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Research Article

Modulars from Nakano onwards

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ABSTRACT. We discuss and compare a number of notions of modulars appeared in literature, among which there is a selection of the well known ones. We highlight the connections between the various definitions and provide several examples, taken from existing literature, recalling known results and completing the picture with some original considerations.

Keywords: Modulars, norms, function norm, function space, modular function spaces, Banach function spaces, Riesz spaces, lattices, Nakano modulars.

2020 Mathematics Subject Classification: 46A80, 46E30.

This paper is dedicated to Professor Francesco Altomare.

1. INTRODUCTION

The Functional Analysis theory is typically built on spaces having at least the structures of vector spaces (essentially infinite-dimensional) endowed with a topology. The notions of Topological Vector space, Metric space, Normed space, Banach space, Hilbert space are a kind of chain of structures, each of them beginning with a set of properties which allows to build an entire theory. The overall result is a branch of Mathematical Analysis of great interest in its own and in applications to several other fields like PDEs. The central notion of norm raises naturally the question of how it can be *generated:* important questions are to establish whether a topology is induced by a norm (see the classical result by Kolmogorov, for instance Swartz [139, Theorem 1 p.182]), and whether a given metric can be derived from some norm (see e.g. Singh, Narang [138]), and, on the other hand, the check that inner products induce a norm ensures that properties defining inner products allow to gain results known for normed spaces. Norms of the most familiar infinite dimensional spaces (such as sequence spaces or Lebesgue spaces over sets in Euclidean spaces) characterize the elements of the spaces they generate in the sense that an element belongs to the space if and only if its norm is finite.

The relevance of the notion we are going to treat in this paper is that it allows to characterize easily certain sets of functions, namely, the sets of functions on which a certain functional, built through a so-called *modular,* is finite. Several norms are modulars (in Preface of the book Kozlowski [95] we read: *roughly speaking, modulars are the functionals that generalize norms*), but the heart of the matter is that modulars are not necessarily norms, and this may happen even for functionals characterizing – through its finiteness – vector spaces: in such case, the crucial role of the modular is that it allows to define a new functional (on the set where it is defined if it is a vector space, otherwise in its linear hull) which is a norm. Hence again, modulars represent

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a way to answer to the question of how generate a norm. The most popular (we could also say: *historical*) example is that one of Orlicz spaces: if $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable set with positive measure and if Φ is a non-negative, convex, increasing function over $[0,\infty[$, vanishing at zero, then the condition

$$\rho(f) = \int_\Omega \Phi(|f(x)|)dx < \infty$$

defines a set of measurable functions on Ω; but, in general, ρ does not enjoy the properties of a norm. In the case of functions Φ satisfying the Δ_2 condition (see e.g. Rao, Ren [134, Theorem 2 p.46] for details), the set $\{f : \rho(f) < \infty\}$, endowed with usual addition and scalar multiplication, is a vector space; but ρ is never a norm, unless $\Phi(t) = ct$ for some $c > 0$. The well known norm of classical Orlicz spaces is maybe the most standard example of norm built from a modular (in this case from ρ).

Overall, a natural question is to establish which properties should define a modular ρ so that one can build, starting from ρ, a structure of normed space.

While the classical structures (topological spaces, metric spaces, normed and Banach spaces, etc.) studied in Functional Analysis are defined in standard ways, the literature containing the notion of modular starts frequently from sets of axioms which may have differences among authors. This is not a serious problem, because each treatise must be built from all their necessary prerequisites. However, a comparison between the various notions seems missing at our knowledge, because authors are mostly interested in deriving their particular results. The goal of this paper is to give a contribution to this apparent lack of investigation, which may be of help for researchers which try to extend results previously known in classical structures to the context of modular spaces and need to choose a suitable set of properties from which to derive their results.

The plan of the paper is the following: in Section 2, we recall the definitions of normed and Riesz spaces: this will be of help when we will need to specify the domains of the modulars we will consider; *some* of the modulars introduced by Nakano are recalled and studied in Section 3, which contains Theorem 3.1 about how to build, starting from a modular, a structure of normed space; we will devote Section 4 to modulars introduced after Nakano by various authors; some of these notions – besides, of course, those ones introduced by Nakano – are among the most quoted in literature. They will be not presented in chronological order, because it will be privileged someway the mathematical sequence of the structures, even if the chain of modulars we will consider cannot be ordered rigorously: from the logical point of view the various notions of modular spaces are someway pairwise slanting. We hope that this paper will raise new questions, even from old literature, never studied in a systematic way; few ideas will be listed in the final Section 5.

We will see that the notion of modular is strongly linked, not only historically, to ordered vector spaces. In Subsection 2.2, we will recall that this topic has been within the interests of Professor Francesco Altomare, to which this paper is dedicated.

We close this section pointing out that the term *modular* is used also in contexts different from that one considered in this paper: for instance, it is used in abstract set function theory (see the comment on the property (P.3.1.11) below) and in the framework of finite lattices (see e.g. Section 4 in Kohonen [84]).

2. Normed spaces and Riesz spaces

In next subsection, we recall the standard notion of norm (see e.g. Dunford, Schwartz [46, p.59] or Megginson [118, p.ix or Definition 1.2.1 p.9]), which represents in fact the most important category of one of the next sets of modulars.

2.1. **Normed spaces.** Let $\mathcal{R}$ be a real or complex vector space. A functional

(2.1) $$\|\cdot\| : \mathcal{R} \to [0, +\infty[$$

is said to be a *norm* if it satisfies the following properties ($f, g \in \mathcal{R}$, $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}$):
(P.2.1.1) $\|f\| = 0 \Leftrightarrow f = 0$,
(P.2.1.2) $\|\lambda f\| = |\lambda|\|f\|$,
(P.2.1.3) $\|f + g\| \leqslant \|f\| + \|g\|$.
Norms appear in every book dealing with Functional Analysis and/or Function spaces and their applications; normed spaces are not a topic of interest of this paper. However, when we think to the norm as functional, it may be of interest to highlight some interesting properties: for instance, the triangle inequality can be strengthened. In this framework, we mention just two papers about inequalities for the norm, see Maligranda [111, 112].

2.2. **Riesz spaces.** The set of the real numbers $\mathbb{R}$ may be enriched by the structure of complete ordered field, but it is also the substructure for an ordered *vector* field over $\mathbb{R}$ itself. In other familiar vector spaces order relations can be defined, but the compatibility with operations must be lost: for instance, it is well known from undergraduate Calculus that in $\mathbb{R}^2$ one cannot define at all an order compatible with the structure of field. However, a partial order compatible with the structure of vector space can be defined (see e.g. Example 2.1 below). One gets this way an important structure, which can be built also starting from other well known sets: for instance, the real vector space $\mathcal{M}$ of the Lebesgue measurable real-valued functions defined in $(0,1) \subset \mathbb{R}$. We are going to recall the definition of this structure.

The *Riesz spaces* (see Nakano [126, p.9], who called them "semi-ordered linear spaces"; for a modern exposition see e.g. Luxemburg, Zaanen [107, p.48]) are real vector spaces $\mathcal{R}$, where a binary relation (= subset of $\mathcal{R} \times \mathcal{R}$), denoted by $\geqslant$, is defined (for semiorders see also Luce [104, Section 2 p.181]; roughly speaking, we stress that elements f, g such that both $f \leqslant g$, $g \leqslant f$ do not hold, are allowed), satisfying a set of properties, which in turn is divided in other sets of properties with appropriate terminology. The various definitions are collected in the following scheme, where $\alpha \in \mathbb{R}$:

$$
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f \geqslant g\,,\ g \geqslant h \ \Rightarrow f \geqslant h \\
f \geqslant f \\
f \geqslant g\,,\ g \geqslant f \ \Rightarrow f = g
\end{array}\right\}\begin{array}{c}\text{partial}\\ \text{ordering}\end{array} \\
f \geqslant g\,,\ h \in \mathcal{R} \ \Rightarrow f + h \geqslant g + h \\
f \geqslant g\,,\ \alpha \geqslant 0 \ \Rightarrow \alpha f \geqslant \alpha g
\end{array}\right\}\begin{array}{c}\text{(partially)}\\ \text{ordered}\\ \text{vector}\\ \text{space}\end{array} \\
\left.\begin{array}{l}
\forall f, g \ \exists f \vee g \in \mathcal{R} \ : \ f \vee g \geqslant f, f \vee g \geqslant g \\
h \geqslant f \vee g \ \ \forall h \ : \ h \geqslant f, h \geqslant g
\end{array}\right\}\begin{array}{c}\text{lattice}\\ \text{property}\end{array}
\end{array}\right\}\begin{array}{c}\text{(semiordered)}\\ \text{vector lattice}\\ \textit{or}\\ \text{lattice}\\ \text{ordered}\\ \text{vector space}\\ \textit{or}\\ \textbf{Riesz space}\\ \textit{or}\\ \text{semiordered}\\ \text{linear space}\end{array} \\
\text{There exists a norm } \|\cdot\| \text{ such that, setting} \\
\text{“absolute value of } f\text{”} = |f| := f \vee (-f) \in \mathcal{R}\,, \\
|f| \geqslant |g| \ \Rightarrow \|f\| \geqslant \|g\| \quad \text{(lattice norm property)} \\
\text{(note that this implies } \|\, f \,\| = \|\, |f| \,\|\text{, see (P.3.1.9) below)}
\end{array}\right\}\begin{array}{c}\text{normed}\\ \text{Riesz}\\ \text{space}\end{array} \\
\text{The norm } \|\cdot\| \text{ is complete}
\end{array}\right\}\begin{array}{c}\text{Banach}\\ \text{lattice}\end{array}
$$

The element $f \vee g$ is, by definition, the *least upper bound* of the set $\{f, g\}$; its existence for all sets of two elements is equivalent to the existence of the greatest lower bound $f \wedge g$ for all sets of two elements, because it can be easily shown that $(-f) \wedge (-g) = -f \vee g$ (see e.g. Aliprantis, Border [11, Theorem 8.6 (1.) p.318]). For a treatise on lattices and order, see Davey, Priestley [40].

Example 2.1. *Examples of Banach lattices.* Banach lattices are common in Analysis. $\mathbb{R}$ with usual operations, as real vector space, is a Banach lattice, the standard $\vee$ being the maximum of two real numbers. Also, the Euclidean space $\mathbb{R}^n$ under the componentwise ordering (see e.g. Zaanen [149, Example 1.2 (i) p.2])

$$x = (x_1, \dots, x_n) \geqslant y = (y_1, \dots, y_n) \quad \Leftrightarrow \quad x_i \geqslant y_i \ \forall i = 1, \dots, n$$

is a Riesz space under

$$x \vee y = (\max\{x_1, y_1\}, \dots, \max\{x_n, y_n\})$$

(see e.g. Aliprantis, Border [11, Example 8.1, n.1, p.313]), and with the Euclidean norm it becomes a Banach lattice (see e.g. Aliprantis, Border [11, Example 9.1 p.348]).

Also, classical Lebesgue spaces are examples of Banach lattices (see e.g. Meyer-Nieberg [121, Example (v) p.9]), under the ordering defined by $f \geqslant g$ whenever $f(x) \geqslant g(x)$ almost everywhere; this is stated also in Altomare, Campiti [13, (1.2.39) p.30].

Moreover, the Banach space of all real valued continuous functions on a compact Hausdorff space X, endowed with the pointwise order and the supremum norm, is a Banach lattice (see e.g. Meyer-Nieberg [121, Example (ii) p.8]). For other examples, see Chill, F., Król [32, Remark 4.4 p.522] and Szankowski [140]. □

The reader can find several examples of Riesz spaces in Luxemburg, Zaanen [107, Example 11.2 p.48], Zaanen [149, Example 4.2 p.13 and Example 7.3 p.28]. Other interesting examples are in Aliprantis, Border [11, Examples 8.1 p.313].

We remark that properties of ordered vector spaces play a role in Korovkin-type approximation theory, a subject in which Professor Francesco Altomare gave important contributions (see the treatise Altomare, Campiti [13]).

We close this section showing, through next examples, that structures having the properties above are not only abstract, but they really exist. In particular, we are going to see that any additional property required in the chain of notions above is necessary, because it is not a consequence of the previous ones.

Example 2.2. *There exist ordered sets (i.e., sets with partial ordering) without a real vector space structure.* $\mathbb{N}$, the natural numbers with the usual order, constitute an ordered set. Obviously with the usual notion of addition $\mathbb{N}$ cannot have a real vector spaces structure (because, for instance, the opposite of *1* is missing), but we can assert that it cannot exist any notion of sum and scalar multiplication which gives to $\mathbb{N}$ a real vector space structure. In fact, $\alpha 1$ should belong to $\mathbb{N}$ for every $\alpha \in \mathbb{R}$, but this cannot happen because $\mathbb{N}$ has not the cardinality of the continuum. □

Example 2.3. *There exist (real) ordered sets without the lattice property.* Consider for instance $C^1([0,2])$ as the real vector space of all continuously differentiable real-valued functions on the closed interval $[0,2]$: it is a real vector space with the usual pointwise sum and pointwise scalar multiplication. It has a partial ordering with $\geqslant$ defined pointwise. However, there exist couples of functions in $C^1([0,2])$ without a least upper bound (in $C^1([0,2])$). Take for instance $f(x) = x$, $g(x) = 2 - x$. Then, a $C^1(0,2)$ function $f \vee g$ satisfying the properties $f \vee g \geqslant f$, $f \vee g \geqslant g$, $h \geqslant f \vee g \ \forall h : h \geqslant f, h \geqslant g$ does not exist. In fact, on the contrary, from $f \vee g \geqslant f$, $f \vee g \geqslant g$ it would be $(f \vee g)(x) \geqslant \max\{x, 2 - x\}$ and any function h of the type

$$h(x) = \alpha(x)(f \vee g)(x) + (1 - \alpha(x)) \max\{x, 2 - x\},$$

where $\alpha \in C^1(0,2)$, $0 \leqslant \alpha(x) \leqslant 1$, $\alpha \not\equiv 1$, $\alpha(1) = 1$ (hence $\alpha'(1) = 0$; say, $\alpha(x) = \exp(-(x-1)^2)$) would be such that

$$h \in C^1(0,2), \quad \max\{x, 2 - x\} \leqslant h(x) \leqslant (f \vee g)(x), \quad h \neq f \vee g$$

which is absurd. □

Example 2.4. *There exist a Riesz space without the lattice norm property.* An example of Riesz space is the Sobolev space $W^{1,p}(\Omega)$, that we consider here for $1 \leqslant p \leqslant \infty$ and $\Omega \subset \mathbb{R}^n$ $(n \geqslant 2)$ open bounded set. It is the real vector space of the real valued functions in $L^p(\Omega)$ whose weak derivatives of first order exist in $L^p(\Omega)$; endowed with the norm (here, as usual, $f = D^0 f$)

$$\|f\|_{W^{1,p}(\Omega)} = \sum_{0 \leqslant |\beta| \leqslant 1} \|D^\beta f\|_{L^p(\Omega)} \quad \text{if} \quad 1 \leqslant p \leqslant \infty,$$

it becomes a Banach space (see e.g. Brezis [28, Proposition 9.1 p.264], Gilbarg, Trudinger [66, Section 7.5 p.153], Adams, Fournier [2, Theorem 3.3 p.60]). Moreover, under the ordering defined by $f \geqslant g$ whenever $f(x) \geqslant g(x)$ almost everywhere, it becomes a Riesz space: in fact, it is well known that the positive part of a weakly differentiable function is again a weakly differentiable function, and $Df^+ = \chi_{\{f>0\}} Df$ (see e.g. Gilbarg, Trudinger [66, Lemma 7.6 p.152]), from which $f \in W^{1,p}(\Omega)$ entrains $f^+, f^- \in W^{1,p}(\Omega)$, and therefore from

$$f \vee g = [(f-g) \vee 0] + g = (f-g)^+ + g,$$

it is immediate to realize that the lattice property holds: if $f, g \in W^{1,p}(\Omega)$, then also $f \vee g \in W^{1,p}(\Omega)$. On the other hand, the lattice norm property does not hold. We can verify this statement on a particular case. Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$, $f(x,y) \equiv 1$, $g(x,y) \equiv x$. Then, $f, g \in W^{1,p}(\Omega)$, $|f| \geqslant |g|$; but

$$\|f\|_{W^{1,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \left\|\frac{\partial f}{\partial x}\right\|_{L^p(\Omega)} + \left\|\frac{\partial f}{\partial y}\right\|_{L^p(\Omega)} = \|1\|_{L^p(\Omega)}$$

and

$$\|g\|_{W^{1,p}(\Omega)} = \|g\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial x}\right\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial y}\right\|_{L^p(\Omega)} = \|g\|_{L^p(\Omega)} + \left\|\frac{\partial g}{\partial x}\right\|_{L^p(\Omega)} = \|x\|_{L^p(\Omega)} + \|1\|_{L^p(\Omega)},$$

hence $\|f\|_{W^{1,p}(\Omega)} < \|g\|_{W^{1,p}(\Omega)}$. The conclusion is that $W^{1,p}(\Omega)$ is not a Banach lattice (for a delicate result in this direction see Pełczyński, Wojciechowski [131]). We stress, however, that there are situations where the lattice norm property is not needed for the whole Sobolev space, but just for the space to which f and $|Df|$ belong (see the recent study in Jain, Molchanova, Singh, Vodopyanov [74], where a characterization in terms of the boundedness of the maximal operator is proved).

We recall here also another example, which is in a finite dimensional vector space. Consider the following example from Chill, F., Król [32, Remark2.3(a) p.513]. If we equip the Riesz space $\mathbb{R}^2$ with either of the norms

$$N_1(x_1, x_2) := \begin{cases} |x_1| + |x_2| & \text{if } x_1 x_2 \geq 0 \\ \sup\{|x_1|, |x_2|\} & \text{if } x_1 x_2 < 0 \end{cases}$$

or

$$N_2(x_1, x_2) := \begin{cases} \sup\{|x_1|, |x_2|\} & \text{if } x_1 x_2 \geq 0 \\ |x_1| + |x_2| & \text{if } x_1 x_2 < 0 \end{cases},$$

since $N_i(1,-1) \neq N_i(1,1) = N_i(|(1,-1)|)$, then $(\mathbb{R}^2, N_i)$, $i = 1,2$, do not enjoy the lattice norm property. □

Example 2.5. *There exist a normed Riesz space whose norm does not satisfy the completeness property.* A first, immediate, example is $\mathbb{Q}$, the set of rational numbers endowed with usual operations. A second example is $C([0,1])$, the real vector space of all continuous real-valued functions on the closed interval $[0,1]$: it is a real vector space with the usual pointwise sum and pointwise scalar multiplication, and it has a partial ordering with $\geqslant$ defined pointwise. It is a normed space under the L^1 lattice norm $\|f\| := \int_0^1 |f(x)|dx$, but such norm is not complete (see e.g. Aliprantis, Border [11, p.348]): setting for instance $f_n(x) = \min\{x^{-1/2}, n\}$, $f_n(0) = 0$, the sequence (f_n) is a Cauchy sequence not converging to any element in $C([0,1])$. However, as written above, endowed with the standard supremum norm, it is a Banach lattice.

Another example is the following. Let $\mathcal{R}$ be the space of sequences $f = (a_1, a_2, \ldots, a_n, \ldots)$, where all the a_i's are real numbers, and $a_i \neq 0$ for only finitely many values of i. It is a real vector space with the usual pointwise sum and pointwise scalar multiplication, and it has a partial ordering with $\geqslant$ defined pointwise. If $f = (a_1, a_2, \ldots, a_n, \ldots)$ and $g = (b_1, b_2, \ldots, b_n, \ldots)$, then clearly

$$f \vee g = (\max\{a_1, b_1\}, \max\{a_2, b_2\}, \ldots, \max\{a_n, b_n\}, \ldots).$$

Setting $\|f\| = \max_i \{|a_i|\}$, we obtain a Riesz space, however, the norm is not complete because, for instance, setting

$$f_n = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, 0, 0, \ldots\right) \quad n \in \mathbb{N},$$

the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence not converging to any element in $\mathcal{R}$. □

3. Some Nakano modulars

Modulars were historically introduced by Hidegorô Nakano. The reader interested in the life and in the scientific activity of Hidegorô Nakano should consult the beautiful exposition by Maligranda [113]. Nakano introduced and studied modulars in various different frameworks: he introduced modulars on *universally continuous semi-ordered linear spaces* (actually called *Dedekind complete Riesz spaces*), and gave a generalized version for *general semi-ordered linear spaces* (actually called *Riesz spaces*, see Nakano [126]). In a subsequent paper, he introduced a still more general definition for real vector spaces without assuming the existence of an order (Nakano [127]). Here, we will limit ourselves to these three notions (so that we can deserve attention to notions introduced later) which do not cover all variants, appeared already in the pioneering book Nakano [126] (which are, for instance, complete modulars, monotone complete modulars, simple modulars, semi-simple modulars, singular modulars, linear modulars, etc.) or in other papers (see e.g. Nakano [129, 130]).

For our goals, it is convenient to fix a unique notation for the discussion about the various notions of modulars. It is therefore natural that our symbols may differ from the original references.

Modulars introduced by Nakano are particular functionals defined on real vector spaces or on richer structures (as Riesz spaces), which will be denoted always by $\mathcal{R}$. In the sequel, the same symbol will be used also to denote complex vector spaces (in which case *complex* will be specified). Elements of $\mathcal{R}$ will usually be denoted by $f, g, h, \ldots$, and the zero vector in $\mathcal{R}$ will be denoted by 0; greek letters $\alpha, \beta, \ldots$ will be used to denote constants in $\mathbb{R}$ which act as scalar multipliers, and the real number zero will be denoted by 0 (so that it can be easily distinguished by $0 \in \mathcal{R}$). Modulars will be denoted by ρ. Natural numbers are denoted by $\mathbb{N}$ $(= \{1, 2, \ldots\})$.

3.1. **Nakano modulars on Dedekind complete Riesz spaces.** A Riesz space $\mathcal{R}$ is said to be *Dedekind complete* (Nakano used to say *universally continuous,* as recalled in Luxemburg, Zaanen [107, p.124]) if every non-empty subset of $\mathcal{R}$ with an upper bound admits in $\mathcal{R}$ the supremum, i.e., the least upper bound (equivalently, if every non-empty subset of $\mathcal{R}$ with a lower bound admits in $\mathcal{R}$ the infimum). The definition makes sense because the notion of Riesz space requires the existence of a supremum (or of an infimum) only for sets of two elements (which can be shown to be equivalent to the same requirement for finite sets, see Luxemburg, Zaanen [107, (vii) p.56]), and the extension to every non-empty subset is not automatic, as we can see from Example 3.8 below.

Example 3.6. *Some standard examples of Dedekind complete Riesz spaces.* $\mathbb{R}^n$, as Riesz space with the usual coordinatewise ordering, is Dedekind complete (see Zaanen [149, p.65]). More generally, the Riesz space of all real (pointwise) functions on any non-empty set X (if X is finite, consisting of n points, we get back $\mathbb{R}^n$), endowed with pointwise ordering, is Dedekind complete (see e.g. Luxemburg, Zaanen [107, p.466]).

The affine functions on a closed interval $[a, b]$ in $\mathbb{R}$ are a Dedekind complete Banach lattice with the Euclidean norm (see Zaanen [149, p.67]).

Moreover, the classical Lebesgue spaces (on totally $\sigma-$finite measures, i.e., the elements of the Lebesgue spaces are defined on a measure space which is union of a countable collection of sets of finite positive measure) are examples of Dedekind complete Banach lattices, and therefore Dedekind complete Riesz spaces (see e.g. Meyer-Nieberg [121, Example (v) p.9]). Another example is $L^0(X)$, the Riesz space – under the a.e. ordering – of all measurable real-valued functions over a measure space (X, Λ, μ) where μ is non-negative, $\sigma-$additive (equivalently, countably additive, i.e., the measure of the union of a countable collection of pairwise disjoint sets coincides with the sum of the measures) and $\sigma-$finite, with identification of the functions which are equal almost everywhere on X, which is Dedekind complete (see e.g. Boccuto, Riečan, Vrábelová [26, p.37], Luxemburg, Zaanen [107, p.459]). □

In order to understand how abstract Dedekind complete Riesz spaces can be, we exhibit the details of an example of such a structure (we will mainly follow Luxemburg [106, p.110]), whose elements are, in general, not functions but measures.

Example 3.7. *A Dedekind complete Riesz space of measures.* Let X be a non-empty set and let Λ be an algebra (i.e., a non-empty family of subsets closed under finite unions and complementation, that is, if $A, B \in \Lambda$, then $A \cup B \in \Lambda$ and $X \setminus A \in \Lambda$, see e.g. Aliprantis, Border [11, p.129]; for instance, if X is a topological space, one can consider the Borel sets in X) in X whose elements will be called *measurable sets.* Let $C(X, \Lambda)$ be the completion of the real-valued $\Lambda-$step functions (i.e., linear combinations of characteristic functions of measurable sets) with respect to the topology of the uniform convergence, i.e., with respect to the supremum norm. Let $\mathcal{R}$ be the (norm) dual space of $C(X, \Lambda)$, usually denoted by $ba(X, \Lambda)$, i.e., the vector space of the linear and bounded ($\Leftrightarrow$ continuous) operators $X \to \mathbb{R}$ (see e.g. Aliprantis, Border [11, Sect.6.3 p.230]; this norm dual space is not to be confused with the *order* dual of Riesz spaces, made of linear functionals which are bounded in the sense of the order, see e.g. Aliprantis, Border [11, p.327] – the two spaces may be different!). This dual can be represented by the space of the finitely additive set functions μ on Λ (*finitely additive* means that for each finite family of pairwise disjoint sets $\{A_i\}$ whose union belongs to Λ, μ computed in the union equals the sum of the $\mu(A_i)$'s, see Aliprantis, Border [11, p.374]) of finite total variation $\|\mu\|_1$. Here, $\|\mu\|_1$ denotes the total variation of μ over X, defined by

$$\|\mu\|_1 = \lim_{\pi} V(\mu; \pi) = \lim_{\pi} \sum_{k=1}^{n} |\mu(E_k)|,$$

where $\pi = \pi(E_1, \ldots, E_n)$ is a $\Lambda-$partition (note that the limit exists since $V(\mu; \pi)$ is increasing in π). If $f \in C(X, \Lambda)$ and $\mu \in ba(X, \Lambda)$, then the bilinear form determining the duality between $C(X, \Lambda)$ and $ba(X, \Lambda)$ is denoted by

$$< f, \mu >= \int_X f d\mu.$$

It can be proved (see Aliprantis, Border [11, p.374], see also Meyer-Nieberg [121, Example (vi) p.9]) that $\mathcal{R} = ba(X, \Lambda)$ is a Dedekind complete Banach lattice (in particular, a Riesz space) under the ordering defined by $\mu \geqslant \nu$ whenever $\mu(A) \geqslant \nu(A)$ for all $A \in \Lambda$ (see Aliprantis, Border [11, p.314]). The induced supremum is given by

$$\mu \vee \nu(E) = \mu(E) + \sup\{(\nu - \mu)(A) : A \in \Lambda, A \subset E\} = \sup\{\mu(A) + \nu(E \setminus A) : A \in \Lambda, A \subset E\}. \quad \square$$

Example 3.8. *There exists a Riesz space which is not Dedekind complete.* Consider an infinite set X, and let $\mathcal{R}$ be the Riesz space of all real-valued functions defined on X whose range is finite, with the pointwise ordering. Then, $\mathcal{R}$ is not a Dedekind complete Riesz space (see Aliprantis, Burkinshaw [12, Example 2.13(3) p.15] or Luxemburg, Zaanen [107, (iii) p.139]). We give here the full details of the proof. Fix a sequence $(x_i)_{i\in\mathbb{N}}$ of pairwise disjoint elements in X, and consider the subset $\{f_i\}_{i\in\mathbb{N}} \subset \mathcal{R}$, where

$$f_i(x) = \begin{cases} j^{-1} & \text{if} \quad x = x_j \text{ and } j \leqslant i \\ 0 & \text{otherwise} \end{cases}.$$

Clearly, $\{f_i\}_{i\in\mathbb{N}}$ admits an upper bound, because every f_i is smaller than the function identically 1 (which has finite range); however, we can show that the supremum does not exist. In fact, let $g \in \mathcal{R}$ be an upper bound for $\{f_i\}_{i\in\mathbb{N}}$; we show that there exists $h \in \mathcal{R}$ such that $h < g$, h being an upper bound for $\{f_i\}_{i\in\mathbb{N}}$ as well. Since g has a finite range, there exists $k \in \mathbb{N}$ such that k^{-1} does not belong to the range of g. Set

$$h(x) = \begin{cases} g(x) & \text{if} \quad x \neq x_k \\ k^{-1} & \text{if} \quad x = x_k \end{cases}.$$

Since $f_k(x_k) \leqslant g(x_k)$, we have $k^{-1} \leqslant g(x_k)$, and since k^{-1} does not belong to the range of g, we can assert that in fact $k^{-1} < g(x_k)$. It follows that $h < g$. On the other hand, $f_i \leqslant h$ for all $i \in \mathbb{N}$: if $x \neq x_k$, then $f_i(x) \leqslant h(x)$ because it is equivalent to $f_i(x) \leqslant g(x)$; if $x = x_k$ and $i < k$, then $f_i(x_k) = 0 < k^{-1} = h(x_k)$; finally, if $x = x_k$ and $i \geqslant k$, then $f_i(x_k) = k^{-1} = h(x_k)$.

Another example of Riesz space which is not Dedekind complete is $C([0,1])$ (see e.g. Meyer-Nieberg [121, p.7]), which has been considered in Example 2.5: the set $\{x^\alpha\}_{0<\alpha<1} \subset C([0,1])$ is bounded by the constant function 1, but it does not admit a supremum in $C([0,1])$.

Finally, the Riesz space of all real bounded functions f over the interval $[0,1] \subset \mathbb{R}$ such that $f(x) \neq f(0)$ holds for at most countably many x, with pointwise ordering, is not Dedekind complete (see e.g. Luxemburg, Zaanen [107, (ii) p.139] for details). □

In Dedekind complete Riesz spaces, it is defined a monotone convergence (see Nakano [126, p.17]) as follows (here Λ denotes a set of indices):

$$f_\lambda \uparrow_{\lambda\in\Lambda} f \Leftrightarrow \begin{cases} \forall\, \lambda_1, \lambda_2 \in \Lambda \;\exists\, \lambda \in \Lambda \text{ such that } f_\lambda \geqslant f_{\lambda_1} \vee f_{\lambda_2} \\ f = \text{least upper bound of } \{f_\lambda\}_{\lambda\in\Lambda} \end{cases}.$$

A functional ρ defined on a Dedekind complete Riesz space $\mathcal{R}$ is said to be a *(Nakano) modular on a Dedekind complete Riesz space* (see Nakano [126, p.153]) if it satisfies the following properties ($f, f_\lambda, g \in \mathcal{R}$, $\alpha \in \mathbb{R}$):

(P.3.1.1) $0 \leq \rho(f) \leq \infty, \quad \forall f \in \mathcal{R}$,
(P.3.1.2) $\rho(\alpha f) = 0, \quad \forall \alpha \geq 0 \quad \Rightarrow \quad f = 0$,
(P.3.1.3) $\forall\, f \in \mathcal{R}, \quad \exists\, \alpha > 0$ such that $\rho(\alpha f) < \infty$,
(P.3.1.4) $\forall\, f \in \mathcal{R}, \;\; \alpha \to \rho(\alpha f)$ is a convex function,
(P.3.1.5) $|f| \leq |g| \quad \Rightarrow \quad \rho(f) \leq \rho(g)$,
(P.3.1.6) $f \wedge g = 0 \quad \Rightarrow \quad \rho(f+g) = \rho(f) + \rho(g)$,
(P.3.1.7) $0 \leq f_\lambda \uparrow_{\lambda\in\Lambda} f \quad \Rightarrow \quad \rho(f) = \sup_{\lambda\in\Lambda} \rho(f_\lambda)$.

When on $\mathcal{R}$ a modular is defined, $\mathcal{R}$ is said to be a *modulared space*.

Let us record few consequences of this definition of modular:

(P.3.1.8) $\rho(0) = 0$.

Proof. By (P.3.1.3), for some $\alpha > 0$ it is $\rho(\alpha 0) < \infty$, but, since $\alpha 0 = 0$, we have $\rho(0) < \infty$; on the other hand, since $0 \wedge 0 = 0$ and $0 + 0 = 0$, by (P.3.1.6)

$$\rho(0) = \rho(0+0) = \rho(0) + \rho(0) = 2\rho(0),$$

from which $\rho(0) = 0$. □

(P.3.1.9) $\rho(f) = \rho(|f|)$.

Proof. Since $|f| \leqslant |f| \vee (-|f|) = ||f||$, by (P.3.1.5), with g replaced by $|f|$, we have $\rho(f) \leqslant \rho(|f|)$; on the other hand, setting $h = 0$ in the triangle inequality $||f| - |h|| \leqslant |f + h|$ (whose simple proof coincides with the standard one for real numbers, however, for Riesz spaces see e.g. Luxemburg, Zaanen [107, Theorem 12.1 p.62]), we get $||f|| \leqslant |f|$, and again by (P.3.1.5), with f replaced by $|f|$ and g replaced by f, we get the reversed inequality. □

(P.3.1.10) $\lim\limits_{\alpha \to 0} \rho(\alpha f) = 0, \quad \forall f \in \mathcal{R}$.

Proof. For every $f \in \mathcal{R}$, using the compatibility of $\geqslant$ with respect with the addition (included in the definition of order given in the scheme at the beginning of subsection 2.2), we have

$$\begin{aligned} |f| &= f \vee (-f) = f + f \vee (-f) - f = (2f) \vee 0 - f = (2f) \vee 0 + [f \vee 0 - f] - f \vee 0 \\ &= 2(f \vee 0) + [0 \vee (-f)] - (f \vee 0) = (f \vee 0) + (-f) \vee 0 \geqslant 0 + (-f) \vee 0 \\ &= (-f) \vee 0 \geqslant 0. \end{aligned} \tag{3.2}$$

On the other hand, for $\alpha \in \mathbb{R}$,

$$|\alpha f| = (\alpha f) \vee (-(\alpha f)) = (\alpha f) \vee ((-\alpha) f) = |\alpha|(f \vee (-f)) = |\alpha||f|. \tag{3.3}$$

Now, let $0 \leqslant \alpha_1 < \alpha_2 \leqslant 1$, so that $\alpha_2 - \alpha_1 > 0$. Using the compatibility of $\geqslant$ with respect with the scalar multiplication, by (3.2), we have $(\alpha_2 - \alpha_1)|f| \geqslant 0$. Then, by (3.3), using again the compatibility of $\geqslant$ with respect with the addition,

$$|\alpha_1 f| = \alpha_1 |f| \leqslant \alpha_1 |f| + (\alpha_2 - \alpha_1)|f| = \alpha_2 |f| = |\alpha_2 f|$$

from which, by (P.3.1.5), we get that $\alpha \in [0,1] \to \rho(\alpha f)$ is an increasing function. Now, by (P.3.1.8), we have $\rho(0) = 0$, by (P.3.1.3) the same function is finite around $\alpha = 0$, and by (P.3.1.4) the same function is also continuous. Property (P.3.1.10) is therefore proved. □

Remark 3.1. *The proof above uses implicitly some properties which are true also in the more general framework of ordered vector spaces. Readers interested in a systematic exposition containing a huge sequence of simple propositions may consult, for instance, Luxemburg, Zaanen* [107, Chapter 2, Section 11 p.48].

A less immediate consequence is the identity (see Nakano [126, (12) p.154]).
(P.3.1.11) $\rho(f \vee g) + \rho(f \wedge g) = \rho(f) + \rho(g) \quad$ for $f, g \geqslant 0$.
It should be remarked, here, that in the abstract set function theory (which is not within the topic of this paper), the property (P.3.1.11) alone (where f, g are sets, $\vee$ is the union and $\wedge$ the intersection) defines ρ as *modular* (for details see Konig [89, p.11], see also Weber [146]).
By using a very technical result (one of the tools being Zorn's Lemma), Nakano obtains the convexity of ρ in $\mathcal{R}$ (see Nakano [126, Theorem 36.8 p.163]):
(P.3.1.12) $\rho(\alpha f + \beta g) \leqslant \alpha \rho(f) + \beta \rho(g) \quad$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$
and also, in the same statement,
(P.3.1.13) $\rho(f + g) \geqslant \rho(f) + \rho(g) \geqslant \rho(f - g) \quad$ for $f, g \geqslant 0$.
A description of the whole theory built by Nakano and the several contributions concerning vector lattices are out of the goal of this paper; the reader interested in this topic is warmly invited to read the already mentioned nice exposition by Lech Maligranda [113], full of historical details. We mention here also Koshi, Shimogaki [93], where the authors introduced the $quasi-modular$ spaces, weakened the theory of modular spaces and summarized the work in

Nakano [126]. The same authors then built a theory developed in Koshi, Shimogaki [92], Koshi [91]. See also Yamamuro [147].

For our purposes, we just highlight that essentially the model example of modular satisfying properties (P.3.1.1)-(P.3.1.7) is the following, highlighted by Nakano himself in the introduction in [126, p.4]. If $\Phi(\xi, t)$ is measurable as a function of t for $0 \leqslant t \leqslant 1$, and non-decreasing, convex as a function of $\xi \geqslant 0$, $\Phi(0, t) = 0$, $\Phi(\xi, t) = \lim_{\varepsilon \to 0+} \Phi(\xi - \varepsilon, t)$ for $\xi > 0$, and $\Phi(\alpha_t, t) < \infty$ for some $\alpha_t > 0$, then the class $\mathcal{R}$ of all real a. e. measurable functions φ on $[0, 1]$ such that

$$\int_0^1 \Phi(\alpha|\varphi(t)|, t) d\mu_t < \infty$$

for some $\alpha > 0$, is a Dedekind complete Riesz space, and

$$\rho(\varphi) := \int_0^1 \Phi(|\varphi(t)|, t) d\mu_t < \infty \tag{3.4}$$

is a modular on $\mathcal{R}$. In particular, $\mathcal{R}$ can be the Lebesgue space $L^1(\Omega, \mu)$. For another, less standard, example, see Albrycht, Orlicz [10].

Example 3.9. *There exist norms which are (Nakano) modulars (on Dedekind complete Riesz space).* The model example discussed above, in the case $\Phi(\xi, t) = \xi$, reduces $\mathcal{R}$ to the Lebesgue space $L^1(\Omega, \mu)$, and the corresponding modular ρ is the standard norm in $L^1(\Omega, \mu)$. A still more particular, nevertheless relevant, example is that one of μ equal to the sum of a finite number n of Dirac masses: in this case the set of the a.e. measurable functions can be identified with $\mathbb{R}^n$. The identification holds also as Riesz spaces: the a.e. ordering of measurable functions corresponds to the componentwise ordering, and the modular ρ, which is the L^1 norm, corresponds to the so-called *Taxicab norm* or *Manhattan norm.* □

Example 3.10. *There exist norms which are not (Nakano) modulars (on Dedekind complete Riesz space).* Norms of well known spaces whose elements are real measurable functions, in general, are *not* modulars. The key point is property (P.3.1.6), which is a fundamental tool to get the convexity property (P.3.1.12), but it throws away some important norms, for instance, the norm in $L^p(\Omega, \mu)$, $1 < p < \infty$: in fact, if for instance $A, B \subset \Omega$ are disjoint and $\mu(A) = \mu(B) = 1$, then $\chi_A \wedge \chi_B = 0$ and

$$\begin{aligned} \rho(\chi_A + \chi_B) &= \|\chi_A + \chi_B\|_{L^p(\Omega,\mu)} = (\mu(A) + \mu(B))^{1/p} = 2^{1/p} \\ &\neq 2 = \mu(A) + \mu(B) = \|\chi_A\|_{L^p(\Omega,\mu)} + \|\chi_B\|_{L^p(\Omega,\mu)} = \rho(\chi_A) + \rho(\chi_B), \end{aligned}$$

and therefore (P.3.1.6) is not satisfied. As in the previous example, the consideration of the case of measures which are sums of Dirac masses tells that when $n \geq 2$ the Euclidean norm is not a modular on $\mathbb{R}^n$, in the sense introduced in Nakano [126]. □

3.2. **Nakano modulars on Riesz spaces.** Still in the same Nakano [126], there is a notion of modular over Riesz spaces which does not satisfy necessarily the Dedekind completeness assumption, which clearly plays its role in property (P.3.1.7). Since such property had a role in the proof of convexity, then the one-dimensional convexity (P.3.1.4) is changed definitively in the whole convexity of the modular, and this had some consequences also on other properties. We are going to list the resulting set of properties, which define modulars that Nakano called *general modulars.* A functional ρ defined on a Riesz space $\mathcal{R}$ is said to be a *(Nakano) modular on Riesz space* (see Nakano [126, p.271]) if it satisfies the following properties ($f, f_\lambda, g \in \mathcal{R}$, $\alpha \in \mathbb{R}$):

(P.3.2.1) $0 \leq \rho(f) \leq \infty, \quad \forall f \in \mathcal{R}$,

(P.3.2.2) $\rho(\alpha f) = 0, \quad \forall \alpha \geq 0 \quad \Rightarrow \quad f = 0$,

(P.3.2.3) $\forall\, f \in \mathcal{R}, \quad \exists\, \alpha > 0$ such that $\rho(\alpha f) < \infty$,

(P.3.2.4) $\rho(\alpha f + \beta g) \leqslant \alpha\rho(f) + \beta\rho(g)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$,

(P.3.2.5) $|f| \leq |g| \quad \Rightarrow \quad \rho(f) \leq \rho(g)$,

(P.3.2.6) $\rho(f + g) \geqslant \rho(f) + \rho(g) \geqslant \rho(f \vee g)$ for $f, g \geqslant 0$, (P.3.2.7) $\sup_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$.

The consequences (P.3.1.8), (P.3.1.9), (P.3.1.10) are still true with almost the same proofs. It is worth to make a comment on (P.3.2.7). Given $f \in \mathcal{R}$, if for some $0 \leqslant \alpha < 1$ we have $\rho(\alpha f) = \infty$, then (P.3.2.7) is not a condition on ρ, because (P.3.2.7) is satisfied: in fact, in all Riesz spaces it is known that $|\alpha f| = |\alpha||f|$ (the proof in (3.3) does not use the Dedekind completeness), and therefore, by (P.3.1.9), since $\alpha \geqslant 0$,

$$\rho(\alpha f) = \rho(|\alpha f|) = \rho(|\alpha||f|) = \rho(\alpha|f|) \leqslant \rho(|f|) = \rho(f).$$

On the other hand, if for all $0 \leqslant \alpha < 1$ we have $\rho(\alpha f) < \infty$, then

$$\sup_{0\leqslant\alpha<1} \rho(\alpha f) \leqslant \sup_{\substack{|f|\geqslant|g|\\ \rho(g)<\infty}} \rho(g) \leqslant \rho(f),$$

hence, (P.3.2.7) is not weaker than

$$\sup_{\substack{|f|\geqslant|g|\\ \rho(g)<\infty}} \rho(g) = \rho(f)$$

which is a condition on ρ called by Nakano *modular continuity* (see Nakano [126, p.182] and Nakano [126, p.192]).

When considering Nakano modulars on Riesz spaces, the gain is, for instance, the possibility to consider the modular (3.4) restricted to the Riesz space of all real-valued functions whose range is finite, with the pointwise ordering, considered in Example 3.8. However, the gain of new structures still leaves out, in general, several norms. In fact, a norm $\|\cdot\|$ satisfying (P.3.2.6) and, at the same time, the triangle inequality, necessarily must satisfy

$$\|f+g\| = \|f\| + \|g\| \quad \text{for } f, g \geqslant 0,$$

which is not satisfied, for instance, by the norm in $L^p(\Omega,\mu)$ when $p > 1$ (as we saw in Example 3.10). Moreover, it would be good to have a notion of modular which admits norms on structures not enjoying the requirements imposed by Riesz spaces, for instance, the real vector space $C^1([0,1])$ considered in Example 2.3 (as ordered vector space which is not a Riesz space) which is a normed space (see e.g. Kufner, John, Fučik [101, (1) p.25]) when endowed with the (standard) norm

$$\|f\|_{C^1([0,1])} = \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)|.$$

The hole is filled again by Nakano, in a 1951 paper, as we are going to see.

3.3. **Nakano modulars on real vector spaces.** A functional ρ defined on a real vector space $\mathcal{R}$ is said to be a *(Nakano) modular on real vector space* (see Nakano [127]) if it satisfies the following properties ($f, g \in \mathcal{R}$):

(P.3.3.1) $0 \leq \rho(f) \leq \infty$,

(P.3.3.2) $\rho(f) = \rho(-f)$,

(P.3.3.3) $\exists f \in \mathcal{R},\ \exists \alpha > 0$ such that $\rho(\alpha f) < \infty$,

(P.3.3.4) $\rho(\alpha f) = 0, \quad \forall \alpha > 0 \quad \Leftrightarrow \quad f = 0$,

(P.3.3.5) $\sup_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$,

(P.3.3.6) ρ is convex: $0 \leqslant \alpha \leqslant 1 \ \Rightarrow\ \rho(\alpha f + (1-\alpha)g) \leqslant \alpha\rho(f) + (1-\alpha)\rho(g) \qquad (0 \cdot \infty = 0)$.

Remark 3.2. *Unfortunately reference is not easily accessible, and properties (P.3.3.1)-(P.3.3.6) are taken from the review MR44048 (13,362a) in MathSciNet, where the list of properties is given in a style raising doubts. This means that our translation into a precise style could be not faithful with the original source. The sentence "$m(\lambda x) = \infty$ for all positive λ does not occur for any x" appearing in the review could have different interpretations; maybe the original source intends, "For any x, $m(\lambda x) = \infty$ for all positive λ does not occur", because the existence of x in (P.3.3.3) is trivial (hence, there would be no reason to have* *this extra property) in the case $\rho(0) = 0$, hence implicit in (P.3.3.4). However, the whole question is not really important, because we will never use it and we stated it just to try to respect the historical value of this notion. We observe that in Nakano's terminology (see e.g. [129]), an element $x \in \mathcal{R}$ is said to be finite whenever $\rho(\lambda x) < \infty$ for all $\lambda > 0$, is said to be a null element if $\rho(\lambda x) = 0$ for all $\lambda > 0$, and a modular ρ is said to be pure if 0 is the only null element: when applying results from a set of axioms, it may be worth to choose properties which avoid unpleasant situations (for instance: all elements must be finite and/or the modular must be pure, etc.).*

Proposition 3.1. *Norms $\|\cdot\|$ on a real vector space $\mathcal{R}$ are (Nakano) modulars (on real vector space).*

Proof. Property (P.3.3.1) follows directly from (2.1). Property (P.3.3.2) follows applying (P.2.1.2) with $\alpha = -1$: $\|f\| = |-1|\|f\| = \|-f\|$. Property (P.3.3.3) holds choosing any $f \in \mathcal{R}$, any $\alpha \in \mathbb{R}$, because by (2.1) the norm is always finite. About property (P.3.3.4), if $\|\alpha f\| = 0 \quad \forall \alpha \geq 0$, then by (P.2.1.2) we have $\alpha\|f\| = 0 \quad \forall \alpha \geq 0$, which means that $\|f\| = 0$. By (P.2.1.1), we get $f = 0$. The viceversa comes directly again from (P.2.1.1). Property (P.3.3.5) is consequence of (P.2.1.2):

$$\sup\{\|\lambda f\| \ : 0 \leqslant \lambda < 1\} = \sup\{\lambda\|f\| \ : 0 \leqslant \lambda < 1\} = \|f\|.$$

Finally, convexity follows by (P.2.1.3) and (P.2.1.2): for $0 \leqslant \alpha \leqslant 1$, $f, g \in \mathcal{M}$,

$$\|\alpha f + (1-\alpha)g\| \leqslant \|\alpha f\| + \|(1-\alpha)g\| = \alpha\|f\| + (1-\alpha)\|g\|.$$

□

As a consequence of Proposition 3.1, the norms of several well known spaces highly used in Analysis and applications are modulars, for instance, Musielak-Orlicz spaces (and therefore Orlicz spaces and variable Lebesgue spaces with their weighted versions), Lorentz spaces, grand and small Lebesgue spaces, spaces of continuous differentiable functions, Hölder-continuous differentiable functions, Morrey and Campanato spaces, Sobolev spaces. Such spaces are treated in many books, an incomplete list being Adams, Fournier [2], Bennett, Sharpley [24], Brezis [28], Castillo, Rafeiro [31], Cruz-Uribe, F. [38], Cruz-Uribe, Martell, Pérez [39], Demengel, Demengel [41], Diening, Harjulehto, Hästö, Růžička [43], Edmunds, Evans [47], Edmunds, Triebel [49], Fiorenza [64], Genebashvili, Gogatishvili, Kokilashvili, Krbec [65], Harjulehto, Hästö [69], Haroske [70], Haroske, Triebel [71], Kokilashvili, Krbec [85], Kokilashvili, Meskhi, Rafeiro, Samko [86, 87], Kufner [100], Kufner, John, Fučik [101], Lindenstrauss, Tzafriri [102, 103], Maligranda [110], Maz'ja [116], Mendez, Lang [119], Meskhi [120], Musielak [124], Pick, Kufner, John, Fučik [132], Rakotoson [133], Rao, Ren [134], Schmeisser, Triebel [137], Triebel [142], Triebel [143], Triebel [141], Turett [144]. We mention here also other nonstandard norms, which are obtained as roots of polynomials (Anatriello, F., Vincenzi [15]) or as fixed points (F., Talponen [63]).

Example 3.11. *There exist (Nakano) modulars (on real vector space) which are not norms.* If $\mathcal{R}$ is a normed space with norm $\|\cdot\|$, setting $\rho(f) = \|f\|^2$, we have a modular which is not a norm. The fact that it is not a norm is a consequence of (P.2.1.2): since

$$\rho(\alpha f) = \|\alpha f\|^2 = \alpha^2\|f\|^2 = \alpha^2\rho(f),$$

then ρ cannot satisfy (P.2.1.2) if $\alpha \neq 0, 1, -1$. On the other hand, ρ is a modular: the proof of properties (P.3.3.1)-(P.3.3.6) using (2.1), (P.2.1.1)-(P.2.1.3) is immediate. The reader may check that, in general, the square can be replaced by any nondecreasing, convex function on $[0, +\infty[$ assuming value 0 in the origin and not identically 0. □

Norms are also special *quasinorms*, however, in next example, we will see that quasinorms are not necessarily modulars. Incidentally, we recall that the notion of modular in Nakano [129] includes quasinorms, but we will not deal with it in this paper.

Example 3.12. *There exist quasinorms, i.e., functionals $\rho : \mathcal{R} \to [0, \infty[$ such that for some $C > 0$, for every $f, g \in \mathcal{R}$: (j) $\rho(f) = 0$ if and only if $f = 0$, (jj) $\rho(\lambda f) = |\lambda|\rho(f)$ for all $\lambda \in \mathbb{R}$, (jjj) $\rho(f+g) \leqslant$*

$C(\rho(f)+\rho(g))$ *which are not (Nakano) modulars (on real vector space).* We consider the following example, borrowed from Anatriello, F., Vincenzi [15, Example 2.1 p.4]: set $\mathcal{R} = L^1(0,1)$ and

$$\rho(f) = (2 - \sin(\pi|\mathrm{supp} f|)) \int_0^1 |f(x)|dx,$$

where $|\mathrm{supp} f|$ denotes the measure of the support of f. It is easy to check that it is a quasinorm, using the fact that the factor of the integral on the right hand side is in the interval $[1,2]$. On the other hand, ρ is not a (Nakano) modular because it is not convex: in fact,

$$\rho\left(\frac{1}{2}\chi_{(0,\frac{1}{2})} + \frac{1}{2}(-\chi_{(\frac{1}{2},1)})\right) = 2\int_0^1 \frac{1}{2}\chi_{(0,\frac{1}{2})}(x) + \frac{1}{2}\chi_{(\frac{1}{2},1)}(x)dx = 1;$$

on the other hand,

$$\frac{1}{2}\rho\left(\chi_{(0,\frac{1}{2})}\right) + \frac{1}{2}\rho\left(-\chi_{(\frac{1}{2},1)}\right) = \frac{1}{2}\int_0^1 \chi_{(0,\frac{1}{2})}(x)dx + \frac{1}{2}\int_0^1 \chi_{(\frac{1}{2},1)}(x)dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

□

Example 3.13. *There exist (Nakano) modulars (on real vector space) for which $\rho(f) = 0$ does not imply $f = 0$.* Let $\mathcal{M}$ be the vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in the set of the real numbers $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f) = \begin{cases} 0 & \text{if} \quad \operatorname{ess\,sup}|f| \leqslant 1 \\ \infty & \text{if} \quad \operatorname{ess\,sup}|f| > 1 \end{cases}.$$

It is easy to check that ρ is a modular. Moreover, ρ vanishes on any function whose modulus is bounded by 1. We stress that this example is standard, and it appears in literature also with minor variations (see e.g. Koshi [90], Bachar, Mendez, Bounkhel [17, (2)]). □

The importance of Nakano modulars relies upon the following result, appeared in a primitive form in Nakano [126, Theorem 43.6 p.192]. The heart of the matter is that properties (P.3.3.1)-(P.3.3.6) guarantee the existence of a vector subspace on which a certain functional, whose expression is given explicitly, is a norm. The expression in (3.6) below is usually said to be *Luxemburg norm* (because it is known from the celebrated Luxemburg's thesis, see [105, Definition 1 p.43], given in the restricted framework of Orlicz spaces), however, following Maligranda (see [113] and [108, Comment 2]), we call it Luxemburg-Nakano norm. However, it should be stressed also what it is remarked in Diening, Harjulehto, Hästö, Růžička [43, p.25] (see also Maligranda [73]), namely, that the Luxemburg norm has the structure of the Minkowski functional introduced in Kolmogoroff [88] long before; in Diening, Harjulehto, Hästö, Růžička [43, Remark 2.1.16] it is shown that the proof of the fact that it is a norm comes from a more general (and nowadays classical) statement of Functional Analysis (see e.g. Schechter [136, 12.29.g p.317]).

Theorem 3.1. *Let ρ be a (Nakano) modular on a real vector space $\mathcal{R}$, and set*

$$\widetilde{\mathcal{R}} := \left\{ f \in \mathcal{R} : \textit{the set } \left\{ \lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty \right\} \textit{ is non-empty} \right\}.$$

The following statements hold:

(i) *$\widetilde{\mathcal{R}}$ is a vector subspace of $\mathcal{R}$ (in particular, $\widetilde{\mathcal{R}}$ is non-empty).*

(ii) $\widetilde{\mathcal{R}} = \left\{ f \in \mathcal{R} : \lim\limits_{\lambda\to 0+} \rho(\lambda f) = 0 \right\}.$

(iii) *For every $\alpha > 0$,*

$$\widetilde{\mathcal{R}} = \left\{ f \in \mathcal{R} : \textit{the set } \left\{ \lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha \right\} \textit{ is non-empty} \right\}. \tag{3.5}$$

(iv) *The functional* $\|\cdot\|_\alpha : \widetilde{\mathcal{R}} \to [0,+\infty[$ *defined by*

$$\|f\|_\alpha := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\} \quad \textit{(Luxemburg-Nakano norm)} \tag{3.6}$$

is a norm on $\widetilde{\mathcal{R}}$.

(v) *For all* $f \in \widetilde{\mathcal{R}}$, *we have* $\|f\|_\alpha = \inf\limits_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\}$.

(vi) *The norms* $\|f\|_\alpha$ *are pairwise equivalent, and if* $\alpha > \beta > 0$, *then for all* $f \in \widetilde{\mathcal{R}}$

$$\|f\|_\alpha \leqslant \|f\|_\beta \leqslant \frac{\alpha}{\beta}\|f\|_\alpha. \tag{3.7}$$

(vii) *The functional* $|||\cdot|||_\alpha : \widetilde{\mathcal{R}} \to [0,+\infty[$ *defined by*

$$|||f|||_\alpha := \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} \quad \textit{(Amemiya norm)} \tag{3.8}$$

is a norm on $\widetilde{\mathcal{R}}$.

(viii) *The norms* $\|\cdot\|_\alpha$ *and* $|||\cdot|||_\alpha$ *are equivalent, and for all* $f \in \widetilde{\mathcal{R}}$

$$\|f\|_\alpha \leqslant |||f|||_\alpha \leqslant 2\|f\|_\alpha.$$

Proof of **(i)**. By (P.3.3.4), if $f = 0$, then $\rho(0) = \rho(1f) = 0$, hence $\widetilde{\mathcal{R}}$ is non-empty and we have also that $0 \in \widetilde{\mathcal{R}}$. Now, we show that $\widetilde{\mathcal{R}}$ is a subspace of $\mathcal{R}$. If $f \in \widetilde{\mathcal{R}}$ is such that for some $\lambda > 0$ we have $\rho(f/\lambda) < \infty$, then for every $\alpha > 0$ also $\alpha f \in \widetilde{\mathcal{R}}$, because we have $\rho(\alpha f/(\alpha\lambda)) < \infty$. On the other hand, let $f, g \in \widetilde{\mathcal{R}}$ be such that $\rho(f/\lambda_1) < \infty$, $\rho(g/\lambda_2) < \infty$, where $\lambda_1, \lambda_2 > 0$. By the convexity property (P.3.3.6), we have

$$\rho\left(\frac{f+g}{\lambda_1+\lambda_2}\right) = \rho\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{f}{\lambda_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{g}{\lambda_2}\right) \leqslant \frac{\lambda_1}{\lambda_1+\lambda_2}\rho\left(\frac{f}{\lambda_1}\right) + \frac{\lambda_2}{\lambda_1+\lambda_2}\rho\left(\frac{g}{\lambda_2}\right) < \infty,$$

hence also the sum of elements of $\widetilde{\mathcal{R}}$ belongs to $\widetilde{\mathcal{R}}$. □

Proof of **(ii)**. Let $f \in \widetilde{\mathcal{R}}$, hence $\rho(f/\lambda_0) < \infty$ for some $\lambda_0 > 0$. Recalling again that by (P.3.3.4) $\rho(0) = \rho(10) = 0$, by the convexity property (P.3.3.6), for every $0 \leqslant \lambda \leqslant 1/\lambda_0$, we have

$$0 \leqslant \rho(\lambda f) = \rho\left(\lambda\lambda_0\frac{f}{\lambda_0} + (1-\lambda\lambda_0)0\right) \leqslant \lambda\lambda_0\rho\left(\frac{f}{\lambda_0}\right) + (1-\lambda\lambda_0)0,$$

from which, letting $\lambda \to 0$, we get

$$\widetilde{\mathcal{R}} \subset \left\{f \in \mathcal{R} : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\right\}.$$

Viceversa, if $f \in \mathcal{R}$ is such that

$$\lim_{\lambda\to 0+} \rho(\lambda f) = 0,$$

then for λ small we have

$$\rho(\lambda f) < 1 < \infty,$$

hence, we get also

$$\widetilde{\mathcal{R}} \supset \left\{f \in \mathcal{R} : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\right\}.$$

Part **(ii)** is therefore proved. □

Remark 3.3. *Equality* **(ii)** *is currently used in literature. For instance, recently, it has been stated in Costarelli, Vinti* [36, p.9, after (3)].

Remark 3.4. *In principle, the same proof of Part* **(ii)** *could have been used to prove (P.3.1.10), because we just used convexity (which is still true, see (P.3.1.12)). However, we observed that the proof of (P.3.1.12) is very technical; in the case of Part* **(ii)***, convexity is directly in the assumption (P.3.3.6).*

Proof of **(iii)**. Equality (3.5) follows from **(ii)** and the definition of $\widetilde{\mathcal{R}}$: in fact, if for some $f \in \mathcal{R}$ the set $\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\}$ is non-empty, then f is such that $\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\}$ is non-empty, therefore in (3.5) the $\supset$ holds; on the other hand, if $f \in \widetilde{\mathcal{R}}$, by **(ii)**, we have

$$\lim_{\lambda \to 0+} \rho(\lambda f) = 0 \tag{3.9}$$

and therefore the definition of limit gives that for every $\alpha > 0$ there exists $\lambda_0 > 0$ such that $\rho(\lambda_0 f) < \alpha$, hence

$$\frac{1}{\lambda_0} \in \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

from which (3.5) follows. ☐

Proof of **(iv)**. We begin by showing that $\|\cdot\|_\alpha$ satisfies (P.2.1.1). We have

$$\|0\|_\alpha = \inf\left\{\lambda > 0 : \rho\left(\frac{0}{\lambda}\right) \leqslant \alpha\right\},$$

and since for every $\lambda > 0$

$$\rho\left(\frac{0}{\lambda}\right) = 0 < \alpha,$$

we have $\|0\|_\alpha = 0$. On the other hand, let $f \in \widetilde{\mathcal{R}}$ be such that $\|f\|_\alpha = 0$, so that

$$\rho\left(\frac{f}{\lambda}\right) \leqslant \alpha \quad \forall \lambda > 0.$$

Fix $\beta > 0$, and let $\lambda > 0$ be such that $0 < \beta\lambda < 1$. By the convexity property (P.3.3.6), we have

$$\rho(\beta f) = \rho\left(\beta\lambda\frac{f}{\lambda} + (1 - \beta\lambda)0\right) \leqslant \beta\lambda\rho\left(\frac{f}{\lambda}\right) + (1 - \beta\lambda)0 \leqslant \beta\lambda \cdot \alpha, \tag{3.10}$$

from which, letting $\lambda \to 0$, we get $\rho(\beta f) = 0$ for all $\beta > 0$. By property (P.3.3.4), we conclude that $f = 0$.

We now show (P.2.1.2). Fix $f \in \widetilde{\mathcal{R}}$, so that $\|f\|_\alpha$ is well defined and finite. If $\lambda = 0$, we have to show that $\|0f\|_\alpha = 0\|f\|_\alpha$, i.e., $\|0\|_\alpha = 0$, but this is already known from (P.2.1.1). If $\lambda \neq 0$, by property (P.3.3.2),

$$\begin{aligned}
\|\lambda f\|_\alpha &:= \inf\left\{\mu > 0 : \rho\left(\frac{\lambda f}{\mu}\right) \leqslant \alpha\right\} \\
&= \inf\left\{\mu > 0 : \rho\left(\frac{|\lambda| f}{\mu}\right) \leqslant \alpha\right\} \\
&= \inf\left\{\mu|\lambda| > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \alpha\right\} \\
&= |\lambda| \inf\left\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \alpha\right\} \\
&= |\lambda| \|f\|_\alpha.
\end{aligned}$$

Finally, we show that $\|\cdot\|_\alpha$ satisfies (P.2.1.3). At first, we observe that for every $f \in \widetilde{\mathcal{R}}$, $f \neq 0$,

$$\rho\left(\frac{f}{\|f\|_\alpha}\right) \leqslant \alpha : \tag{3.11}$$

in fact, on the contrary, by property (P.3.3.5), it would exist $0 < \lambda < 1$ such that

$$\rho\left(\frac{\lambda f}{\|f\|_\alpha}\right) > \alpha,$$

hence $\|f\|_\alpha/\lambda > \|f\|_\alpha$ does not belong to the set defining $\|f\|_\alpha$, which is absurd. Now let $f, g \in \widetilde{\mathcal{R}}$, $f, g \neq 0$ (otherwise (P.2.1.3) is trivially true). We have

$$\begin{aligned}\rho\left(\frac{f+g}{\|f\|_\alpha+\|g\|_\alpha}\right) &= \rho\left(\frac{\|f\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\frac{f}{\|f\|_\alpha}+\frac{\|g\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\frac{g}{\|g\|_\alpha}\right)\\ &\leqslant \frac{\|f\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\rho\left(\frac{f}{\|f\|_\alpha}\right)+\frac{\|g\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\rho\left(\frac{g}{\|g\|_\alpha}\right)\\ &\leqslant \frac{\|f\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\alpha+\frac{\|g\|_\alpha}{\|f\|_\alpha+\|g\|_\alpha}\alpha=\alpha\end{aligned}$$

from which $\|f+g\|_\alpha \leqslant \|f\|_\alpha + \|g\|_\alpha$. □

Remark 3.5. *Recently, it has been shown that* (3.11) *holds for every convex pseudomodulars on real vector spaces which are left lower semicontinuous (see F., Talponen* [63, Proposition 1.2] *for details); for Musielak-Orlicz spaces see Harjulehto, Hästö* [69, Lemma 3.2.3 p.53], *for variable Lebesgue spaces see Cruz-Uribe, F.* [38, Proposition 2.21 p.24]*).*

Proof of **(v)**. Let us set temporarily (it is the symbol of norm without α)

$$\|f\| = \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\}.$$

Fix $f \in \mathcal{R}$, and let us split the positive λ's into two sets. If $\lambda > 0$ is such that $\rho(\lambda f) \leqslant \alpha$, then

$$\rho\left(\frac{f}{1/\lambda}\right) \leqslant \alpha,$$

from which we get $\|f\|_\alpha \leqslant 1/\lambda$. On the other hand, if $\lambda > 0$ is such that $\alpha < \rho(\lambda f) < \infty$, then by the convexity property (P.3.3.6), we have

$$\rho\left(\frac{f}{\rho(\lambda f)/(\alpha\lambda)}\right) = \rho\left(\frac{\alpha\lambda f}{\rho(\lambda f)}\right) \leqslant \frac{\alpha}{\rho(\lambda f)}\rho(\lambda f) = \alpha,$$

and therefore in this case $\|f\|_\alpha \leqslant \rho(\lambda f)/(\alpha\lambda)$. Note that the same inequality is obviously true if $\rho(\lambda f) = \infty$. Overall, in any case, for every $\lambda > 0$, we get $\|f\|_\alpha \leqslant \|f\|$.

Viceversa, it will be sufficient to show that $\|f\|$ is smaller than any positive number which is in the set defining $\|f\|_\alpha$. Let us denote such generic positive number by $1/\mu$, so that

$$\rho\left(\frac{f}{1/\mu}\right) \leqslant \alpha. \tag{3.12}$$

We have

$$\|f\| = \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\} \leqslant \max\left\{\frac{1}{\mu}, \frac{\rho(\mu f)}{\alpha\mu}\right\} \leqslant \frac{1}{\mu}.$$

Since the infimum of all $1/\mu$ satisfying (3.12) is $\|f\|_\alpha$, we get also the inequality $\|f\| \leqslant \|f\|_\alpha$. □

Proof of **(vi)**. Fix $f \in \widetilde{\mathcal{R}}$. If $\alpha > \beta > 0$, then clearly

$$\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \beta\right\} \subset \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

hence

$$\inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \beta\right\} \geqslant \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

i.e.,

$$\|f\|_\alpha \leqslant \|f\|_\beta.$$

On the other hand, if

$$\lambda \in \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \alpha\right\},$$

then

$$\rho\left(\frac{\beta}{\alpha\lambda} f\right) = \rho\left(\frac{\beta}{\alpha}\frac{f}{\lambda} + \left(1 - \frac{\beta}{\alpha}\right)0\right) \leqslant \frac{\beta}{\alpha}\rho\left(\frac{f}{\lambda}\right) + \left(1 - \frac{\beta}{\alpha}\right)0 \leqslant \frac{\beta}{\alpha}\alpha = \beta,$$

hence

$$\frac{\alpha\lambda}{\beta} \in \left\{\mu > 0 : \rho\left(\frac{f}{\mu}\right) \leqslant \beta\right\},$$

from which

$$\|f\|_\beta \leqslant \frac{\alpha\lambda}{\beta}.$$

Passing to the infimum over λ, we get the right wing inequality in (3.7). □

Proof of **(vii)**. We begin by showing that $|||\cdot|||_\alpha$ satisfies (P.2.1.1). If $f = 0$, then $|||f|||_\alpha = 0$: in fact,

$$|||0|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda 0)}{\alpha\lambda} = \inf_{\lambda>0} \frac{1}{\lambda} = 0.$$

On the other hand, if $f \in \widetilde{\mathcal{R}}$ is such that $|||f|||_\alpha = 0$, we have

$$0 = |||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} \geqslant \inf_{\lambda>0} \max\left\{\frac{1}{\lambda}, \frac{\rho(\lambda f)}{\alpha\lambda}\right\} = \|f\|_\alpha \geqslant 0, \tag{3.13}$$

hence $\|f\|_\alpha = 0$, from which we already showed in **(iv)** that $f = 0$. Property (P.2.1.2) follows observing that

$$\begin{aligned}|||\lambda f|||_\alpha &= \inf_{\mu>0} \frac{\alpha + \rho(\mu\lambda f)}{\alpha\mu} = \inf_{\mu>0} \frac{\alpha + \rho(\mu|\lambda| f)}{\alpha\mu} = \inf_{\mu>0} \frac{\alpha + \rho(\mu f)}{\alpha(\mu/|\lambda|)} \\ &= |\lambda| \inf_{\mu>0} \frac{\alpha + \rho(\mu f)}{\alpha\mu} = |\lambda| |||f|||_\alpha.\end{aligned}$$

Finally, we show that $|||\cdot|||_\alpha$ satisfies (P.2.1.3). At first, we observe that for every $f, g \in \widetilde{\mathcal{R}}$, for arbitrary $\varepsilon > 0$ there exist $\lambda, \mu > 0$ such that

$$\frac{\alpha + \rho(\lambda f)}{\alpha\lambda} < |||f|||_\alpha + \varepsilon, \qquad \frac{\alpha + \rho(\mu g)}{\alpha\mu} < |||g|||_\alpha + \varepsilon,$$

and therefore, by the convexity property (P.3.3.6), we have

$$
\begin{aligned}
|||f+g|||_\alpha &\leqslant \frac{\alpha+\rho\left(\frac{\lambda\mu}{\lambda+\mu}(f+g)\right)}{\alpha\frac{\lambda\mu}{\lambda+\mu}} \\
&= \frac{\lambda+\mu}{\alpha\lambda\mu}\left[\alpha+\rho\left(\frac{\mu}{\lambda+\mu}\lambda f+\frac{\lambda}{\lambda+\mu}\mu g\right)\right] \\
&\leqslant \frac{\lambda+\mu}{\alpha\lambda\mu}\left[\alpha+\frac{\mu}{\lambda+\mu}\rho(\lambda f)+\frac{\lambda}{\lambda+\mu}\rho(\mu g)\right] \\
&= \frac{1}{\mu}+\frac{1}{\lambda}+\frac{\rho(\lambda f)}{\alpha\lambda}+\frac{\rho(\mu g)}{\alpha\mu} < |||f|||_\alpha+|||g|||_\alpha+2\varepsilon,
\end{aligned}
$$

from which $|||f+g|||_\alpha \leqslant |||f|||_\alpha+|||g|||_\alpha$. □

Proof of **(viii)**. Estimates in the chain (3.13) show already that not only for $f=0$, but for all $f\in\widetilde{\mathcal{R}}$, we have

$$\|f\|_\alpha \leqslant |||f|||_\alpha.$$

On the other hand, for all $f\in\widetilde{\mathcal{R}}$, by **(v)**, we have

$$|||f|||_\alpha = \inf_{\lambda>0}\frac{\alpha+\rho(\lambda f)}{\alpha\lambda} \leqslant \inf_{\lambda>0} 2\max\left\{\frac{1}{\lambda},\frac{\rho(\lambda f)}{\alpha\lambda}\right\} = 2\|f\|_\alpha.$$

□

Remark 3.6. *It can happen that the vector subspace $\widetilde{\mathcal{R}}$ is strictly contained in $\mathcal{R}$: for instance, in the case of the modular in Example 3.13, we have $\widetilde{\mathcal{R}}=L^\infty(0,1)\subsetneq\mathcal{M}=\mathcal{R}$. Of course, if one introduces modulars imposing the further condition that* (3.9) *must hold on the whole vector space, then $\widetilde{\mathcal{R}}=\mathcal{R}$ (see e.g. Biegert* [25, (M4) p.295]*).* □

Theorem 3.1, especially in the case $\alpha=1$, is well known and repetitively quoted and proved in literature, often with some variants in the assumptions and/or with only some of the implications included in our exposition. For instance, it appears in Diening, Harjulehto, Hästö, Růžička [43, Theorem 2.1.7 p.24], where essentially **(i)-(iv)** are proved. In Harjulehto, Hästö [69, Lemma 3.1.3. p.48] the result is proved in the framework of Generalized Orlicz spaces: in this case the interest is in the assumptions on the modular, written in terms of properties of $\Phi-$functions. For the case of variable Lebesgue spaces, see e.g. Cruz-Uribe, F. [38, Theorem 2.17]. In Maligranda [110, Theorem 1.2 p.5] (see also Maligranda [113, Theorem 4 p.125], Musielak [124, Theorem 1.5 p.3], Bardaro, Musielak, Vinti [23, Theorem 1.1(b) p. 4]) the assertions in Theorem 3.1 are analyzed in the case of a weaker assumption of convexity (introduced in Musielak, Orlicz [125]; we will consider it later, see property (P.4.1.4)): in such case **(ii)** is not necessarily true and the two vector subspaces of $\mathcal{R}$ are denoted with different symbols. Equality **(ii)** appears in Mendez, Lang [119, Lemma 1.3.1 p.28], and in the same reference Proposition 1.3.2 contains the proof of **(iv)**, which appears also in Edmunds, Mendez, Lang [48, Proposition 1.3 p.11]. The idea to introduce the parameter α in the statement of Theorem 3.1 is inspired by Miranda [122, (49.7) p.265, proof in p. 266], stated for Orlicz spaces; the same trick has been used also more recently, see e.g. Greco, Iwaniec, Moscariello [68, Lemma 4.2]: the advantage is to get a "clean" Hölder inequality (this is explicitly remarked in Miranda [122, (49.IV) p.270]).

Theorem 3.1 opens the way to define the norm of some of the familiar function spaces. If it is applied to a modular which is already a norm, one gets again the same norm (multiplied by a constant if $\alpha\neq 1$): setting, say, $\rho(f)=\|f\|$ in

$$\|f\|_\alpha = \inf\left\{\lambda>0:\rho\left(\frac{f}{\lambda}\right)\leqslant\alpha\right\},$$

one gets

$$\|f\|_\alpha = \inf\left\{\lambda > 0 : \left\|\frac{f}{\lambda}\right\| \leqslant \alpha\right\} = \inf\left\{\lambda > 0 : \left\|\frac{f}{\alpha}\right\| \leqslant \lambda\right\} = \frac{1}{\alpha}\|f\|,$$

and the same happens substituting ρ with $\|\cdot\|$ in

$$|||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} :$$

in fact,

$$|||f|||_\alpha = \inf_{\lambda>0} \frac{\alpha + \rho(\lambda f)}{\alpha\lambda} = \inf_{\lambda>0} \frac{\alpha + \|\lambda f\|}{\alpha\lambda} = \inf_{\lambda>0} \frac{1}{\lambda} + \frac{\|f\|}{\alpha} = \frac{1}{\alpha}\|f\|.$$

However, the value of Theorem 3.1 is that it is the key to define function spaces from modulars which are not norms. The most "popular" example is that one known as Musielak-Orlicz space (as stressed by Maligranda in [108, Comment 1], they should be called *variable Orlicz spaces* or *Orlicz-Nakano spaces*, because – as highlighted also in Bardaro, Musielak, Vinti [23, Section 1.5] – they were introduced in Nakano [126]), whose standard norm is the Luxemburg-Nakano norm built from a modular of the type

$$\rho(f) = \int_\Omega \Phi(x, f)d\mu.$$

Musielak-Orlicz spaces never lost their interest among researchers: we mention for instance the recent research Youssfi, Ahmida [148] on approximation results, and applications in Ahmida, Chlebicka, Gwiazda, Youssfi [4], Ahmida, F., Youssfi [5]. The example of Musielak-Orlicz spaces is not only popular, but in some sense is *the* example of modular, because under suitable assumptions, modulars have an integral representations of this type (see Drewnowski, Orlicz [45] for details; see also Kranz, Wnuk [98]).

Note that also other norms are defined using the same machinery, for instance, the Orlicz-Lorentz spaces, which are a common generalization of the Orlicz spaces and the Lorentz spaces (see part 4 of Mastyło [115], Maligranda [109], Kaminska [75, 76, 77], Montgomery-Smith [123], Kamińska, Leśnik, Raynaud [78]).

Moreover, we observe that one could introduce the grand Lebesgue spaces over a (Lebesgue) measurable set $\Omega \subset \mathbb{R}^n$, $n \geqslant 1$, $0 < |\Omega| < \infty$ (see F., Formica, Gogatishvili [53] and the more recent papers Farroni, F., Giova [50], Di Fratta, F., Slastikov [42], F., Formica [52]), as the set of the real valued, measurable functions such that

$$\rho(f) = \sup_{0<\varepsilon<p-1} \frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon} dx < \infty;$$

the Luxemburg-Nakano norm build from this modular gives back the usual norm (see F., Giannetti [55, Remark 4.3])

$$\|f\|_{L^{p)}(\Omega)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{|\Omega|} \int_\Omega |f(x)|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}.$$

We recall that still in the same paper by Nakano (Nakano [127]), other norms appear ("first norm", "second norm"), whose expressions involve the so-called *modular bounded linear functionals over* $\mathcal{R}$.

A comparison between the (Nakano) modulars on Riesz spaces and the (Nakano) modulars on real vector spaces is in order. In the former notion, property (P.3.2.5) requires the structure of Riesz space, because the absolute value of some $f \in \mathcal{R}$ is defined in the terms of the order, while the latter requires just the general structure of vector space, which is a minimum requirement to state the convexity property (P.3.3.6).

After a look at the properties defining the two notions and the consequences of the first one, it is immediate to realize that:

Proposition 3.2. *If $\mathcal{R}$ is a Riesz space and if ρ satisfies the properties defining the (Nakano) modulars on Riesz spaces, then ρ satisfies also the properties defining the (Nakano) modulars on real vector spaces.*

We close this section quoting the existence of another notion of modular, again by Nakano, again on real vector spaces (besides the already mentioned Nakano [129]): according to Musielak [124, p.164], Nakano in his second book [128, Sect.78 p.204] gave a notion of modular, which is more restrictive with respect to that one analyzed here. In fact, it is assumed that for every $f \in \mathcal{R}$ there exists a $\lambda > 0$ such that $\rho(\lambda f) < \infty$; therefore, for instance, the Example 3.13 would be excluded (because, for instance, $f(x) = 1/x$ would be such that $\rho(\lambda f) = \infty$ for all $\lambda > 0$) and this restriction is not necessary to get Theorem 3.1.

4. Some post Nakano modulars

4.1. **Musielak-Orlicz modulars: a way to weaken convexity.** A functional ρ defined on a real vector space $\mathcal{R}$ is said to be a *(Musielak-Orlicz) modular* (see Musielak, Orlicz [125]) if it satisfies the following properties ($f, g \in \mathcal{R}$):

(P.4.1.1) $0 \leqslant \rho(f) \leqslant \infty$,

(P.4.1.2) $\rho(f) = \rho(-f)$,

(P.4.1.3) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0$,

(P.4.1.4) $0 \leqslant \alpha \leqslant 1 \;\Rightarrow\; \rho(\alpha f + (1-\alpha)g) \leqslant \rho(f) + \rho(g)$.

It must be noted that the original definition is with (P.4.1.1) replaced by $-\infty \leqslant \rho(f) \leqslant \infty$, but the authors proved (see 1.02(a) therein) that from (P.4.1.2) and (P.4.1.4) one gets that $\rho(f) \geqslant 0$, therefore the notion given in this paper is equivalent to the original one.

Proposition 4.3. *Norms $\|\cdot\|$ on a real vector space $\mathcal{R}$ are (Musielak-Orlicz) modulars.*

Proof. Property (P.4.1.1) follows directly from (2.1). Property (P.4.1.2) follows applying (P.2.1.2) with $\alpha = -1$: $\|f\| = |-1|\|f\| = \|-f\|$. Property (P.4.1.3) coincides with (P.2.1.1). Finally, property (P.4.1.4) follows by (P.2.1.3) and (P.2.1.2): for $0 \leqslant \alpha \leqslant 1$, $f, g \in \mathcal{M}$,

$$\|\alpha f + (1-\alpha)g\| \leqslant \|\alpha f\| + \|(1-\alpha)g\| = \alpha\|f\| + (1-\alpha)\|g\| \leqslant \|f\| + \|g\|.$$

□

Example 4.14. *There exist (Musielak-Orlicz) modulars which are not norms.* If $\mathcal{R}$ is a normed space with norm $\|\cdot\|$, setting $\rho(f) = \|f\|^2$, we have a modular which is not a norm. The fact that it is not a norm has been shown in Example 3.11. On the other hand, ρ is a modular: the proof of properties (P.4.1.1)-(P.4.1.4) is immediate. The reader may check that, in general, the square can be replaced by any increasing, convex function on $[0, +\infty[$ assuming value 0 in the origin. □

The original paper Musielak, Orlicz [125] contains a list of examples of modulars. More examples are e.g. in Maligranda [110] (where the second chapter is entirely dedicated to examples of modulars) and the list in Bardaro, Musielak, Vinti [23, Example 1.5 p.5] (we highlight the interesting (e) in p.7, due to the same authors, mentioned again in the final Section 5).

Next two examples show that the class of Musielak-Orlicz modulars and the class of Nakano modulars (on real vector space) are not comparable with respect to inclusion.

Example 4.15. *There exist Nakano modulars (on real vector space) which are not Musielak-Orlicz modulars.* In Example 3.13, we have seen that there exist Nakano modulars for which $\rho(f) = 0$ does not imply $f = 0$, i.e., such that (P.4.1.3) does not hold. □

Example 4.16. *There exist Musielak-Orlicz modulars which are not Nakano modulars (on real vector space).* Set $\mathcal{R} = \mathbb{R}$, $\rho(x) := \sqrt{|x|}$ for every $f \in \mathbb{R}$. Clearly ρ is a Musielak-Orlicz modular; in particular, (P.4.1.4) holds because for $0 \leqslant \alpha \leqslant 1$, $x, y \in \mathcal{R}$,

$$\rho(\alpha x + (1-\alpha)y) = \sqrt{|\alpha x + (1-\alpha)y|} \leqslant \sqrt{\alpha|x| + (1-\alpha)|y|} \leqslant \sqrt{|x|} + \sqrt{|y|} = \rho(x) + \rho(y).$$

On the other hand, ρ is not a Nakano modular, because the convexity property (P.3.3.6) is lost. □

The previous example shows that Musielak-Orlicz modulars may loose convexity, which has been used in Theorem 3.1 to prove that the Luxemburg-Nakano norm satisfies the properties of the norm. And in fact, we can consider the following:

Example 4.17. *There exist Musielak-Orlicz modulars ρ such that*

$$[f] := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant 1\right\} \tag{4.14}$$

is not a norm. Let $\mathcal{M}$ be the real vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f) = \int_0^1 \sqrt{|f(x)|}dx, \qquad f \in \mathcal{M}.$$

After the chain of inequalities in the previous example, clearly ρ is a modular. However, in this case

$$[f] = \left(\int_0^1 \sqrt{|f(x)|}dx\right)^2$$

which is not a norm, because the triangle inequality (P.2.1.3) fails (see e.g. Castillo, Rafeiro [31, p.51 and Theorem 3.79 p.124], where the authors proved also that $\{[f] < \infty\}$ is not normalizable). More generally, one can consider modulars of the type $\rho(f) = \|f^{1/q}\|_{L^p(0,1)}$, with $q > p$. □

On the other hand, convexity is not necessary, for a modular ρ, to get that (4.14) is a norm: the Musielak-Orlicz modular in Example 4.16 is not convex, nevertheless, in this case (4.14) gives $[f] = |f|$ (see also the example in Maligranda [110, Remark 5 p.8]).

Musielak-Orlicz modulars are the starting point of a rich theory developed in the book by Musielak [124] (see also Maligranda [110]), where the definition has been extended to complex vector spaces, replacing property (P.4.1.2) $\rho(f) = \rho(-f)$ with $\rho(e^{it}f) = \rho(f)$ for all $t \in \mathbb{R}$. They owe its success from a result, proved in the original paper Musielak, Orlicz [125], analogous to Theorem 3.1: from (P.4.1.1)-(P.4.1.4), hence even without the convexity property (P.3.3.6) (replaced by the weaker property (P.4.1.4)), it is possible to build, on the vector subspace of $\mathcal{R}$

$$\widetilde{\mathcal{R}} = \left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\}, \tag{4.15}$$

a kind of "norm", called *$F-$norm* (originated by Mazur, Orlicz [117, 1.82 p.105] in the framework of Orlicz spaces; in Maligranda [113, p.128] it is called *Mazur-Orlicz $F-$norm*), defined by

$$\|f\|_\rho := \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \leqslant \lambda\right\}. \tag{4.16}$$

This functional, which is a modified version of the Luxemburg-Nakano, satisfies *almost* all the properties of a norm (see e.g. Maligranda [110, Theorem 1.1 p.2], Bardaro, Musielak, Vinti [23, Theorem 1.1(a) p.4], Rolewicz [135, Theorem 1.2.4 p.8]; a version for quasi-modular spaces is in Koshi, Shimogaki [92]). For this reason, quite frequently, in literature, the notion of modular is given in the Musielak-Orlicz sense, and therefore, in particular, with the weaker version of convexity (P.4.1.4) (see e.g. Maligranda [110, p.1], Rolewicz [135, p.6], Abdou, Khamsi [1, Definition 2.1 p.4047], Mantellini, Vinti [114], etc.). The missing property is the homogeneity property (P.2.1.2): in fact, at first, in Maligranda [110, Example 1 p.4] it is observed that setting $\rho(f) = \|f\|$, where $\|\cdot\|$ is some norm, then $\|f\|_\rho = \|f\|^{1/2}$ (which clearly does not satisfy (P.2.1.2)). Moreover, it must be noted also that in general **(ii)** of Theorem 3.1 does not hold for Musielak-Orlicz modulars: a careful look at the proof tells that, while the inclusion

$$\left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\} \subset \left\{f \in \mathcal{R} : \text{the set } \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\} \text{ is non-empty}\right\} \tag{4.17}$$

is immediate (see also Maligranda [110, Property 2 p.2]), the opposite inclusion has been proved in **(ii)** using convexity, which is now missing. The fact that the inclusion (4.17) can be proper is shown by the following:

Example 4.18. *There exist Musielak-Orlicz modulars ρ such that*

$$\left\{f \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda f) = 0\right\} \subsetneq \left\{f \in \mathcal{R} : \textit{the set } \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\} \textit{ is non-empty}\right\}.$$

Set $\mathcal{R} = \mathbb{R}$ and

$$\rho(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ 1 + x^2 & \text{if} \quad x \neq 0 \end{cases}, \qquad x \in \mathbb{R}.$$

Note that ρ is not convex and that properties (P.4.1.1)-(P.4.1.4) are satisfied: the first three are immediate; about (P.4.1.4) it suffices to consider the three cases

⋆ $xy = 0$: if, say, $y = 0$, since $\rho(0) = 0$, the property is reduced to $0 \leqslant \alpha \leqslant 1 \;\Rightarrow\; \rho(\alpha x) \leqslant \rho(x)$, which is true because, if $x \neq 0$, the inequality is equivalent to $1 + (\alpha x)^2 \leqslant 1 + x^2$; if $x = 0$, it is reduced to $0 \leqslant 0$;

⋆ $xy > 0$: it suffices to recall that $x \to 1 + x^2$ is convex;

⋆ $xy < 0$: in this case there are two possibilities: if $\alpha x + (1 - \alpha)y \neq 0$, then again it suffices to recall that $x \to 1 + x^2$ is convex; otherwise, the inequality to be proved is reduced, taking into account that $\rho(0) = 0$, to $0 \leqslant \rho(x) + \rho(y)$.

The two sets to be analyzed are different, because clearly

$$\left\{x \in \mathcal{R} : \lim_{\lambda \to 0+} \rho(\lambda x) = 0\right\} = \{0\}$$

and

$$\left\{x \in \mathcal{R} : \text{the set } \left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) < \infty\right\} \text{ is non-empty}\right\} = \mathbb{R}.$$

□

The existence of examples like Example 4.18 motivates the fact that in the literature concerning Musielak-Orlicz modulars the vector space on which norms are considered is that one given in (4.15).

We close this section recalling that Musielak-Orlicz modulars are not only a tool to build norms, but there are contexts where modulars are of interest in their own. For instance, we mention the modular inequalities studied in Cruz-Uribe, Di Fratta, F. [37]: the replacement of norms with modulars, in Harmonic Analysis, has often the effect to restrict the validity of certain inequalities to a smaller set of functions.

4.2. **Q-quasi convex Musielak-Orlicz modulars: the role of Q-quasi convexity.** The notion of Q-quasi convexity in the framework of modular spaces goes back to Bardaro, Musielak, Vinti [22], where the authors considered Musielak-Orlicz modulars ρ on the vector space of the $\mu-$measurable complex-valued functions over a measure space (X, Λ, μ) with $\sigma-$finite measure, with equality $\mu-$a.e. . They imposed on ρ the condition

$$\rho\left(\int_X p(t)h(t)d\mu(t)\right) \leqslant Q \int_X p(t)\rho(Qh(t))d\mu(t)$$

satisfied for some $Q \geqslant 1$, for every $p(\cdot) \in L^1(X)$, $p(\cdot) \geqslant 0$, $\int_X p(t)d\mu(t) = 1$, and every $h(\cdot)$ $\mu-$measurable complex-valued functions over (X, Λ, μ) (note that in Bardaro, Musielak, Vinti [22] the function h is written with a second variable, because the paper concerns double integrals – in fact, in such paper the authors extend the Fubini-Tonelli identity for double integral to the more general context of modulars, obtaining inequalities which are then applied to linear and nonlinear integral operators).

The generalization to the abstract setting is the notion of Q-*quasi convex Musielak-Orlicz modular* (see Bardaro, Mantellini [21]), where Q$\geqslant$ 1 is the parameter involved in the property (additional to (P.4.1.1)-(P.4.1.4))

(Q) $0 \leqslant \alpha \leqslant 1 \;\Rightarrow\; \rho(\alpha f + (1-\alpha)g) \leqslant Q\alpha\rho(Qf) + Q(1-\alpha)\rho(Qg)$.

It is clear that 1-quasi convex modulars are convex and that the greater is Q, the weaker is the condition (for a detailed study of properties quasiconvex functions on $[0,\infty[$ see Gogatishvili, Kokilashvili [67, Section 1 p.646]). However, whatever Q$\geqslant$ 1 is given, Q-quasi convex modulars are an important selection of Musielak-Orlicz modulars: in fact, while in general **(ii)** of Theorem 3.1 does not hold for Musielak-Orlicz modulars, the Q-quasi convexity ensures that **(ii)** of Theorem 3.1 is still true, namely,

$$\left\{f \in \mathcal{R} : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\right\} = \left\{f \in \mathcal{R} : \text{the set } \left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) < \infty\right\} \text{ is non-empty}\right\}$$

(see e.g. Bardaro, Mantellini [21] and references therein). Moreover, it must be stressed that 1-quasi convex modulars, i.e., the class of the convex Musielak-Orlicz modulars, is not comparable with the class of the Nakano modulars on real vector spaces (note that modulars in both classes enjoy the standard convexity property): in fact in Example 4.15, (which goes back to Example 3.13), we saw that there exist Nakano modulars which are not Musielak-Orlicz modulars; on the other hand, we can consider the following:

Example 4.19. *There exist 1-quasi (hence Q-quasi, for any given Q$\geqslant$ 1) convex modulars, i.e., convex Musielak-Orlicz modulars which are not Nakano modulars on real vector spaces.* Let $\mathcal{M}$ be the real vector space of the Lebesgue measurable functions defined in the real interval $(0,1)$, with values in $\mathbb{R}$ (i.e., almost everywhere finite), and let us set

$$\rho(f) = \begin{cases} \operatorname{ess\,sup}|f| & \text{if} \quad \operatorname{ess\,sup}|f| < 1 \\ \infty & \text{if} \quad \operatorname{ess\,sup}|f| \geqslant 1 \end{cases}.$$

Then, ρ is a convex Musielak-Orlicz modular, but it is not a Nakano modular on real vector spaces, because the property (P.3.3.5) $\sup\limits_{0\leqslant\alpha<1} \rho(\alpha f) = \rho(f)$ is missing (it suffices to consider $f \equiv 1$). □

For applications of Q-quasi convex modulars, see e.g. Bardaro, Musielak, Vinti [23], Bardaro, Mantellini [20] (see also Bardaro, Boccuto, Dimitriou, Mantellini [18] for *Q-quasi semiconvex modulars*).

We mention also that Bardaro and Mantellini introduced also another class of abstract modular spaces, which we will not treat in this exposition: they are generated by modulars defined on the vector space of measurable real functions defined on a locally compact Hausdorff topological space (see Bardaro, Mantellini [19] for details).

4.3. **Luxemburg Banach function spaces: a selection of norms of spaces of functions.** A functional ρ defined on $L^0_+(X)$, the cone of the non-negative elements of $L^0(X)$ (which in turn is the vector space of the $\mu-$measurable real-valued functions over a complete measure space (X, Λ, μ) – *complete* means that $\mu(E) = 0$ implies $F \in \Lambda$ for any set $F \subset E$), $\sigma-$additive and $\sigma-$finite, with identification of the functions which are equal almost everywhere on X) is said to be a *Banach function norm* (see Luxemburg [105]) if it satisfies the following properties ($f, g \in L^0_+(X)$):

(P.4.3.1) $0 \leqslant \rho(f) \leqslant \infty$,

(P.4.3.2) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0\,,$

(P.4.3.3) $\rho(f+g) \leqslant \rho(f) + \rho(g)\,,$

(P.4.3.4) $\rho(\alpha f) = \alpha\rho(f), \quad \forall\alpha \geq 0 \quad (0\cdot\infty = 0)$,

(P.4.3.5) $f_n \in L^0_+(X)\ (n \in \mathbb{N})\,,\ f_n \uparrow f$ a.e. $\quad \Rightarrow \quad \rho(f_n) \uparrow \rho(f)$, (P.4.3.6) $E \subset X$ "bounded" $\Rightarrow$ $\rho(\chi_E) < \infty$,

(P.4.3.7) $E \subset X$ "bounded" $\Rightarrow$ $\exists\, c_E \geqslant 0$ such that $\int_E f d\mu \leqslant c_E\, \rho(f), \quad \forall f \in L^0_+(X)$.

Here, as usual, χ_E denotes the characteristic function of E. Moreover, E "bounded"set in a measure space (which cannot mean that E is contained in a ball, since we don't assume that X is a metric space and therefore balls are not defined) means that if X is the union of a fixed, once for all, increasing sequence of sets X_n, $(n \in \mathbb{N})$ of finite measure μ, then there exists $m \in \mathbb{N}$ such that $E \subset X_m$.

Now, let $\mathcal{R}$ be the complex vector space of the $\mu-$measurable complex-valued functions on X such that $\rho(|f|) < \infty$ (here, of course, $|\cdot|$ denotes the modulus in $\mathbb{C}$). While ρ is not defined on a vector space, the functional $\rho(|\cdot|)$ is defined on the complex (and therefore also real) vector space $\mathcal{R}$ (for complex vector spaces which can be considered also real vector spaces the reader may consult Brezis [28, Section 11.4 p.361]), and using (P.4.3.1)-(P.4.3.4) it is immediate to realize that it is a norm on $\mathcal{R}$ (this is stated also, for instance, in Bennett, Sharpley [24, Theorem 1.4 p.3]). Properties (P.4.3.5)-(P.4.3.7) impose further conditions on the norm, which allow to build the theory begun with the Luxemburg's thesis [105] and described in several treatises (for instance, Bennett, Sharpley [24]).

We already noticed, in Proposition 3.1, that all norms are Nakano modulars on real vector spaces. Therefore, it is legitimate to include the functional $\rho(|\cdot|)$ among special Nakano modulars. Analogously, by Proposition 4.3, the functional $\rho(|\cdot|)$ is also a special Musielak-Orlicz modular. The theory goes on setting $\|f\| := \rho(|f|)$ for all $f \in \mathcal{R}$ and the resulting normed spaces, widely known as *Banach function spaces,* include several classical Banach spaces of functions, some of them listed in next example.

Example 4.20. *Examples of Banach function spaces.* Several function spaces treated in the books listed after Proposition 3.1 are Banach function spaces: many are classical, such as Lebesgue, Lorentz, Orlicz spaces (which include the Zygmund spaces and the space denoted by EXP, which is the Orlicz space generated by $\Phi(t) = e^t - 1$) and the Musielak-Orlicz spaces, which include the weighted Lebesgue spaces and the variable exponent Lebesgue spaces (see e.g. Cruz-Uribe, F. [38]). We mention here also the Orlicz-Lorentz spaces, already considered after Theorem 3.1. Of still high interest we mention the grand Lebesgue spaces (see the survey F., Formica, Gogatishvili [53]; a detailed proof of the properties of Banach function spaces is in Anatriello [14]) and small Lebesgue spaces (see e.g. F. [51], F., Rakotoson [61], Capone, F. [30], F., Krbec, Schmeisser [60]). Grand and small Lebesgue spaces stimulated the introduction of several variants and generalizations, such as the weighted grand Lebesgue spaces (see e.g. F., Gupta, Jain [56], F., Kokilashvili [58]), the weighted grand variable Lebesgue spaces (see e.g. F., Kokilashvili, Meskhi [59] and references therein), the $G\Gamma$ spaces introduced in F., Rakotoson [62], which are special cases of the $G\Gamma$ spaces with double weights (see F., Formica, Gogatishvili, Kopaliani, Rakotoson [54], Ahmed, F., Formica, Gogatishvili, Rakotoson [3]). We close the (obviously incomplete) list mentioning the (maybe most) important spaces $L^1 + L^\infty$ and $L^1 \cap L^\infty$: all rearrangement-invariant Banach function spaces X over resonant measure spaces satisfy

$$L^1 \cap L^\infty \;\subset\; X \;\subset\; L^1 + L^\infty$$

(see Bennett Sharpley [24, Theorem 6.6] for details; see also Chill, F., Król [32]). □

It must be stressed that Banach function spaces are, in some sense, the "nicest" Banach spaces of functions, because properties (P.4.3.5)-(P.4.3.7) allow to prove in a unified theory several results (concerning, for instance, separability, duality, reflexivity); however, they do not cover all possible Banach spaces of functions (in spite of the standard name "Banach function spaces"), as we are going to see in next

Example 4.21. *Examples of Banach spaces of functions which are not "Banach function spaces".* An important class of Banach spaces of functions is that of the Sobolev spaces. They are not Banach function spaces: in fact, property (P.4.3.5) applied to the sequence $f, g, \dots, g, \dots$ entrains that $|f| \leqslant |g| \;\Rightarrow \|f\| \leqslant \|g\|$, but this implication is generally not true (see Example 2.4, when we showed that the lattice norm property does not hold). More generally, other examples are all the proper, closed Banach subspaces of Banach function spaces: for instance, if $\Omega \subset \mathbb{R}^n$, the Banach space exp, defined as the closure in $L^\infty(\Omega)$ in EXP, is not a Banach function space.

We quote also the John-Nirenberg BMO space and other BMO-like spaces like the recent ones introduced in Bourgain, Brezis, Mironescu [27] (see also D'Onofrio, Greco, Perfekt, Sbordone, Schiattarella [44]), which are Banach spaces whose elements are measurable functions modulo constants, and only representatives from each equivalence class belong to L^0. □

4.4. **Kozlowski modular function spaces: modulars for applications of function space theory.** In order to treat problems linked to nonlinear operators (e.g., to find a maximal domain of continuity, or to establish the existence of fixed points, or to find conditions for the extension of functions of several complex variables to holomorphic functions, etc.), in 1988 Kozlowski introduced the modular function spaces, i.e., a class of function spaces defined through modulars (hence definitively something more concrete with respect to the abstract theory built from modulars on fairly general structures) having the properties necessary to develop both a general theory and tools for several applications. Starting from integrals, which can be seen as functionals depending both on functions and sets, now modulars are defined on the pair $(\mathcal{E}, \Sigma)$. Here, $\mathcal{E}$ denotes the vector space of all $\mathcal{P}-$simple functions, i.e., (finite) linear combinations of characteristic functions of pairwise disjoint sets in a non-trivial $\delta-$ring (a ring closed with respect to countable intersections) $\mathcal{P}$ of subsets of a non-empty set X, with values in a Banach space $(S, |\cdot|)$. On the other hand, $\Sigma \supset \mathcal{P}$ denotes the smallest $\sigma-$algebra of subsets of X having the properties:

$$(*)_1 \qquad E \cap A \in \mathcal{P}; \quad \forall E \in \mathcal{P},\ A \in \Sigma,$$

$$(*)_2 \qquad \text{there exists a non-decreasing sequence } X_1 \subset X_2 \subset \cdots,\ X_i \in \mathcal{P};\ X = \bigcup_{i=1}^{\infty} X_i.$$

A functional $\rho : (\mathcal{E}, \Sigma) \to [0, \infty]$ is said to be a *(Kozlowski) function modular* (see Kozlowski [94]) if it satisfies the following properties ($f, g \in \mathcal{E}$, $E, E_1, \ldots, E_n, \ldots, F \in \Sigma$):

(P.4.4.1) $0 \leqslant \rho(f, E) \leqslant \infty$,
(P.4.4.2) $\rho(0, E) = 0$,
(P.4.4.3) $|f(x)| \leqslant |g(x)|,\ \forall x \in E \quad \Rightarrow \rho(f, E) \leqslant \rho(g, E)$,
(P.4.4.4) $\rho(f, \cdot)$ is a $\sigma-$submeasure:
(i) $\rho(f, \emptyset) = 0$, (ii) $\rho(f, E) \leqslant \rho(f, F)$ if $E \subset F$, (iii) $\rho(f, \cup E_n) \leqslant \sum \rho(f, E_n)$,
(P.4.4.5) $\lim_{\alpha \to 0+} \overline{\rho}_\alpha(E) := \lim_{\alpha \to 0+} \sup\{\rho(g, E) : g \in \mathcal{E},\ |g(x)| \leqslant \alpha\ \forall x \in E\} = 0$,
(P.4.4.6) $(\ \exists\, \alpha > 0 : \overline{\rho}_\alpha(E) = 0\ ) \quad \Rightarrow \quad (\ \overline{\rho}_\beta(E) = 0,\ \forall\, \beta > 0\ )$,
(P.4.4.7) For every $\alpha > 0$, $\overline{\rho}_\alpha$ is order continuous on $\mathcal{P}$:
for each sequence $(E_n) \subset \Sigma$ such that $E_n \searrow \emptyset$, $\lim_{n\to\infty} \overline{\rho}_\alpha(E_n) = 0$.

Next result, in line with the statements given for the previous modulars, is the essence of Kozlowski [94, Theorem 2.5 p.91]:

Theorem 4.2. *Let $M(X, S)$ be the vector space consisting of all measurable functions $f : X \to S$, i.e., of the functions f for which there exists a sequence of $\mathcal{P}-$simple functions (f_n) such that $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$, with identification of functions which differ only on $\rho-$null sets (i.e., sets on which $\overline{\rho}_\alpha$ is zero for all $\alpha > 0$).*

The functional

$$f \in M(X, S) \quad \to \quad \rho(f, X) := \sup\{\rho(g, X) : g \in \mathcal{E}, |g| \leqslant |f| \text{ in } E\} \in [0, \infty] \tag{4.18}$$

is a Musielak-Orlicz modular.

It must be noticed that by Musielak-Orlicz modular it must be intended the aforementioned extension, made in the Musielak's book [124], of the original Musielak-Orlicz modulars to complex vector spaces.

Theorem 4.2 inserts Kozlowski modulars into the Musielak-Orlicz theory, so that now it is automatically defined the *modular function space* as $\widetilde{\mathcal{R}}$ defined in (4.15) (setting $\mathcal{R} = M(X,S)$ therein), endowed with the F-norm (4.16). According to one of the main features of the Musielak-Orlicz theory, in general the theory is not affected by convexity, even if in some occasions this assumption allows to rephrase/improve the results. As a consequence, from a general perspective, without the explicit addition of assumptions, the modular (4.18) is not a Nakano modular.

After the definition of (Kozlowski) function modular, a natural question arises, namely, how to build such kind of modulars. Roughly speaking, $\rho(f,E)$ mimics the integral of $|f|$ over E, but in general, if ρ is a Nakano modular on $M(X,S)$, the functional

$$\widehat{\rho}\ :(f,E)\ni(\mathcal{E},\Sigma)\ \to\ \widehat{\rho}(f,E):=\rho(f\chi_E)\in[0,\infty] \tag{4.19}$$

is not necessarily a Kozlowski modular: it suffices to consider, for instance, Sobolev spaces norms (see Example (2.4)): since they are norms, they fit into the category of Nakano modulars on real vector spaces (see Proposition 3.1), however, as shown in Example 2.4, the corresponding $\widehat{\rho}$ does not satisfy property (P.4.4.3). In Kozlowski [96] the author uses the trick (4.19) to build modulars, starting from the following notion.

A nontrivial functional ρ defined on $\mathcal{M}_\infty$, the space of all extended measurable functions, i.e., all functions $f : X \to [-\infty,\infty]$ such that there exists a sequence $\{g_n\} \subset \mathcal{E}$, $|g_n| \leqslant |f|$ and $g_n(x) \to f(x)$ for all $x \in X$, is said to be a *regular convex function modular* if it satisfies the following properties ($f, g \in \mathcal{M}_\infty$):

(P.4.4.**i**) $0 \leqslant \rho(f) \leqslant \infty$,
(P.4.4.**ii**) $\rho(f) = \rho(-f)$,
(P.4.4.**iii**) $\rho(f) = 0 \quad \Leftrightarrow \quad f = 0$,
(P.4.4.**iv**) ρ is convex: $0 \leqslant \alpha \leqslant 1 \ \Rightarrow\ \rho(\alpha f + (1-\alpha)g) \leqslant \alpha\rho(f) + (1-\alpha)\rho(g) \qquad (0\cdot\infty = 0)$,
(P.4.4.**v**) ρ is monotone: $|f(x)| \leq |g(x)|,\ \forall x \in X \quad \Rightarrow \quad \rho(f) \leq \rho(g)$,
(P.4.4.**vi**) ρ is orthogonally subadditive: $A, B \in \Sigma,\ A \cap B \neq \emptyset \Rightarrow \rho(f\chi_{A\cup B}) \leq \rho(f\chi_A) + \rho(f\chi_B)$,
(P.4.4.**vii**) ρ has the Fatou property: $f_n \in \mathcal{M}_\infty\ (n \in \mathbb{N})\,,\ f_n(x) \uparrow f(x),\ \forall x \in X \Rightarrow \rho(f_n) \uparrow \rho(f)$,
(P.4.4.**viii**) ρ is order continuous in $\mathcal{E}$: $f_n \in \mathcal{M}_\infty\ (n \in \mathbb{N}),\ |f_n(x)| \downarrow 0,\ \forall x \in X \ \Rightarrow\ \rho(f_n) \downarrow 0$.

The following result (see Kozlowski [96, p.479, after Definition 2.2]) is a method to build modulars:

Proposition 4.4. *Let $\mathcal{M}$ be the vector space consisting of all functions $f \in \mathcal{M}_\infty$ which are $\rho-$a.e. finite, i.e., finite up to a $\rho-$null set ($A \in \Sigma$ is $\rho-$null if $\rho(f\chi_A) = 0$ for every $f \in \mathcal{E}$), with identification of functions which differ only on $\rho-$null sets. If ρ is a regular convex function modular, then the functional in* (4.19)*:*

$$\widehat{\rho}\ :(f,E)\ni(\mathcal{M},\Sigma)\ \to\ \widehat{\rho}(f,E):=\rho(f\chi_E)\in[0,\infty]$$

is a Kozlowski modular.

Regular convex function modulars are particular Nakano modulars on real vector spaces, hence Kozlowski modulars built from Proposition 4.4 are an important category of modulars, which benefit either the theory shown in Kozlowski [95], either, for instance, Theorem 3.1. It is worth to mention, here, that regular convex function modulars generate in a natural way a quite general structure called *modulated topological vector space*, introduced very recently in Kozlowski [97].

Example 4.22. *Examples of Modular function spaces.* Musielak-Orlicz spaces (see Kozlowski [95, Sect. 4.1p.86]), defined through the modular

$$\rho(f,E) = \int_E \phi(x,|f(x)|)d\mu,$$

where μ is a $\sigma-$additive measure on (X, Σ) and $\phi = \phi(x, u)$ is measurable and locally integrable in $x \in X$, continuous in $u \geqslant 0$, and such that $\phi(x, 0) = 0$ for every $x \in X$, $\phi(x, \infty) = \infty$, and, finally, $\phi(\cdot, u) > 0$ $\rho-$a.e. for every $u > 0$ are Modular function spaces. Moreover, one can consider generalizations of Musielak-Orlicz spaces (see Kozlowski [95, Sect. 4.2.1 p.92]) defined through

$$\rho(f, E) = \sup_{\mu} \int_E \phi(x, |f(x)|) d\mu,$$

where μ varies in a family of $\sigma-$additive measures on (X, Σ); the theory includes Lorentz type L^p-spaces (see Kozlowski [95, Sect. 4.2.2 p.93]) defined through

$$\rho(f, E) = \sup_{z} \int_E |f(x)|^p z(x) d\mu,$$

where μ is a fixed measure and z varies in a family of non-negative $\mu-$measurable functions; moreover, the theory includes also countably modulared spaces (see Kozlowski [95, Sect. 4.2.3 p.93]), whose modulars are defined, for instance, as suprema of modulars, and, for instance, also a class of Fenchel-Orlicz spaces (see Kozlowski [95, Sect. 4.2.4 p.94]), which are Orlicz-like spaces constituted by functions Banach-space valued.

Modular function spaces have an extensive applications to Fixed Point Theory: see the pioneering paper Khamsi, Kozlowski, Reich [82] and e.g. Khamsi, Kozlowski [81], Al-Mezel, Al-Solamy, Ansari [9], Alfuraidan, Khamsi, Manav [8], Alfuraidan, Bachar, Khamsi [7] and references therein.

4.5. **Chistyakov modular metric spaces: structures born from modulars on arbitrary sets.** In the opposite direction with respect to the previous class of modulars, there exist questions where the notion of modular over vector spaces (or, in fact, *any* algebraic structure) is restrictive and therefore it may be of help an abstract notion of modular acting on arbitrary sets, which leads to an extension of the theories built by Nakano and Musielak-Orlicz.

Let X be a non-empty set. A functional

$$w : (0, \infty) \times X \times X \to [0, \infty]$$

is said to be a *metric modular on* X (see Chistyakov [33, 34, 35]) if it satisfies the following properties:

(P.4.5.1) Given $x, y \in X$,

$$w(\lambda, x, y) = 0 \text{ for all } \lambda > 0 \Leftrightarrow x = y,$$

(P.4.5.2) $w(\lambda, x, y) = w(\lambda, y, x)$ for all $\lambda > 0, x, y \in X$,

(P.4.5.3) $w(\lambda + \mu, x, y) \leqslant w(\lambda, x, z) + w(\mu, y, z)$ for all $\lambda, \mu > 0, x, y, z \in X$.

Now fix an element $x_0 \in X$ arbitrarily. The subsets of X of the type

$$X_w := \{y \in X : \lim_{\lambda \to \infty} w(\lambda, x, y) = 0\}$$

are said to be *modular sets*. Endowed with the metric given by

$$d_w^0(x, y) = \inf\{\lambda > 0 : w(\lambda, x, y) \leqslant \lambda\},$$

X_w becomes a metric space. Moreover, setting

$$d_w^1(x, y) = \inf_{\lambda > 0} (\lambda + w(\lambda, x, y)),$$

also d_w^1 is a metric and $d_w^0 \leqslant d_w^1 \leqslant 2d_w^0$ on $X_w \times X_w$. The analogy with Theorem 3.1 is evident; for examples, variants and applications see Chistyakov [33, 34], Ansari, Demma, Guran, Lee, Park [16], Aksoy, Karapinar, Erhan, Rakočević [6]. For a survey on Generalized metric spaces, see Khamsi [80].

We close this subsection mentioning the paper Turkoglu, Manav [145], where a new type of modular metric space has been introduced.

4.6. **A recent class of Banach-function-norm-like modulars.** A classical result in Sobolev space theory states that in general, if $\Omega \subset \mathbb{R}^n$ is an open set and $N \subset \Omega$ is a closed set of zero Lebesgue measure,

$$W_0^{1,p}(\Omega) \neq W_0^{1,p}(\Omega \setminus N).$$

At a first look this is surprising, because functions in Sobolev spaces are defined a.e. and the class of measurable functions on Ω coincides with the class of measurable functions on $\Omega \setminus N$. But recalling that functions in Sobolev spaces with zero boundary values are defined as approximations of regular functions which attain value zero on the boundary, then it is clear that even a very *small* set N – even a single point – forces a decay of regular functions on N and the approximation recognizes, or does not recognize, such decay depending on the topology. The interplay between the smallness of N and the topology has its heart in the notion of capacity, and in fact the precise result is the following (see e.g. Heinonen, Kilpeläinen, Martio [72, Theorem 2.43 p.51], Kilpeläinen, Kinnunen, Martio [83, Theorems 4.6, 4.8, Remark 4.2(4)]):

$$W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega \setminus N)$$

(i.e., the closure in $W^{1,p}(\Omega)$ of the space of C^1 functions in Ω coincides with the closure of the space of C^1 functions having compact support in $\Omega \setminus N$) if and only if $cap_p(N) = 0$, where

$$cap_p(N) = \inf \left\{ \int_{\mathbb{R}^n} |u|^p dx + \int_{\mathbb{R}^n} |\nabla u|^p dx : u \in W^{1,p}(\mathbb{R}^n),\ u = 1 \text{ in an open set containing } N \right\}.$$

This statement can be generalized in several different ways: since Lebesgue spaces are particular Orlicz, Lorentz or variable exponent Lebesgue spaces, similarly one can consider the corresponding Sobolev spaces with zero boundary values and prove analogous *removability* results for sets with zero capacity. The whole question has been investigated in a quite general framework in F., Giannetti [55], where the Lebesgue norm (which appears either in the norm in Sobolev spaces and in the definition of capacity) has been replaced by a functional more general than a norm of a Luxemburg Banach function space, namely, a modular. Unfortunately the minimal requirements to impose to modulars, requirements needed for the extension of such classical result, do not match in any notion of modular considered before: all of them have some extra and/or missing property. For instance, the regular convex function modulars considered in Section 4.4 are convex, while for the removability result it is needed just the orthogonal subadditivity. The notion of modular introduced in F., Giannetti [55, Section 2] – which looks very close to that one of Banach function norm, but which allows, for instance, suitable powers of norms (see F., Giannetti [55, Example 2.7]) – is the following: let $\Omega \subset \mathbb{R}^n$ be an open set and let $\mathcal{M}(\Omega)$ be the set of all measurable, real valued functions with respect to the Lebesgue measure, defined on Ω. Given a mapping $\rho_X(\cdot) : \mathcal{M}(\Omega) \to [0, \infty]$, the set

$$X(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_X(u) < \infty\}$$

is a modular function space over Ω if the pair $(X(\Omega), \rho_X)$ satisfies the following properties for all $u,\ v \in \mathcal{M}(\Omega)$:

i $\rho_X(u) = \rho_X(|u|)$ and $\rho_X(u) = 0$ if and only if $u \equiv 0$,
ii $|u| \leqslant |v|$ a.e. $\Rightarrow \rho_X(u) \leqslant \rho_X(v)$,
iii $\rho_X(u + v) \leqslant \rho_X(u) + \rho_X(v) \quad \forall u, v : uv \equiv 0$,
iv if $E \subset \Omega$ is measurable set and $|E| < \infty$, then $\rho_X(\chi_E) < \infty$,
v $|u_j| \uparrow |u|$ a.e. $\Rightarrow \ \rho_X(u_j) \uparrow \rho_X(u)$,
vi $\forall k > 1,\ \exists\, c_k > 1 : \rho_X(ku) \leqslant c_k \rho_X(u)$.

A link with the Banach function spaces in the sense of Bennett, Sharpley is the following: adding the assumption of the convexity of ρ_X and the imbedding in L^1, one gets a full set of axioms which, analogously to Theorem 3.1, allows to build a Banach function space (see F., Giannetti [55, Proposition 4.1] for details).

We close this subsection recalling that property ***vi*** is well known in Orlicz spaces theory, and that in the framework of modular spaces it has been considered also in Krbec [99], where an interpolation method in modular spaces has been built, generalizing the well known K-method (see e.g. Bennett, Sharpley [24], Maligranda [110], Triebel [142]). We note that modulars in Krbec [99] must be convex and do not satisfy necessarily the Fatou property ***v***.

5. Hints for further research

1. For every set of properties defining a modular, prove independence. Namely, for each property, find an example of functional which is *not* a modular and which satisfies *all* the other properties.

2. For each couple of distinct modulars, are there nontrivial additional properties to impose to a modular, so that one notion fits into the other? Results of this type seem missing, even for the popular Musielak-Orlicz notion of modular. Answers could be different in the case of finite/infinite dimensional vector spaces.

3. Recently, in F., Jain [57], it has been shown that if $\rho(f) = \|f\|_{L^1(0,\ell)}$ and $\psi : [0,\ell] \to [0,\infty[$ is absolutely continuous, nondecreasing, and such that $\psi(\ell) > \psi(0)$, $\psi(t) > 0$ for $t > 0$, then

$$\rho\left(\frac{\psi'(\cdot)}{\psi(\cdot)^2}\int_0^{\cdot} f^*(s)\psi(s)ds\right) \approx \rho(f),$$

where by f^* we denote the decreasing rearrangement of f. It would be interesting to extend the validity of these two inequalities to some class of modulars.

4. In Example 4.17, we showed Musielak-Orlicz modulars for which the Luxemburg-Nakano norm is not a norm; however, after Example 4.17 we observed that convexity is not necessary for having that the Luxemburg-Nakano norm is a norm. Find a necessary and sufficient condition for a Musielak-Orlicz modular so that the Luxemburg-Nakano norm is a norm.

5. Different modulars may generate the same Luxemburg-Nakano norm: are there criteria to characterize the whole class of modulars, for a given norm?

6. Is the sum of modulars a modular? Are suprema of modulars, modulars? Is the multiple of a modular, a modular? Is a functional equivalent to a modular, a modular? These questions could be posed by each category of modulars, and in case of negative answers it would be interesting to know conditions on modulars so that the answers become positive. In particular, such kind of questions can be posed for the *grand modulars* introduced in Farroni, F., Giova [50, 3.11 p.762].

7. In literature, several particular Musielak-Orlicz modulars are well known (e.g., those ones generating Orlicz spaces, variable Lebesgue spaces, weighted Lebesgue spaces, double-phase functionals, etc.), frequently applied in several contexts (say, Harmonic Analysis, PDEs, etc.). A maybe less explored functional, in applications, is a functional of the following type, which combines Orlicz and variable exponents:

$$f \to \int_\Omega \Phi(|f|^{p(x)})dx.$$

This functional is suggested by a look at Bardaro, Musielak, Vinti [23, Example 1.5(e) p.7] and it appears e.g. in an estimate for the local maximal operator (see Capone, Cruz-Uribe, F. [29]).

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