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Parallel curves in Minkowski 3-space

3-boyutlu Minkowski uzayında paralel eğriler

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Abstract
For a curve α on the plane, there exists the curve β at a fixed distance r (except for some degenerate cases). The curve β can be alternatively produced as an envelope of circles with the radius r moving along the curve α. As a result, when this structure is translated to three-dimensional space two parallel curves are obtained. In this study defining some time-like parallel curves on 3-dimensional Minkowski space, some important theorems about these are stated and proved.

Keywords: Minkowski 3-space, Parallel curves, Time-like curves

Öz
Düzlemdeki her α eğrisi için belli bir r mesafesinde bulunan β eğrisi mevcuttur (bazı dejenere durumlar hariç). β eğrisi, merkezleri α eğrisi boyunca hareket eden r yarıçaplı dairelerin zarfı olarak alternatif şekilde üretilebilir. Sonuç olarak bu yapı, üç boyutlu uzay üzerine taşınırsa iki paralel eğri elde edilir. Bu çalışmada 3-boyutlu Minkowski uzayında time-like paralel eğriler tanımlanarak bunlarla ilgili bazı önemli teoremler ifade ve ispat edildi.

***Anahtar kelimeler**: 3-boyutlu Minkowski uzayı, Paralel eğriler, Time-like eğriler*

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1. Introduction
1. Giriş

Generating parallel curves on parametric surfaces is an important issue in many industrial settings. The goal of a given initial curve (called a base curve or generator) on a parametric surface is to obtain curves on the surface that are parallel to the generator. Parallel curves mean the curves that are at a given distance from the generator and the distance is measured point by point along certain characteristic curves (on the surface) orthogonal to the generator. Computing these parallel curves is not so easy for ordinary situations, and they generally have computing difficulties. In fact, only partial, incomplete solutions have been reported so far in the literature. Gálvez et al. (2014) introduced a simple but efficient method to fill this gap. Compared with other techniques, the most important feature of their method is its generality: It can be successfully applied to any differentiable parametric surface and to any kind of characteristic curves on surfaces (Gálvez et al., 2014). Parallel curves in the Lorentz plane are defined by Karacan and Bukcu (2008) and they give the relations between the curvatures of these curves.

Chrastinová (2007) examined parallel curves in E^3 for a curve on the plane. According to Chrastinova, there exist two curves as P_+, P_- at a fixed distance r (except for some degenerate cases). The curves P_+, P_- alternatively can be generated as an envelope of circles of with the radius r whose centers move along the curve P. This structure is translated to three-dimensional space and, eventually, two parallel curves are found as well. Keskin et al. (2016) have defined some classifications of parallel curves of adjoint. The Bishop equations of parallel curves are studied according to Bishop frame in Euclidean 3-space is studied and a new characterization of the parallel curve is obtained by using Bishop frame in E³ in (Körpinar et al., 2013).

2. Materials and methods
2. Materyal ve metot

The Minkowski $3-$space E_1^3 is Euclidean $3-$space E^3 provided with the standard flat metric given by

$$\langle , \rangle_L = -dx_1 + dx_3 + dx_3$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $\langle , \rangle_L$ is an indefinite metric, recall that a vector $v \in E_1^3$ can have one of three Lorentz causal characters: It can be space-like if $\langle v, v\rangle_L > 0$ or $v = 0$, time-like if $\langle v, v\rangle_L < 0$ and null (light-like) if $\langle v, v\rangle_L = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in E_1^3 can locally be space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are space-like, time-like or null (light-like) respectively (Kühnel, 2006; Ali & Lopez, 2011; Lopez, 2014).

Minkowski space is originally obtained from the relativity theory of physics. In fact, a time-like curve corresponds to the path of an observer moving at less than the speed of light. Denote with $\{T, N, B\}$ the moving Frenet frame along the curve $\alpha(s)$ in Minkowski space E_1^3. If α is time-like curve, then the Frenet formula is given by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ .N \\ B \end{bmatrix} \tag{1}$$

where $\langle T, T\rangle = -1, \langle N, N\rangle = 1, \langle B, B\rangle = 1, \langle N, N\rangle = 1, \langle T, N\rangle = \langle N, B\rangle = \langle T, B\rangle = 0.$

Definition 2.1. The Lorentz sphere with center a and the radius $r \in \mathbb{R}$ in Minkowski 3-space E_1^3 is defined by

$S_1^2 = \{v \in E_1^3 : \langle v - a, v - a\rangle_L = r^2\}$ (Petrović-Torgašev & Šućurović, 2001).

Theorem 2.1. Let $\alpha(s)$ be a unit speed time-like curve given with curvature $\kappa=\kappa(s)$ in E_1^3. Then the curve α lies on the Lorentz sphere with center a and the radius $r \in \mathbb{R}$ in E_1^3 if and only if $\kappa \neq 0$ is constant and

$a(s)=\alpha(s)-m_2N-m_3B$ where $m_2=-\dfrac{1}{\kappa}, m_3=\pm\sqrt{r^2-\dfrac{1}{\kappa^2}}$ (Petrović-Torgašev & Šućurović, 2001).

Let E^2 be a 2-dimensional vector space and let $u=(u_1,u_2)$, $v=(v_1,v_2)$ be two vectors in E^2. The Lorentz scalar product of u and v is defined by $\langle u,v\rangle_L=-u_1v_1+u_2v_2$. $E_1^2=\left(E^2,\langle,\rangle_L\right)$ is called Lorentz plane.

Let $\alpha(s)=\left(\alpha_1(s),\alpha_2(s)\right)$ be a smooth curve on the plane parameterized by the arc-length s. Then $T_\alpha(s)=\left(\alpha_1'(s),\alpha_2'(s)\right)$, $N_\alpha(s)=\left(\alpha_2'(s),\alpha_1'(s)\right)$ are the unit tangent and the unit principal normal vector of the curve $\alpha(s)$. If T_α is time-like and N_α is space-like for the time-like curve α, then the Frenet formula is given by

$$T'=\kappa N, \qquad N'=\kappa T$$

where κ is a curvature of the curve α (Ikawa, 2003).

Definition 2.2. Let $\alpha(s)$ be a plane curve in E_1^2

$$\beta(s^*)=\alpha(s)+rN_\alpha(s), \quad s^*=s^*(s) \tag{2}$$

where N_α is the unit principal normal vector of α and s^* denotes the arc-length along β at a distance r. The curve β is called a parallel curve with α (Karacan & Bükçü, 2008).

Using the Frenet formula, one can verify the differential equation

$$\frac{ds^*}{ds}=1+r\kappa$$

to determine the arc-length s^*, where $\kappa=\kappa(s)$ is a curvature of α. We exclude the points at which $1+r\kappa=0$. This construction is not useful in three dimensions, but equation (2) can be alternatively obtained as follows. For all points of $\alpha(s)$, we can consider the circle of all points of the curve β such that

$$\langle\alpha(s)-\beta,\alpha(s)-\beta\rangle_L=r^2.$$

Then, the envelope of such circles is the locus of all points of the curve β that satisfy the conditions

$$\langle\alpha(s)-\beta,\alpha(s)-\beta\rangle_L=r^2 \tag{3}$$

$$\langle T_\alpha(s),\alpha(s)-\beta\rangle_L=0 \tag{4}$$

However, equation (4) clearly implies $\beta-\alpha(s)=CN_\alpha(s)$ (a multiple of the unit principal normal vector) and then, equation (3) exactly yields equation (2). The alternative construction makes good sense in three dimensions.

3. Results
3. Bulgular

Definition 3.1. Let $p = p(s) = \left(p_1(s), p_2(s), p_3(s)\right)$ be a time-like curve parameterized by the arc-length s in E_1^3. If three equations hold for the curve p as follows:

$$\left\langle p(s)-P, p(s)-P \right\rangle_L = r^2, \tag{5}$$

$$\langle T_p(s), p(s) - P\rangle_L = 0, \tag{6}$$

$$\left\langle T_p^{'}(s), p(s)-P \right\rangle_L + \left\langle T_p(s), T_p(s) \right\rangle_L = 0 \;, \tag{7}$$

then p is called a time-like parallel curve with the point P in E_1^3. Explicit formula for the point P depending on the parameter s is obtained using the Frenet formula (1)

$$p^{'} = T_p, \quad T_p^{'} = \kappa N_p, \quad N_p = \kappa T_p + \tau B_p, \quad B_p = -\tau N_p \tag{8}$$

where T_p, N_p, B_p are the unit tangent, the unit principal normal and the unit binormal vector of the curve p, respectively. Then equation (6) with $p^{'} = T_p$ implies

$$p(s) - P = m_2 N_p + m_3 B_p \tag{9}$$

for appropriate coefficients m_2, m_3. Substituting this formula in equation (7) with $p^{''} = \kappa N_p$, $\left\langle T_p(s), T_p(s) \right\rangle_L = -1$ we get $m_2\kappa - 1 = 0$. Moreover $m_2^2 + m_3^2 = r^2$ follows from equation (5) and then we get $m_3 = \pm\sqrt{r^2 - \dfrac{1}{\kappa^2}}$. In total

$$P = p(s) - \frac{1}{\kappa} N_p \pm \sqrt{r^2 - \frac{1}{\kappa^2}} B_p \tag{10}$$

where $N_p = N_p(s)$, $B_p = B_p(s)$, $\kappa = \kappa(s)$ can be calculated from the Frenet formula. We also obtain the arc-length s^* of the curve P. It is given by the differential equation

$$\frac{ds^*}{ds} = \sqrt{\left(m_3\tau - m_2^{'}\right)^2 + \left(m_3^{'} + m_2\tau\right)^2}$$

where $\tau = \tau(s)$ is a torsion of the curve p and $m_2 = m_2(s), m_3 = m_3(s)$ are the functions mentioned before. In general, we have parallel curve P at a distance r from a given time-like curve $p \in E_1^3$.

Theorem 3.1. Let p be a time-like curve with arc-length s in E_1^3 and conversely parallel to the curve P at a distance r. Then, the curves p, P can be exchanged.

Proof. Assume that the curves $p(s), P(s^*)$ are parameterized by the arc-lengths s, s^* respectively, and $P(s^*)$ is parallel to $p(s)$. Then, the equations (5), (6), (7) hold for the curves p and P, and we can write the following equations

$$\left\langle p(s)-P(s^*),\, p(s)-P(s^*)\right\rangle_L = r^2$$

whence

$$\left\langle T_p(s),\, p(s)-P(s^*)\right\rangle_L - \frac{ds^*}{ds}\left\langle T_P(s^*),\, p(s)-P(s^*)\right\rangle_L = 0$$

and therefore

$$\left\langle T_P(s^*),\, p(s)-P(s^*)\right\rangle_L = 0 \tag{11}$$

by using equation (6). Analogously equation (7) holds

$$\left\langle T_p(s),\, p(s)-P(s^*)\right\rangle_L = 0$$

from which it follows that

$$\kappa\left\langle N_p(s),\, p(s)-P(s^*)\right\rangle_L - 1 - \frac{ds^*}{ds}\left\langle T_p(s), T_P(s^*)\right\rangle_L = 0$$

and therefore

$$\left\langle T_p(s), T_P(s^*)\right\rangle_L = 0 \tag{12}$$

identical by using equation (7). On the other hand, by differentiating with respect to s equation (11)

$$\frac{ds^*}{ds}\left\langle T_P'(s^*),\, p(s)-P(s^*)\right\rangle_L + \left\langle T_P(s^*), T_p(s))\right\rangle_L - \frac{ds^*}{ds}\left\langle T_p(s^*), T_P(s^*)\right\rangle_L = 0$$

we get

$$\left\langle T_P'(s),\, p(s)-P(s^*)\right\rangle_L - \left\langle T_P(s^*), T_P(s^*)\right\rangle_L = 0$$

by applying equation (12). Altogether we have obtained formula

$$\left\langle P(s^*)-p(s), P(s^*)-p(s)\right\rangle_L = r^2 \tag{13}$$

$$\left\langle T_P(s^*), P(s^*)-p(s)\right\rangle_L = 0 \tag{14}$$

$$\left\langle T_P'(s^*), P(s^*)-p(s)\right\rangle_L + \left\langle T_P(s^*), T_P(s^*)\right\rangle_L = 0$$

identical to equations (5), (6) and (7) after replacing the curves $p(s)$ and $P(s^*)$.

Theorem 3.2. Let $p = p(s)$ be a time-like curve in E_1^3. The curves $\mathcal{P}_\pm$ conversely determines the primary curve p.

Proof. Assume that p is a time-like curve in E_1^3 and given the parallel curve $P = P(s)$ at a distance r. Then, the equations as below hold

$$\begin{aligned} &\left\langle p(s)-P, p(s)-P \right\rangle_L = r^2 \\ &\left\langle T, p(s)-P \right\rangle_L = 0 \\ &\left\langle T'(s), p(s)-P \right\rangle_L + \left\langle T,T \right\rangle_L = 0 \end{aligned} \tag{15}$$

where $\{T, N, B\}$ are Frenet vectors of the curve p. From equation (9) we can write

$$P = p + m_2 N + m_3 B \tag{16}$$

which are $m_2 = \frac{1}{\kappa}$ and $m_3 = \pm\sqrt{r^2 - \frac{1}{\kappa^2}}$. It can be obtained equations (17) and (18) by taking derivative of equation (16) with respect to s

$$\dot{P} = (-m_2' + \tau m_3)N + (-m_3' - \tau m_2)B \ , \tag{17}$$

$$\ddot{P} = (-m_2' + \tau m_3)\kappa T + \left((-m_2' + \tau m_3)\right)' - \tau(-m_3' - \tau m_2)N + \left((-m_3' - \tau m_2)' + \tau(-m_2' + \tau m_3)\right)B \ . \tag{18}$$

Assume that $\mathcal{P} = P + aT + bN + cB$, substitution in equation (15) yields the system

$$\begin{aligned} &r^2 = -a^2 + b^2 + c^2, \\ &0 = (-m_2' + \tau m_3)b + (-m_3' - \tau m_2)c, \\ &0 = a\kappa(-m_2' + \tau m_3) - b((-m_2' + \tau m_3)' - \tau(-m_3' - \tau m_2))n - c((-m_3' - \tau m_2)' + \tau(-m_2' + \tau m_3)) \\ &+(-m_2' + \tau m_3)^2 + (-m_3' - \tau m_2)^2 \end{aligned} \tag{19}$$

and one can verify that $a = 0$, $b = m_2$ and $c = m_3$ is a solution. This provides the curve $\mathcal{P} = p$. Another solution of equation (19) can be obtained by substituting $b = \alpha m_2$, $c = \alpha m_3$. Then, equation (19) turns into a system of two equations

$$\alpha^2 - (\frac{a}{r})^2 = 1, \tag{20}$$

$$a\kappa(-m_2' + \tau m_3) + (1-\alpha)\left((m_2' - \tau m_3)^2 + (m_3' + \tau m_2)^2\right) = 0 \tag{21}$$

for the unknown functions α and a, the intersection of a hyperbola with a straight line. There are two points of intersection. We already know the point $a = 0$, $\alpha = 1$. If equations (20) and (21) are solved for the remaining result is given by

$$\alpha = \frac{\kappa^2(-m_2^1 + \tau m_3)^2 + \left(\left(m_2^1 - \tau m_3\right)^2 + \left(m_3^1 + \tau m_2\right)^2\right)^2}{\left(\left(m_2^1 - \tau m_3\right)^2 + \left(m_3^1 + \tau m_2\right)^2\right)^2 - \kappa^2(-m_2^1 + \tau m_3)^2},$$

$$a=\frac{2\kappa(-m_2^1+\tau m_3)\left(\left(m_2^1-\tau m_3\right)^2+\left(m_3^1+\tau m_2\right)^2\right)}{\left(\left(m_2^1-\tau m_3\right)^2+\left(m_3^1+\tau m_2\right)^2\right)^2-\kappa^2(-m_2^1+\tau m_3)^2}.$$

4. Conclusion

4. Sonuç

The definition of a parallel curve defined in the Euclidean 3-space but not in Minkowski 3- space is given. The arc-length of this curve is expressed in terms of curvature and torsion using appropriate functions. Later it was shown that p and P curves can be substituted. In the last theorem, we obtain that other curves $\mathcal{P}_{\pm}$ parallel to two parallel P and p curves, p is primitive time-like curve.

Author contribution

Yazar katkısı

All authors contributed to the manuscript equally. All authors have read and approved the final manuscript.

Declaration of ethical code

Etik beyanı

The authors of this article declare that the materials and methods used in this study do not require ethical committee approval and/or legal-specific permission.

Conflicts of interest

Çıkar çatışması beyanı

The authors declare no conflict of interest.

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