



(k, m) -type Slant Helices According to Parallel Transport Frame in Euclidean 4-Space

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ABSTRACT. In this work, we describe a Frenet frame in 4-dimensional Euclidean space and call this frame as parallel transport frame (PTF). PTF is an alternative approach to defining a moving frame. This frame is obtained by rotating the tangent vector and the first binormal vector of a unit speed curve by an euler angle and then we give curvature functions according to PTF of the curve. Also, we introduce (k, m) -type slant helices according to PTF in Euclidean 4-Space. Additionally, we obtain the characterization of slant helices according to this frame in \mathbb{E}^4 and give an example of our main result.

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1. INTRODUCTION

Some special curves and surfaces are of great importance in differential geometry. Frenet equations and curvatures of the curve play a key role in studies on curves. Frenet equations are used in the construction of many curve theories, and offer interesting results. One of the most important of these curves is the helix curve and the characterizations on the curvature and torsion play an important role to define special curve types such as so-called helices. Helices are defined as curves whose tangent vector makes a constant angle with a fixed direction. The curve is a (k, m) -type slant helix, then there exists a non-zero constant vector field, and with this constant vector field, the vector fields which have the same index of a parallel transport frame make a constant angle (as in the Euclidean space \mathbb{E}^4). Recently, many research papers related to this concept. The slant helix concept in Euclidean 3-space is defined by Izumiya and Takeuchi [7]. M.A. Soliman, N. H. Abdel-All, R. A. Hussien and T. Youssef studied [8] evolution of space curves using type-3 Bishop frame. S. Yilmaz and M. Turgut [10, 11] introduced a new version of Bishop frame and an application to spherical images. M.Y. Yilmaz and M. Bektaş defined (k, m) -type slant helices in 4-dimensional Euclidean space and null curves in Minkowski 4-space [1, 9]. In addition, F. Bulut and M. Bektaş [3] obtained special helices on Equiform differential geometry of spacelike curves in Minkowski space-time. On the other hand, the Bishop frame was introduced by R. L. Bishop in 1975 utilizing parallel vector fields [2].

In the working, the parallel transport frame (PTF) are studied. The aim of this paper is to introduce slant helix in Euclidean 4-Space according to PTF. In particular, the definition of PTF created with a t parameter is granted. Subsequently, important characterizations are given for PTF to be (k, m) -type slant helix in 4-dimensional Euclidean space. Also, it is given an example for the curve in \mathbb{E}^4 .

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2. GEOMETRIC PRELIMINARIES

In this section we present basic tools for the space curves in \mathbb{E}^4 . A detailed information can be found in [5]. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be an arbitrary curve in Euclidean space \mathbb{E}^4 . Let $\mathbb{E}^4 = \{(a_1, a_2, a_3, a_4) | a_1, a_2, a_3, a_4 \in \mathbb{R}\}$ be a 4-dimensional vector space. The standard scalar product in \mathbb{E}^4 given by

$$\langle a, b \rangle = \sum_{k=1}^4 a_k b_k.$$

Then the curve γ is said to be of unit speed (or parametrized by arclength) if it satisfies $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 1$ for any $t \in I$. The norm of a vector $a \in \mathbb{E}^4$ is defined by $\|a\| = \sqrt{|\langle a, a \rangle|}$.

Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ be the moving frame along the unit speed curve γ ; where $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$ denote, the tangent, the principal normal, first and second binormal vector fields of the curve γ , respectively. Then the following Frenet formulas is given in [11]

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}, \tag{2.1}$$

where κ_1, κ_2 and κ_3 denote the first, the second and the third curvature functions according to of γ , respectively. Here, since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ is an orthonormal frame, then we can write

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}_1, \mathbf{B}_1 \rangle = \langle \mathbf{B}_2, \mathbf{B}_2 \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{B}_1 \rangle = \langle \mathbf{T}, \mathbf{B}_2 \rangle = \langle \mathbf{N}, \mathbf{B}_1 \rangle = \langle \mathbf{N}, \mathbf{B}_2 \rangle = \langle \mathbf{B}_1, \mathbf{B}_2 \rangle = 0. \end{aligned}$$

Now, we rotate this frame in \mathbb{E}^4 arbitrarily. So, when we rotate $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ vectors, we get $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}_1^*, \mathbf{B}_2^*\}$ vectors. If we sort the $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ vectors as $\{V_1, V_2, V_3, V_4\}$ vectors in order, then $\{\mathbf{T}^*, \mathbf{N}^*, \mathbf{B}_1^*, \mathbf{B}_2^*\}$ vectors will be $\{V_1^*, V_2^*, V_3^*, V_4^*\}$. The Euler angles are introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system [6]. Alternative forms can represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra. If θ is an Euler angle, using we using θ_{ij} Euler angles between the vector V_i and the vector V_j^* , $1 \leq i, j \leq 4$, then the rotational transform is given by an arbitrary rotation matrix below:

$$\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \\ V_4^* \end{bmatrix} = \begin{bmatrix} \cos\theta_{11} & \cos\theta_{21} & \cos\theta_{31} & \cos\theta_{41} \\ \cos\theta_{12} & \cos\theta_{22} & \cos\theta_{32} & \cos\theta_{42} \\ \cos\theta_{13} & \cos\theta_{23} & \cos\theta_{33} & \cos\theta_{43} \\ \cos\theta_{14} & \cos\theta_{24} & \cos\theta_{34} & \cos\theta_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}.$$

Here, if we leave the \mathbf{N} and \mathbf{B}_2 vectors motionless, we rotate only the vectors \mathbf{T} and \mathbf{B}_1 in the $\{\mathbf{T}, \mathbf{B}_1\}$ plane by an angle θ . Then, we can find

$$\begin{bmatrix} V_1^* \\ V_2^* \\ V_3^* \\ V_4^* \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2} - \theta\right) & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2}\right) & \cos(0) & \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2} + \theta\right) & \cos\left(\frac{\pi}{2}\right) & \cos\theta & \cos\left(\frac{\pi}{2}\right) \\ \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) & \cos(0) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}.$$

PTF of a curve and the relations between the frame and Frenet frame of the curve in 4-dimensional Euclidean space by using the Euler angles is given in [4]. In this section, we proved with this similar thinking the following theorem.

Theorem 2.1. Let $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)\}$ be a Frenet frame along a unit speed curve $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ and $\{\bar{\mathbf{T}}(t), \bar{\mathbf{N}}(t), \bar{\mathbf{B}}_1(t), \bar{\mathbf{B}}_2(t)\}$ denotes the PTF of the curve γ . The relation can be expressed as

$$\begin{aligned} \bar{\mathbf{T}}(t) &= \mathbf{T}(t)\cos\theta(t) + \mathbf{B}_1(t)\sin\theta(t) \\ \bar{\mathbf{N}}(t) &= \mathbf{N}(t) \\ \bar{\mathbf{B}}_1(t) &= -\mathbf{T}(t)\sin\theta(t) + \mathbf{B}_1(t)\cos\theta(t) \\ \bar{\mathbf{B}}_2(t) &= \mathbf{B}_2(t). \end{aligned} \tag{2.2}$$

The alternative parallel transport frame equations are

$$\begin{bmatrix} \bar{\mathbf{T}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix}' = \begin{bmatrix} 0 & \bar{\kappa}_1 & 0 & \tan\theta\bar{\kappa}_3 \\ -\bar{\kappa}_1 & 0 & \bar{\kappa}_2 & 0 \\ 0 & -\bar{\kappa}_2 & 0 & \bar{\kappa}_3 \\ -\tan\theta\bar{\kappa}_3 & 0 & -\bar{\kappa}_3 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{T}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix},$$

where $\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3$ are curvature functions according to PTF of the curve γ and their expression as follows

$$\begin{aligned} \bar{\kappa}_1(t) &= \kappa_1(t)\cos\theta(t) - \kappa_2(t)\sin\theta(t) \\ \bar{\kappa}_2(t) &= \kappa_2(t)\cos\theta(t) + \kappa_1(t)\sin\theta(t) \\ \bar{\kappa}_3(t) &= \kappa_3(t)\cos\theta(t), \end{aligned}$$

where $\theta(t) = \text{constant}$, $\theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Proof. Let $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)\}$ be a Frenet frame and $\{\bar{\mathbf{T}}(t), \bar{\mathbf{N}}(t), \bar{\mathbf{B}}_1(t), \bar{\mathbf{B}}_2(t)\}$ denotes the PTF along a unit speed curve γ in \mathbb{E}^4 . The relation between Frenet frame and PTF as follows

$$\begin{bmatrix} \bar{\mathbf{T}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta(t) & 0 & \sin\theta(t) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta(t) & 0 & \cos\theta(t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Since the matrix in the middle is an orthogonal matrix, the inverse exists and from inverse matrix, we find

$$\begin{aligned} \mathbf{T} &= \bar{\mathbf{T}}\cos\theta(t) - \bar{\mathbf{B}}_1\sin\theta(t) \\ \mathbf{N} &= \bar{\mathbf{N}} \\ \mathbf{B}_1 &= \bar{\mathbf{T}}\sin\theta(t) + \bar{\mathbf{B}}_1\cos\theta(t) \\ \mathbf{B}_2 &= \bar{\mathbf{B}}_2. \end{aligned} \tag{2.3}$$

Differentiating (2.2) with respect to t , we get

$$\begin{aligned} \bar{\mathbf{T}}' &= \mathbf{T}'\cos\theta(t) - \mathbf{T}\sin\theta(t)\theta'(t) + \mathbf{B}_1'\sin\theta(t) + \mathbf{B}_1\cos\theta(t)\theta'(t) \\ \bar{\mathbf{N}}' &= \mathbf{N}' \\ \bar{\mathbf{B}}_1' &= -\mathbf{T}'\sin\theta(t) - \mathbf{T}\cos\theta(t)\theta'(t) + \mathbf{B}_1'\cos\theta(t) - \mathbf{B}_1\sin\theta(t)\theta'(t) \\ \bar{\mathbf{B}}_2' &= \mathbf{B}_2', \end{aligned}$$

where substituting the equation (2.1) instead of $\mathbf{T}'(t), \mathbf{N}'(t), \mathbf{B}_1'(t), \mathbf{B}_2'(t)$, we get

$$\begin{aligned} \bar{\mathbf{T}}'(t) &= \kappa_1\mathbf{N}\cos\theta - \mathbf{T}(t)\sin\theta\theta' + (-\kappa_2\mathbf{N} + \kappa_3\mathbf{B}_2)\sin\theta + \mathbf{B}_1(t)\cos\theta\theta' \\ \bar{\mathbf{N}}'(t) &= \mathbf{N}' = -\kappa_1\mathbf{T} + \kappa_2\mathbf{B}_1 \\ \bar{\mathbf{B}}_1'(t) &= -\kappa_1\mathbf{N}\sin\theta - \mathbf{T}(t)\cos\theta\theta' + (-\kappa_2\mathbf{N} + \kappa_3\mathbf{B}_2)\cos\theta - \mathbf{B}_1(t)\sin\theta\theta' \\ \bar{\mathbf{B}}_2'(t) &= \mathbf{B}_2' = -\kappa_3\mathbf{B}_1 \end{aligned}$$

and (2.3) results are written instead of $\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2$, we have

$$\begin{aligned} \bar{\mathbf{T}}' &= (\kappa_1\cos\theta(t) - \kappa_2\sin\theta(t))\bar{\mathbf{N}} + \theta'(t)\bar{\mathbf{B}}_1 + \kappa_3\sin\theta(t)\bar{\mathbf{B}}_2 \\ \bar{\mathbf{N}}' &= (\kappa_2\sin\theta(t) - \kappa_1\cos\theta(t))\bar{\mathbf{T}} + (\kappa_2\cos\theta(t) + \kappa_1\sin\theta(t))\bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_1' &= -\bar{\mathbf{T}}\theta'(t) - (\kappa_1\sin\theta(t) + \kappa_2\cos\theta(t))\bar{\mathbf{N}} + \kappa_3\cos\theta(t)\bar{\mathbf{B}}_2 \\ \bar{\mathbf{B}}_2' &= -\kappa_3\sin\theta(t)\bar{\mathbf{T}} - \kappa_3\cos\theta(t)\bar{\mathbf{B}}_1, \end{aligned}$$

where $\theta(t) = \text{constant}$, $\theta'(t) = 0$ and $\theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Then, we define this frame as PTF [10]. So, the PTF formulas are given by

$$\begin{bmatrix} \bar{\mathbf{T}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix}' = \begin{bmatrix} 0 & \bar{\kappa}_1 & 0 & \tan\theta\bar{\kappa}_3 \\ -\bar{\kappa}_1 & 0 & \bar{\kappa}_2 & 0 \\ 0 & -\bar{\kappa}_2 & 0 & \bar{\kappa}_3 \\ -\tan\theta\bar{\kappa}_3 & 0 & -\bar{\kappa}_3 & 0 \end{bmatrix} \begin{bmatrix} \bar{\mathbf{T}} \\ \bar{\mathbf{N}} \\ \bar{\mathbf{B}}_1 \\ \bar{\mathbf{B}}_2 \end{bmatrix},$$

where $\theta(t) = \text{constant}$, $\theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and curvatures are defined by

$$\begin{aligned} \overline{\kappa}_1 &= \kappa_1 \cos\theta - \kappa_2 \sin\theta \\ \overline{\kappa}_2 &= \kappa_2 \cos\theta + \kappa_1 \sin\theta \\ \overline{\kappa}_3 &= \kappa_3 \cos\theta. \end{aligned}$$

The proof is completed. □

3. (k, m)-TYPE SLANT HELICES

In this section, we define (k, m)-type slant helices for PTF in \mathbb{E}^4 such as [9].

Definition 3.1 ([9]). Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a regular unit speed curve in \mathbb{E}^4 with PTF $\{V_1, V_2, V_3, V_4\}$. We call γ is a (k, m)-type slant helix if there exists a non-zero constant vector field $\mathbf{U} \in \mathbb{E}^4$ satisfies $\langle V_k, \mathbf{U} \rangle = a$ and $\langle V_m, \mathbf{U} \rangle = b$ (a, b are constants) for $1 \leq k, m \leq 4, k \neq m$. The constant vector \mathbf{U} is on axis of (k, m)-type slant helix [1, 9]. We decompose \mathbf{U} with respect to PTF $\{\overline{\mathbf{T}}(t), \overline{\mathbf{N}}(t), \overline{\mathbf{B}}_1(t), \overline{\mathbf{B}}_2(t)\}$. Here we denote $V_1 = \overline{\mathbf{T}}, V_2 = \overline{\mathbf{N}}, V_3 = \overline{\mathbf{B}}_1, V_4 = \overline{\mathbf{B}}_2$.

Theorem 3.2. Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\overline{\mathbf{T}}(t), \overline{\mathbf{N}}(t), \overline{\mathbf{B}}_1(t), \overline{\mathbf{B}}_2(t)\}$. If the curve γ is a (1, 2)-type slant helix in \mathbb{E}^4 , then we have

$$\langle \overline{\mathbf{B}}_1, \mathbf{U} \rangle = \frac{\overline{\kappa}_1}{\overline{\kappa}_2} c_1$$

and

$$\langle \overline{\mathbf{B}}_2, \mathbf{U} \rangle = \frac{-\overline{\kappa}_1}{\tan\theta\overline{\kappa}_3} c_2,$$

where c_1 are c_2 are constants.

Proof. If the curve γ is a (1, 2)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} , we can write

$$\langle \overline{\mathbf{T}}, \mathbf{U} \rangle = c_1 \tag{3.1}$$

and

$$\langle \overline{\mathbf{N}}, \mathbf{U} \rangle = c_2 \tag{3.2}$$

are constants. Differentiating (3.1) and (3.2) with respect to t , we get

$$\langle \overline{\mathbf{T}}', \mathbf{U} \rangle = 0$$

and

$$\langle \overline{\mathbf{N}}', \mathbf{U} \rangle = 0.$$

Using PTF, we find the following equations:

$$\langle \overline{\kappa}_1 \overline{\mathbf{N}} + \tan\theta\overline{\kappa}_3 \overline{\mathbf{B}}_2, \mathbf{U} \rangle = 0$$

and it follows that

$$\langle -\overline{\kappa}_1 \overline{\mathbf{T}} + \overline{\kappa}_2 \overline{\mathbf{B}}_1, \mathbf{U} \rangle = 0.$$

In that case, we have

$$\overline{\kappa}_1 \langle \overline{\mathbf{N}}, \mathbf{U} \rangle + \tan\theta\overline{\kappa}_3 \langle \overline{\mathbf{B}}_2, \mathbf{U} \rangle = 0 \tag{3.3}$$

$$-\overline{\kappa}_1 \langle \overline{\mathbf{T}}, \mathbf{U} \rangle + \overline{\kappa}_2 \langle \overline{\mathbf{B}}_1, \mathbf{U} \rangle = 0. \tag{3.4}$$

By setting (3.1) in (3.4), we find

$$-\overline{\kappa}_1 c_1 + \overline{\kappa}_2 \langle \overline{\mathbf{B}}_1, \mathbf{U} \rangle = 0.$$

Substituting (3.4) to (3.3), we obtain as below:

$$\overline{\kappa}_1 c_2 + \tan\theta\overline{\kappa}_3 \langle \overline{\mathbf{B}}_2, \mathbf{U} \rangle = 0.$$

Finally, we obtain

$$\langle \overline{\mathbf{B}}_1, \mathbf{U} \rangle = \frac{\overline{\kappa}_1}{\overline{\kappa}_2} c_1$$

and for $\theta(t) = \text{constant}$, $\theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\langle \overline{B}_2, \mathbf{U} \rangle = \frac{-\overline{\kappa}_1}{\tan\theta\overline{\kappa}_3} c_2.$$

The proof is completed. □

Theorem 3.3. *Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\overline{\mathbf{T}}(t), \overline{\mathbf{N}}(t), \overline{\mathbf{B}}_1(t), \overline{\mathbf{B}}_2(t)\}$. Then, there are no (1, 3)-type slant helix in \mathbb{E}^4 .*

Proof. If the curve γ is a (1, 3)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} , we can write as below:

$$\langle \overline{T}, \mathbf{U} \rangle = c_1 \tag{3.5}$$

and

$$\langle \overline{B}_1, \mathbf{U} \rangle = c_3 \tag{3.6}$$

are constants. Differentiating (3.5) and (3.6) with respect to t , we get

$$\langle \overline{T}', \mathbf{U} \rangle = 0$$

and

$$\langle \overline{B}_1', \mathbf{U} \rangle = 0.$$

Using PTF, we obtain the following equations:

$$\overline{\kappa}_1 \langle \overline{N}, \mathbf{U} \rangle + \tan\theta \overline{\kappa}_3 \langle \overline{B}_2, \mathbf{U} \rangle = 0, \tag{3.7}$$

and we have that

$$-\overline{\kappa}_2 \langle \overline{N}, \mathbf{U} \rangle + \overline{\kappa}_3 \langle \overline{B}_2, \mathbf{U} \rangle = 0. \tag{3.8}$$

(We know that \mathbf{U} is constant). By setting equation (3.8) in equation (3.7), we get

$$\left(\tan\theta \frac{\overline{\kappa}_2 \overline{\kappa}_3}{\overline{\kappa}_1} + \overline{\kappa}_3 \right) \langle \overline{B}_2, \mathbf{U} \rangle = 0,$$

it is clear that

$$\tan\theta \frac{\overline{\kappa}_2 \overline{\kappa}_3}{\overline{\kappa}_1} + \overline{\kappa}_3 \neq 0.$$

Then $\langle \overline{B}_2, \mathbf{U} \rangle = 0$, this means that \mathbf{U} is orthogonal to \overline{B}_2 . Then, there are no (1, 3)-type slant helices in \mathbb{E}^4 . □

Theorem 3.4. *Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\overline{\mathbf{T}}(t), \overline{\mathbf{N}}(t), \overline{\mathbf{B}}_1(t), \overline{\mathbf{B}}_2(t)\}$. If the curve γ is a (1, 4)-type slant helix in \mathbb{E}^4 , then there exists a constant such that*

$$\langle \overline{N}, \mathbf{U} \rangle = -\tan\theta \frac{\overline{\kappa}_3}{\overline{\kappa}_1} c_4$$

and

$$\langle \overline{B}_1, \mathbf{U} \rangle = -\tan\theta c_1,$$

where c_1 are c_4 are constants.

Proof. Assume that γ is a (1, 4)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} , we can write the following equations:

$$\langle \overline{T}, \mathbf{U} \rangle = c_1 \tag{3.9}$$

and

$$\langle \overline{B}_2, \mathbf{U} \rangle = c_4 \tag{3.10}$$

are constants. Differentiating (3.9) and (3.10) with respect to t , we get

$$\langle \overline{T}', \mathbf{U} \rangle = 0$$

and

$$\langle \overline{B}_2', \mathbf{U} \rangle = 0.$$

Using PTF, we get

$$\overline{\kappa}_1 \langle \overline{N}, \mathbf{U} \rangle + \tan\theta \overline{\kappa}_3 \langle \overline{B}_2, \mathbf{U} \rangle = 0 \tag{3.11}$$

and

$$\overline{\kappa_3} \langle \overline{B_1}, \mathbf{U} \rangle + \tan\theta \overline{\kappa_3} \langle \overline{T}, \mathbf{U} \rangle = 0. \tag{3.12}$$

By setting (3.10) in (3.11), we have

$$\langle \overline{N}, \mathbf{U} \rangle = -\tan\theta \frac{\overline{\kappa_3}}{\overline{\kappa_1}} c_4.$$

Substituting (3.9) to (3.12), we obtain

$$\langle \overline{B_1}, \mathbf{U} \rangle = -\tan\theta c_1.$$

The proof is completed. □

Theorem 3.5. *Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\overline{T}(t), \overline{N}(t), \overline{B_1}(t), \overline{B_2}(t)\}$. If the curve γ is a (2, 3)-type slant helix in \mathbb{E}^4 , then there exists a constant such as*

$$\langle \overline{T}, \mathbf{U} \rangle = \frac{\overline{\kappa_2}}{\overline{\kappa_1}} c_3$$

and

$$\langle \overline{B_2}, \mathbf{U} \rangle = \frac{\overline{\kappa_2}}{\overline{\kappa_3}} c_2$$

where c_2 are c_3 are constants.

Proof. If the curve γ is a (2, 3)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} , we can write

$$\langle \overline{N}, \mathbf{U} \rangle = c_2 \tag{3.13}$$

is a constant and

$$\langle \overline{B_1}, \mathbf{U} \rangle = c_3 \tag{3.14}$$

is a constant. Differentiating (3.13) and (3.14) with respect to t , we find

$$\langle \overline{N}', \mathbf{U} \rangle = 0$$

and

$$\langle \overline{B_1}', \mathbf{U} \rangle = 0.$$

Using PTF, the following equations can be obtained:

$$-\overline{\kappa_1} \langle \overline{T}, \mathbf{U} \rangle + \overline{\kappa_2} \langle \overline{B_1}, \mathbf{U} \rangle = 0 \tag{3.15}$$

and

$$-\overline{\kappa_2} \langle \overline{N}, \mathbf{U} \rangle + \overline{\kappa_3} \langle \overline{B_2}, \mathbf{U} \rangle = 0. \tag{3.16}$$

By setting (3.14) in (3.15), we get

$$\langle \overline{T}, \mathbf{U} \rangle = \frac{\overline{\kappa_2}}{\overline{\kappa_1}} c_3$$

and substituting (3.13) to (3.16), we have

$$\langle \overline{B_2}, \mathbf{U} \rangle = \frac{\overline{\kappa_2}}{\overline{\kappa_3}} c_2.$$

The proof is completed. □

Theorem 3.6. *Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\overline{T}(t), \overline{N}(t), \overline{B_1}(t), \overline{B_2}(t)\}$. Then, there are no (2, 4)-type slant helix in \mathbb{E}^4 .*

Proof. If the curve γ is a (2, 4)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} . We can write the following equations:

$$\langle \bar{N}, \mathbf{U} \rangle = c_2 \tag{3.17}$$

and

$$\langle \bar{B}_2, \mathbf{U} \rangle = c_4 \tag{3.18}$$

are constants. By differentiating (3.17) and (3.18) with respect to t , we get the following equations:

$$\langle \bar{N}', \mathbf{U} \rangle = 0$$

and

$$\langle \bar{B}_2', \mathbf{U} \rangle = 0.$$

Using PTF, we find

$$-\bar{\kappa}_1 \langle \bar{T}, \mathbf{U} \rangle + \bar{\kappa}_2 \langle \bar{B}_1, \mathbf{U} \rangle = 0, \tag{3.19}$$

$$\tan\theta \bar{\kappa}_3 \langle \bar{T}, \mathbf{U} \rangle + \bar{\kappa}_3 \langle \bar{B}_1, \mathbf{U} \rangle = 0. \tag{3.20}$$

Substituting (3.19) to (3.20), we obtain as follows:

$$\left(\tan\theta \frac{\bar{\kappa}_2}{\bar{\kappa}_1} + 1 \right) \bar{\kappa}_3 \langle \bar{B}_1, \mathbf{U} \rangle = 0,$$

it is clear that

$$\tan\theta \bar{\kappa}_3 \frac{\bar{\kappa}_2}{\bar{\kappa}_1} + \bar{\kappa}_3 \neq 0.$$

Then $\langle \bar{B}_1, \mathbf{U} \rangle = 0$, this means that \mathbf{U} is orthogonal to \bar{B}_1 . Then, there are no (2, 4)-type slant helices in \mathbb{E}^4 . □

Theorem 3.7. *Let γ be a regular curve in \mathbb{E}^4 with PTF $\{\bar{\mathbf{T}}(t), \bar{\mathbf{N}}(t), \bar{\mathbf{B}}_1(t), \bar{\mathbf{B}}_2(t)\}$. If the curve γ is a (3, 4)-type slant helix in \mathbb{E}^4 , then there exists a constant such that*

$$\langle \bar{T}, \mathbf{U} \rangle = -\frac{c_3}{\tan\theta}$$

and

$$\langle \bar{N}, \mathbf{U} \rangle = \frac{\bar{\kappa}_3}{\bar{\kappa}_2} c_4$$

where c_3, c_4 are constants.

Proof. If the curve γ is a (3, 4)-type slant helix in \mathbb{E}^4 , then for a constant field \mathbf{U} , we can write as follows:

$$\langle \bar{B}_1, \mathbf{U} \rangle = c_3 \tag{3.21}$$

is a constant and

$$\langle \bar{B}_2, \mathbf{U} \rangle = c_4 \tag{3.22}$$

is a constant. By differentiating (3.21) and (3.22) with respect to t , we have the following equations:

$$\langle \bar{B}_1', \mathbf{U} \rangle = 0$$

and

$$\langle \bar{B}_2', \mathbf{U} \rangle = 0.$$

Using PTF, we find as below:

$$-\tan\theta \bar{\kappa}_3 \langle \bar{T}, \mathbf{U} \rangle - \bar{\kappa}_3 \langle \bar{B}_1, \mathbf{U} \rangle = 0 \tag{3.23}$$

and

$$-\bar{\kappa}_2 \langle \bar{N}, \mathbf{U} \rangle + \bar{\kappa}_3 \langle \bar{B}_2, \mathbf{U} \rangle = 0. \tag{3.24}$$

Substituting (3.21) to (3.23) and for $\theta(t) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ we can write

$$\langle \bar{T}, \mathbf{U} \rangle = -\frac{c_3}{\tan\theta},$$

and by setting (3.22) in (3.24), we obtain

$$\langle \bar{N}, \mathbf{U} \rangle = \frac{\bar{\kappa}_3}{\bar{\kappa}_2} c_4.$$

The proof is completed. □

Example 3.8. Let $\gamma(t) = \left(\cos \frac{\sqrt{2}t}{\sqrt{3}}, \sin \frac{\sqrt{2}t}{\sqrt{3}}, -\cos \frac{t}{\sqrt{3}}, -\sin \frac{t}{\sqrt{3}} \right)$ be a regular unit speed curve in \mathbb{E}^4 with Frenet frame $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)\}$. Frenet frame formulas can be expressed as:

$$\begin{aligned} \mathbf{T}(t) &= \left(\frac{1}{\sqrt{3}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \cos \frac{\sqrt{2}t}{\sqrt{3}} \right) \\ \mathbf{N}(t) &= \left(-\frac{2}{\sqrt{5}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{2}{\sqrt{5}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, \frac{1}{\sqrt{5}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, \frac{1}{\sqrt{5}} \sin \frac{\sqrt{2}t}{\sqrt{3}} \right) \\ \mathbf{B}_1(t) &= \left(-\frac{1}{\sqrt{3}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \cos \frac{\sqrt{2}t}{\sqrt{3}} \right) \\ \mathbf{B}_2(t) &= \left(-\frac{1}{\sqrt{5}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{1}{\sqrt{5}} \sin \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{2}{\sqrt{5}} \cos \frac{\sqrt{2}t}{\sqrt{3}}, -\frac{2}{\sqrt{5}} \sin \frac{\sqrt{2}t}{\sqrt{3}} \right), \end{aligned}$$

where

$$\begin{aligned} \kappa_1 &= \frac{1}{\sqrt{3}}, \\ \kappa_2 &= -\frac{\sqrt{2}}{3\sqrt{5}}, \\ \kappa_3 &= \frac{\sqrt{2}}{\sqrt{5}}. \end{aligned}$$

Equation (2.3) results are written instead of $\{\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}_1(t), \mathbf{B}_2(t)\}$ for $\theta(t) = \frac{\pi}{3}$, we find $\{\bar{\mathbf{T}}(t), \bar{\mathbf{N}}(t), \bar{\mathbf{B}}_1(t), \bar{\mathbf{B}}_2(t)\}$ and their curvature functions expression as follows

$$\begin{aligned} \bar{\kappa}_1 &= \frac{\sqrt{15} + 1}{\sqrt{30}}, \\ \bar{\kappa}_2(t) &= \frac{3\sqrt{5} - \sqrt{2}}{6\sqrt{5}}, \\ \bar{\kappa}_3(t) &= \frac{1}{\sqrt{10}}. \end{aligned}$$

4. CONCLUSION

In this study, we construct (k, m)-type slant helices using a PTF equations in \mathbb{E}^4 . (1, k)-type for $2 \leq k \leq 4$ slant helices do not exist according to Frenet frame in \mathbb{E}^4 obtained in [9]. On the contrary, in our paper, we have investigated a PTF in \mathbb{E}^4 and furthermore, we have showed that according to PTF in 4-dimensional Euclidean space \mathbb{E}^4 , (k, m)-type (m is constant) for $1 \leq k \leq 4, k \neq m$ (m is constant) slant helices exist and but (1, 3)-type, (2, 4)-type slant helices do not exist.

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CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

AUTHORS CONTRIBUTION STATEMENT

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

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