})j+(l_{n+3}+l_{n-3})k)$$
$$=(-1)^n\mathcal{T}'_n+(-1)^{n+1}+5f_n i+(-1)^{n+1}+5f_n j+(-1)^{n+1}+10f_n k,$$

which completes the proof.

Example 1. Let $\mathcal{T}_0$, $\mathcal{T}_1$ and $\mathcal{T}_2$ be the Fibonacci tessarines such that

$\mathcal{T}_0=i+j+2k,$

$\mathcal{T}_1$=1+i+2j+3k,

$\mathcal{T}_2$=1+2i+3j+5k.

Considering the Eqn. (11) and Eqn. (13), we have

$$\mathcal{T}_0\,\mathcal{T}_2-\mathcal{T}_1^2=(i+j+2k)(1+2i+3j+5k)-(1+i+2j+3k)^2$$
$$=-(4+2i+6j+3k)=-(2\mathcal{T}_1+2+2j-3k)$$

and

$$\mathcal{T}_1\,\mathcal{T}_3-\mathcal{T}_2^2=(1+i+2j+3k)(2+3i+5j+8k)-(1+2i+3j+5k)^2$$
$$=4+2i+6j+3k=2\mathcal{T}_1+2+2j-3k.$$

4. Some Applications on the Fibonacci Tessarines

Definition 3. The n^{th} Fibonacci tessarine vector $\vec{\mathcal{T}}_{\mathrm{n}}$ and the n^{th} Lucas tessarine vector $\vec{\mathcal{T}}'_{\mathrm{n}}$ are defined as

$$\vec{\mathcal{T}}_{\mathrm{n}}=f_{n+1}i+f_{n+2}\,j+f_{n+3}k$$

and

$$\vec{\mathcal{T}}_{\mathrm{n}}' = l_{n+1}i + l_{n+2}\,j + l_{n+3}k,$$

where f_n is the n^{th} Fibonacci number and l_n is the n^{th} Lucas number. Also i, j and k are arbitrary units which satisfy the relations;

$$i^2 = -1, j^2 = +1, k^2 = -1 \text{ and } ij = ji = k\,.$$

Definition 4. [13] Let $\vec{\mathcal{T}}_{\mathrm{n}}$ and $\vec{\mathcal{T}}_{\mathrm{m}}$ be Fibonacci tessarine vectors. The dot product and the cross product of these vectors are defined by

$$< \vec{\mathcal{T}}_n, \vec{\mathcal{T}}_m >= f_{n+1}f_{m+1} + f_{n+2}f_{m+2} + f_{n+3}f_{m+3}$$

and

$$\vec{\mathcal{T}}_n \times \vec{\mathcal{T}}_m = \det\begin{bmatrix} i & j & k \\ f_{n+1} & f_{n+2} & f_{n+3} \\ f_{m+1} & f_{m+2} & f_{m+3} \end{bmatrix}$$
$$= i(f_{n+2}f_{m+3} + f_{m+2}f_{n+3}) - j(f_{n+1}f_{m+3} + f_{m+1}f_{n+3})$$
$$+k(f_{n+1}f_{m+2} + f_{m+1}f_{n+2}).$$

Some examples of Fibonacci tessarine and Lucas tessarine vectors can be given easily as;

$$\vec{\mathcal{T}}_0 = 1i + 1j + 2k,\ \vec{\mathcal{T}}_1 = 1i + 2j + 3k \text{ and } \vec{\mathcal{T}}_2 = 2i + 3j + 5k$$

and

$$\vec{\mathcal{T}}_0' = 1i + 3j + 4k,\ \vec{\mathcal{T}}_1' = 3i + 4j + 7k \text{ and } \vec{\mathcal{T}}_2' = 4i + 7j + 11k.$$

Theorem 1. Let $\vec{\mathcal{T}}_{\mathrm{n}}$ and $\vec{\mathcal{T}}_{\mathrm{n+1}}$ be Fibonacci tessarine vectors. The dot product and the cross product of these vectors are as follows

$$< \vec{\mathcal{T}}_{\mathrm{n}}, \vec{\mathcal{T}}_{\mathrm{n+1}} > = \frac{4}{5}l_{2n+5} + \frac{(-1)^n}{5}$$

and

$$\vec{\mathcal{T}}_{\mathrm{n}} \times \vec{\mathcal{T}}_{\mathrm{n+1}} = \vec{\mathcal{T}}_{\mathrm{2n+1}}' - 2f_{2n+4}(i + j) + \frac{(-1)^n}{5}(i + 2j - 2k).$$

Proof. Using the Eqn. (7) and Eqn. (10), we obtain,

$$< \vec{\mathcal{T}}_n, \vec{\mathcal{T}}_{n+1} > = f_{n+1}f_{n+2} + f_{n+2}f_{n+3} + f_{n+3}f_{f_{n+4}}$$
$$= \frac{1}{5}(l_{2n+3} - (-1)^{n+1}l_1 + l_{2n+5} - (-1)^{n+2}l_1 + l_{2n+7} - (-1)^{n+3}l_1)$$
$$= \frac{4}{5}l_{2n+5} + \frac{(-1)^n}{5}$$

and the cross product of $\vec{\mathcal{T}}_n$ and $\vec{\mathcal{T}}_{n+1}$ is given by

$$\vec{\mathcal{T}}_{\mathrm{n}} \times \vec{\mathcal{T}}_{\mathrm{n+1}} = i(f_{n+2}f_{n+4} + f_{n+3}^2) - j(f_{n+1}f_{n+4} + f_{n+2}f_{n+3}) + k(f_{n+1}f_{n+3} + f_{n+2}^2)$$

Finally, from Eqn. (7), we have

$$\begin{aligned}\vec{\mathcal{T}}_{\mathrm{n}} \times \vec{\mathcal{T}}_{\mathrm{n+1}} &= \frac{2}{5}(l_{2n+6}i - l_{2n+5}j + l_{2n+4}k) - \frac{1}{5}(-1)^n(i + 2j - 2k) \\ &= \frac{2}{5}[(l_{2n+6} + l_{2n+2} - l_{2n+2})i + (-l_{2n+5} + l_{2n+3} - l_{2n+3})j + l_{2n+4}k)] \\ &+ \frac{(-1)^n}{5}(i + 2j - 2k) \\ &= \vec{\mathcal{T}}'_{2\mathrm{n+1}} - 2f_{2n+4}(i + j) + \frac{(-1)^n}{5}(i + 2j - 2k),\end{aligned}$$

where $\vec{\mathcal{T}}'_{2\mathrm{n+1}}$ is a Lucas tessarine vector.

Example 2. Let $\vec{\mathcal{T}}_1$ and $\vec{\mathcal{T}}_2$ be Fibonacci tessarine vectors such that $\vec{\mathcal{T}}_1$ = (1, 2, 3), $\vec{\mathcal{T}}_2$ = (2, 3, 5). The dot product and cross product of these vectors are

$$\begin{aligned}< \vec{\mathcal{T}}_1, \vec{\mathcal{T}}_2 > &= f_2f_3 + f_3f_4 + f_4f_5 \\ &= \frac{4}{5}l_7 + \frac{(-1)^1}{5} = 23\end{aligned}$$

and

$$\vec{\mathcal{T}}_1 \times \vec{\mathcal{T}}_2 = \vec{\mathcal{T}}'_3 - 2if_6(i + j) - \frac{1}{5}(i - 2j - 2k).$$

Example 3. Let $\vec{\mathcal{T}}_{n-1}$, $\vec{\mathcal{T}}_n$ and $\vec{\mathcal{T}}_{n+1}$ be Fibonacci tessarine vectors. For $n \geq 1$, the Cassini identity for Fibonacci tessarine vectors is given by

$$\vec{\mathcal{T}}_{\mathrm{n-1}}\vec{\mathcal{T}}_{\mathrm{n+1}} - \vec{\mathcal{T}}_n^2 = (-1)^n\left(\vec{\mathcal{T}}'_0 + 3 - \frac{4}{5}k\right).$$

Proof. Now, we will give proof of identities $\vec{\mathcal{T}}_{n-1}$, $\vec{\mathcal{T}}_{n+1}$ and $\vec{\mathcal{T}}_n^2$, we obtain

$$\begin{aligned}\vec{\mathcal{T}}_{n-1}\vec{\mathcal{T}}_{n+1} &= (f_ni + f_{n+1}j + f_{n+2}k)(f_{n+2}i + f_{n+3}j + f_{n+4}k) \\ &= -f_nf_{n+2} + f_{n+1}f_{n+3} - f_{n+2}f_{n+4} + i(f_{n+1}f_{n+4} + f_{n+2}f_{n+3}) \\ &\quad -j(f_nf_{n+4} + f_{n+2}^2) + k(f_{n+3}f_n + f_{n+1}f_{n+2})\end{aligned}$$

and

$$\vec{\mathcal{T}}_n^2 = (f_{n+1}i + f_{n+2}j + f_{n+3}k)(f_{n+1}i + f_{n+2}j + f_{n+3}k)$$
$$= -f_{n+1}^2 + f_{n+2}^2 - f_{n+3}^2 + 2if_{n+2}f_{n+3} - 2jf_{n+1}f_{n+3} + 2kf_{n+1}f_{n+2}.$$

From the Eqn. (3) and Eqn. (7), doing necessary calculations, we have

$$\vec{\mathcal{T}}_{\mathrm{n}-1}\,\vec{\mathcal{T}}_{\mathrm{n}+1} - \vec{\mathcal{T}}_n^2 = (-1)^n\left[\left(3 + i + 3j - \frac{4}{5}k\right) - 4k + 4k\right]$$
$$= (-1)^n\left(\vec{\mathcal{T}}_0' + 3 - \frac{4}{5}k\right).$$

5. Conclusion

In this paper, we first describe the concepts of Fibonacci tessarine and Lucas tessarine with coefficients from the Fibonacci and Lucas numbers. We introduce many identities that have an important place in the literature on tessarines.

Acknowledgement

The authors would like to thank the referees for their significant suggestions and comments, which improve the paper.

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