Pamukkale Üniversitesi

Sosyal Bilimler Enstitüsü Dergisi

Pamukkale University Journal of Social Sciences Institute

ISSN 1308-2922 E-ISSN 2147-6985

**Article Info/Makale Bilgisi**
**√Received/Geliş:**11.01.2021 **√Accepted/Kabul:**29.03.2021
**DOI:10.30794/pausbed.858658**
**Araştırma Makalesi/ Research Article**

Doğan, O. ve Taşpınar, S. (2021). "Testing Homoskedasticity in Cross-sectional Spatial Autoregressive Models" *Pamukkale Üniversitesi Sosyal Bilimler Enstitüsü Dergisi*, Sayı 46, Denizli, ss. 497-512.

# TESTING HOMOSKEDASTICITY IN CROSS-SECTIONAL SPATIAL AUTOREGRESSIVE MODELS

**Osman DOĞAN*, Süleyman TAŞPINAR****

**Abstract**

In this paper, we suggest an outer-product-of-gradient (OPG) variant of Lagrange multiplier (LM) test statistic for testing homoskedasticity in cross-sectional spatial autoregressive models. We use the OPG method to estimate the asymptotic variance of the score functions in a quasi-maximum likelihood (QML) setting. We use the OPG variance estimate to develop a robust test statistic in the local presence of spatial parameters. Under some general assumptions, we establish the asymptotic distribution of our test statistic under the null and local alternative hypotheses. In a Monte Carlo simulation, we investigate the finite sample size and power properties of our test. Our simulation results show that our tests perform well in finite samples.

**Keywords:** *Spatial autoregressive models, Homoscedasticity, Heteroscedasticity, OPG test, Inference.*

**Jel Codes:** C12, C13, C15.

## YATAY KESİT MEKÂNSAL OTOREGRESİF MODELLERDE EŞİT YAYILIMIN TEST EDİLMESİ

**Öz**

Bu çalışmada, yatay-kesit mekânsal otoregresif modellerde eşit yayılımı test etmek için Lagrange çarpanı (LM) test istatistiğinin bir dış ürün gradyanı (OPG) varyantını öneriyoruz. Skor fonksiyonlarının asimptotik varyansını sözde-maksimum olabilirlik (QML) metodu altında tahmin etmek için OPG yöntemini kullanıyoruz. Mekansal parametrelerin yerel varlığında dirençli bir test istatistiği geliştirmek için OPG varyans tahmincisini kullanıyoruz. Bazı genel varsayımlar altında, test istatistiğimizin asimptotik dağılımını sıfır ve yerel alternatif hipotezler altında gösteriyoruz. Bir Monte Carlo simülasyonunda, testimizin sonlu örnek boyutunu ve güç özelliklerini araştırıyoruz. Simülasyon sonuçlarımız, testlerimizin sonlu örneklerde iyi çalıştığını göstermektedir.

**Anahtar Kelimeler:** *Mekansal otoregresif modeller, Eşit yayılım, Farklı yayılım, OPG testleri, Çıkarım.*

**Jel Kodları:** C12, C13, C15.

*Assist. Prof. Dr., Kütahya Dumlupinar University, Faculty of Economics and Administrative Sciences, KÜTAHYA.
e-posta: yselim.duger@dpu.edu.tr, (https://orcid.org/0000-0003-3523-9671)

**1.Introduction**

In this paper, we consider a spatial model that has a spatial lag in the dependent variable and the disturbance terms, and develop an outer-product-of-gradient (OPG) variant of Lagrange multiplier (LM) test statistic for testing the presence of homoskedasticity in the disturbance terms. As in Breusch & Pagan (1980), we assume that heteroskedasticity is induced by some exogenous variables through an unknown function such that the variance of the ith disturbance term has the form $\sigma_i^2 = \sigma_0^2 h(z_i'\alpha_0)$, where $h$ is an unknown function and $z_i$ is the $p \times 1$ vector of heteroskedasticity inducing exogenous variables with the matching parameter vector $\alpha_0$. Under the assumption that $h(0) = 1$, the presence of homoskedasticity can be tested by testing the null hypothesis $H_0^\alpha: \alpha_0 = 0$. We use the quasi-maximum likelihood (QML) framework to develop our suggested OPG test for testing $H_0^\alpha: \alpha_0 = 0$, and then establish its asymptotic distribution under the null and the local alternative hypotheses.

Our approach is based on the fact that the score-type functions of spatial models forms a martingale difference array. This inherent martingale structure allows us to estimate the asymptotic variance of score-type functions by the OPG method. We use this OPG estimate of variance term to form a score-type test statistic for testing $H_0^\alpha: \alpha_0 = 0$. To make our statistic robust to the presence of spatial dependence in the dependent variable and/or the disturbance terms, we first show how to adjust the standard score functions. We then determine the asymptotic variance of the adjusted score functions to form a quadratic test statistic that has an asymptotic chi-squared distribution with $p$ degrees of freedom under the null hypothesis. Our suggested test has two important properties: (i) it is a valid test in the local presence of spatial parameters, and (ii) its computation only requires the OLS estimator of a linear regression model. We design a Monte Carlo simulation to investigate the finite sample size and power properties of our test statistic. The results show that our test statistic has good finite sample properties.

In the literature, the martingale structure of the score-type functions in spatial models is explored in Kelejian & Prucha (2001, 2010) to develop a central limit theorem for spatial processes. Born & Breitung (2011) use the OPG variance estimate of score-type functions to derive simple test statistics for testing the presence of a spatial lag term in the spatial auto-regressive and spatial error models.[1] The simple test statistics suggested in Born & Breitung (2011) are equivalent to the one-directional LM tests derived in Burridge (1980) and Anselin (1988) when the disturbance terms are homoskedastic. To improve the finite sample size and power properties of LM tests, Baltagi & Yang (2013a, 2013b) suggest the standardized OPG variants of one-directional LM tests by correcting the mean and the variance of the standard LM test statistics. In the context of standard panel data models, the literature shows that we can improve the performance of the standard LM tests through the standardizing process, especially when the asymptotic critical values are used to implement the test (Baltagi et al., 1992; Koenker, 1981; Moulton & Randolph, 1989). The simulation results in Baltagi & Yang (2013a, 2013b) show that the standardized OPG variants for testing spatial dependence can also perform relatively better in finite sample. Jin & Lee (2018) suggest the OPG variants of $C(\alpha)$-type tests (Neyman, 1959) in the ML and generalized method of moments (GMM) settings for testing the presence of spatial dependence in cross-sectional spatial autoregressive models.

As shown in the preceding paragraph, the bulk of literature on testing in spatial econometrics focuses on testing the presence of spatial dependence in spatial models. Testing homoskedasticity in the disturbance terms of a spatial model has not received much attention in the spatial econometric literature. However, it is important to test the presence homoskedasticity in spatial models, since most of the estimators suggested in the literature are formulated under the assumption that the disturbance terms are homoskedastic (Kelejian & Prucha, 2010; Lee, 2004, 2007; Lin & Lee, 2010; Taşpınar et al., 2019). Anselin (1988) specifies heteroskedasticity through a skedastic function and develops an LM test for testing the presence of homoskedasticity in cross-sectional spatial autoregressive models. Baltagi et al. (2020) suggest an adjusted quasi-score method for constructing diagnostic tests for homoskedasticity in spatial cross-sectional, static and dynamic panel models. In comparison to the testing approaches in Anselin (1988) and Baltagi et al. (2020), our approach is relatively simple since the computation of our test does not require the estimation of spatial parameters.

[1] There is a growing literature on spatial econometric models. Among others , see Anselin ( 1988), LeSage & Pace (2009) and Elhorst (2014).

The rest of this paper is organized as follows. In Section 2, we state our spatial model and show how heteroskedasticity is induced through some exogenous variables. In Section 3, we show how our OPG variant of LM test can be systematically derived under some general assumptions. We establish the asymptotic distribution of our test under both null and local alternative hypotheses. In Section 4, we describe our Monte Carlo design and report the simulation results. In Section 5, we conclude and suggest some directions for future studies. Some technical details and simulation results are collected in an appendix.

## 2. Model Specification

We consider the following cross-sectional spatial autoregressive model.

$$Y = \lambda_0 WY + X\beta_0 + U, \quad U = \rho_0 MU + V, \tag{2.1}$$

where $Y = (y_1, y_2, \cdots, y_n)'$ is the $n \times 1$ vector of observations on the dependent variable, $X$ is the $n \times k$ matrix of observations on the exogenous variables with matching parameter vector $\beta_0$, $U = (u_1, u_2, \dots, u_n)'$ is the $n \times 1$ vector of regression disturbance terms, and $V = (v_1, v_2, \cdots, v_n)'$ is the $n \times 1$ vector of innovation (disturbance) terms. The spatial weights matrices that specify spatial dependence among spatial units are denoted by $W$ and $M$. The scalar parameters $\lambda_0$ and $\rho_0$ in (2.1) are called the spatial parameters as they measure the degree of spatial dependence among the elements of $Y$ and $U$, respectively. As in Baltagi et al. (2020), we assume that the elements of $V$ have independent distributions with mean zero and variances specified by the skedastic function $\sigma_i^2 = \sigma_0^2 h(z_i'\alpha_0)$, where $h$ is an unknown smooth function and $z_i$ is the $p \times 1$ vector of heteroskedasticity variables with the matching parameter vector $\alpha_0$. The model is stated with the true parameter vector $\theta_0 = (\beta_0', \sigma_0^2, \lambda_0, \rho_0, \alpha_0')'$, and we use $\theta = (\beta', \sigma^2, \lambda, \rho, \alpha')'$ to denote any other parameter vector values in the parameter space.

When $\alpha_0 = 0$ holds, we get $\sigma_i^2 = \sigma_0^2 h(0)$. To get a homoskedastic model when $\alpha_0 = 0$, following Baltagi et al. (2020), we further assume that the heteroskedasticity function $h$ satisfies $h(0) = 1$. Then, the null hypothesis for testing homoskedasticity against heteroskedasticity can be stated as

$$H_0^{\alpha}: \alpha_0 = 0. \tag{2.2}$$

Our goal, as shown in the next section, is to develop a robust OPG test in the local presence of $\lambda_0$ and $\rho_0$. Let $\phi_0 = (\lambda_0, \rho_0)'$. Consider the null and local alternative hypotheses $H_0^{\phi}: \phi_0 = 0$ and $H_1^{\phi}: \phi_0 = \delta_{\phi}/\sqrt{n}$, where $\delta_{\phi}$ is a non-stochastic vector of constants. Our suggested test statistic for testing $H_0^{\alpha}$ will be valid irrespective of whether $H_0^{\phi}$ or $H_1^{\phi}$ holds.

## 3.The Robust OPG Test

Let $S(\lambda) = I_n - \lambda W$, $R = I_n - \rho M$ and $H(\alpha) = \text{Diag}(h(z_1'\alpha), h(z_2'\alpha), \cdots, h(z_n'\alpha))$, where $\text{Diag}(\cdot)$ forms a diagonal matrix from a given input vector. Then, the likelihood function of the model can be derived as

$$\ln L(\theta) = -\frac{n}{2}\ln(2\pi\sigma^2) + \ln|S(\lambda)| + \ln|R(\rho)| - \frac{1}{2}\ln|H(\alpha)| - \frac{1}{2\sigma^2}V'(\theta)H^{-1}(\alpha)V(\theta), \tag{3.1}$$

where $V(\theta) = R(\rho)S(\lambda)Y - R(\rho)X\beta$ and $|\cdot|$ denotes the determinant operator. Let $S_a(\theta) = \frac{\partial \ln L(\theta)}{\partial a}$ for $a \in \{\beta_0, \sigma_0^2, \lambda_0, \rho_0, \alpha_0\}$ and $\dot{h}(x) = \frac{dh(x)}{dx}$. Using (3.1), under both $H_0^{\alpha}$ and $H_0^{\phi}$, it can be shown that

$$S_{\beta}(\theta_0) = \frac{1}{\sigma_0^2}X'V(\theta_0),\ S_{\sigma^2}(\theta_0) = \frac{1}{2\sigma_0^4}V'(\theta_0)V(\theta_0) - \frac{n}{2\sigma_0^2},$$

$$S_{\lambda}(\theta_0) = \frac{1}{\sigma_0^2}V'(\theta_0)WY,\ S_{\rho}(\theta_0) = \frac{1}{\sigma_0^2}V'(\theta_0)MV(\theta_0), \tag{3.2}$$

$$S_\alpha(\theta) = \frac{1}{2\sigma_0^2}\dot{h}(0)\sum_{i=1}^{n}( v_i^2 - \sigma_0^2)z_i,$$

where $\dot{h}(0)$ is a constant term which will vanish from our test statistic. Let $S(\theta_0) = \left(S'_\beta(\theta_0), S_{\sigma^2}(\theta_0), S_\lambda(\theta_0), S_\rho(\theta_0), S'_\alpha(\theta)\right)'$. Then, the score functions in (3.2) can be expressed as

$$S(\theta_0) = \begin{pmatrix} \Pi_1' V(\theta_0) \\ V'(\theta_0)\Phi_1 V(\theta_0) - E(V'(\theta_0)\Phi_1 V(\theta_0)) \\ V'(\theta_0)\Phi_2 V(\theta_0) - E(V'(\theta_0)\Phi_2 V(\theta_0)) + V'(\theta_0)\Pi_2 \\ V'(\theta_0)\Phi_3 V(\theta_0) - E(V'(\theta_0)\Phi_3 V(\theta_0)) \\ \frac{1}{2\sigma_0^2}\dot{h}(0)\sum_{i=1}^{n}( v_i^2 - \sigma_0^2)z_i \end{pmatrix}, \tag{3.3}$$

where $\Pi_1 = X/\sigma_0^2$, $\Pi_2 = WX\beta_0/\sigma_0^2$, $\Phi_1 = I_n/2\sigma_0^4$, $\Phi_2 = W/\sigma_0^2$ and $\Phi_3 = M/\sigma_0^2$. For any $n \times n$ matrix $\Phi$, consider the following decomposition: $\Phi = \Phi^u + \Phi^d + \Phi^l$, where $\Phi^u$ is the strictly upper triangular matrix, $\Phi^l$ is the strictly lower triangular and $\Phi^d$ is the diagonal matrix. Thus, the quadratic form $V'(\theta_0)\Phi V(\theta_0)$ can be written as $V'(\theta_0)\Phi V(\theta_0) = V'(\theta_0)(\Phi^u + \Phi^l + \Phi^d)V(\theta_0) = V'(\theta_0)\xi(\theta_0) + V'(\theta_0)\Phi^d V(\theta_0)$, where $\xi(\theta_0) = (\Phi^{u\prime} + \Phi^l)V(\theta_0)$. Thus, we can express the zero-mean quadratic term $V'(\theta_0)\Phi V(\theta_0) - E(V'(\theta_0)\Phi V(\theta_0))$ as

$$V'(\theta_0)\Phi V(\theta_0) - E(V'(\theta_0)\Phi V(\theta_0)) = \sum_{i=1}^{n}(v_i\xi_i + (v_i^2 - \sigma_0^2)\phi_{ii}) = \sum_{i=1}^{n} g_i\,(\theta_0),$$

where $g_i(\theta_0) = (v_i\xi_i + (v_i^2 - \sigma_0^2)\phi_{ii})$, $\xi_i$ is the $i$th element of $\xi(\theta_0)$ and $\phi_{ii}$ is the $(i,i)$th element of $\Phi$. Thus, the zero-mean quadratic terms $V'(\theta_0)\Phi_r V(\theta_0) - E(V'(\theta_0)\Phi_r V(\theta_0))$ in (3.3), for $r = 1,2,3$, can be expressed as $V'(\theta_0)\Phi_r V(\theta_0) - E(V'(\theta_0)\Phi_r V(\theta_0)) = \sum_{i=1}^{n} g_{ri}\,(\theta_0)$, where $g_{ri}(\theta_0) = \left(v_i\xi_{r,i} + (v_i^2 - \sigma_0^2)\phi_{r,ii}\right)$ and $\xi_{r,i}$ is the $i$th element of $\xi_r(\theta_0) = (\Phi_r^{u\prime} + \Phi_r^l)V(\theta_0)$. This exposition indicates that we can express $S(\theta_0)$ in the following way:

$$S(\theta_0) = \sum_{i=1}^{n} g_i\,(\theta_0), \tag{3.4}$$

where $g_i(\theta_0) = \left(g'_{i,\beta}(\theta_0), g_{i,\sigma^2}(\theta_0), g_{i,\lambda}(\theta_0), g_{i,\rho}(\theta_0), g'_{i,\alpha}(\theta_0)\right)'$ with $g_{i,\beta}(\theta_0) = \Pi'_{1i}v_i$, $g_{i,\sigma^2}(\theta_0) = g_{1i}(\theta_0)$, $g_{i,\lambda}(\theta_0) = g_{2i}(\theta_0) + \Pi_{2i}v_i$, $g_{i,\rho}(\theta_0) = g_{3i}(\theta_0)$ and $g_{i,\alpha}(\theta_0) = \frac{1}{2\sigma_0^2}\dot{h}(0)(v_i^2 - \sigma_0^2)z_i$. It follows that the sequence $\{g_i(\theta_0), \mathfrak{F}_i\}$ forms a martingale difference sequence (Baltagi et al., 2020; Jin & Lee, 2019; Kelejian & Prucha, 2001), where $\{\mathfrak{F}_i\}$ is the increasing sequence of $\sigma$-fields generated by $(v_1, \cdots, v_i)$ for $i = 1, \cdots, n$. Therefore, the variance of $S(\theta_0)$, $K = \text{Var}(S(\theta_0)) = \sum_{i=1}^{n} E\,(g_i(\theta_0)g'_i(\theta_0))$, can be estimated by its sample analogue $\widetilde{K} = \sum_{i=1}^{n} g_i\,(\tilde{\theta})g'_i(\tilde{\theta})$, where $\tilde{\theta}$ is a constrained consistent estimator of $\theta_0$ under $H_0^\alpha$ and $H_0^\phi$. That is, $\frac{1}{n}K = \frac{1}{n}\widetilde{K} + o_p(1)$.

Let $J_{ab} = -E\left(\frac{\partial^2 \ln L(\theta_0)}{\partial a \partial b'}\right)$ for $a, b \in \{\beta, \sigma^2, \lambda, \rho, \alpha\}$. Let $\tilde{\theta}$ be the constrained estimator of $\theta_0$ under $H_0^\alpha$ and $H_0^\phi$. Then, the sample analogues of $J_{ab}$ terms under $H_0^\alpha$ and $H_0^\phi$ can be derived as

$$\tilde{J}_{\alpha\beta} = \frac{1}{\tilde{\sigma}^2}((l'_p \otimes \tilde{V}) \odot Z)'X, \quad \tilde{J}_{\alpha\sigma^2} = \frac{1}{2\tilde{\sigma}_0^4}Z'\text{Diag}(\tilde{V}\tilde{V}'),$$

$$\tilde{J}_{\alpha\lambda} = \frac{1}{\tilde{\sigma}^2}Z'\left((WY) \odot \tilde{V}\right), \qquad \tilde{J}_{\alpha\rho} = \frac{1}{\tilde{\sigma}^2}Z'\left((WY - WX\tilde{\beta}) \odot \tilde{V}\right),$$

$$\tilde{J}_{\beta\beta}=\frac{1}{\tilde{\sigma}^2}X'X,\quad \tilde{J}_{\beta\sigma^2}=\frac{1}{\tilde{\sigma}^4}X'\tilde{V},\quad \tilde{J}_{\beta\lambda}=\frac{1}{\tilde{\sigma}^2}X'WY,$$

$$\tilde{J}_{\sigma^2\sigma^2}=\frac{1}{\tilde{\sigma}^6}\|\tilde{V}\|^2-\frac{n}{2\tilde{\sigma}^4},\quad \tilde{J}_{\sigma^2\lambda}=\frac{1}{\tilde{\sigma}^4}\tilde{V}'WY,\quad \tilde{J}_{\sigma^2\rho}=\frac{1}{\tilde{\sigma}^4}\tilde{V}'M\tilde{V},$$

$$\tilde{J}_{\lambda\lambda}=\frac{1}{\tilde{\sigma}^2}\|WY\|^2+\mathrm{tr}(W^2),\quad \tilde{J}_{\lambda\rho}=\frac{1}{\tilde{\sigma}^2}Y'W'M^s\tilde{V},\quad \tilde{J}_{\rho\rho}=\frac{1}{\tilde{\sigma}^2}\|M\tilde{V}\|^2,$$

where $l_p$ is the $p\times 1$ vector of ones, $\tilde{V}=V(\tilde{\theta})=Y-X\tilde{\beta}$, $Z=(z_1,z_2,\cdots,z_n)'$ is the $n\times p$ matrix of observations on the heteroskedastic variables, $A^s=A+A'$ for any square matrix $A$, $\otimes$ denotes the Kronecker product, $\odot$ denotes the Hadamart product, $\mathrm{tr}(\cdot)$ is the trace operator, and $\|\cdot\|$ is the Euclidean norm.

Let $\gamma=(\beta',\sigma^2)'$, and define $J_{a\cdot b}=J_{aa}-J_{ab}J_{bb}^{-1}J_{ba}$ and $J_{ab\cdot c}=J_{ab}(\theta_0)-J_{ac}J_{cc}^{-1}J_{cb}$ for $a,b,c\in\{\gamma,\phi,\alpha\}$. Our suggested test for testing $H_0^{\alpha}$ is formulated with $S_\alpha(\tilde{\theta})$. In order to get an asymptotic null distribution for $S_\alpha(\tilde{\theta})$ that is centered around zero in the local presence of $\phi_0$, we need to adjust this score function. In our setting, it can be shown that the adjusted score function takes the following form (for details see the proof of Theorem 1).

$$S_\alpha^*(\tilde{\theta})=S_\alpha(\tilde{\theta})-\tilde{J}_{\alpha\phi\cdot\gamma}\tilde{J}_{\phi\cdot\gamma}^{-1}S_\phi(\tilde{\theta}). \tag{3.5}$$

Note that $S_\alpha^*(\tilde{\theta})$ reduces to the standard score function $S_\alpha(\tilde{\theta})$ when $\tilde{J}_{\alpha\phi\cdot\gamma}=0$ holds. Assume that $K$ is partitioned into sub-matrices $K_{ab}$ according to the dimensions of $a$ and $b$, where $a,b\in\{\gamma,\phi,\alpha\}$. Then, our suggested robust test statistic is formulated with $S_\alpha^*(\tilde{\theta})$ and is given by

$$T=S_\alpha^{*\prime}(\tilde{\theta})\widetilde{D}_{\alpha\cdot\gamma}^{-1}S_\alpha^*(\tilde{\theta}), \tag{3.6}$$

where

$$\widetilde{D}_{\alpha\cdot\gamma}=\tilde{B}_{\alpha\cdot\gamma}+\tilde{J}_{\alpha\phi\cdot\gamma}\tilde{J}_{\phi\cdot\gamma}^{-1}\tilde{B}_{\phi\cdot\gamma}\tilde{J}_{\phi\cdot\gamma}^{-1}\tilde{J}_{\phi\alpha\cdot\gamma}-\tilde{J}_{\alpha\phi\cdot\gamma}\tilde{J}_{\phi\cdot\gamma}^{-1}\tilde{B}_{\phi\alpha\cdot\gamma}-\tilde{B}_{\alpha\phi\cdot\gamma}\tilde{J}_{\phi\cdot\gamma}^{-1}\tilde{J}_{\phi\alpha\cdot\gamma}, \tag{3.7}$$

with

$$\tilde{B}_{\alpha\cdot\gamma}=\widetilde{K}_{\alpha\alpha}+\tilde{J}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\alpha}-\widetilde{K}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\alpha}-\tilde{J}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\alpha}, \tag{3.8}$$

$$\tilde{B}_{\alpha\phi\cdot\gamma}=\widetilde{K}_{\alpha\phi}-\tilde{J}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\phi}-\widetilde{K}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\phi}+\tilde{J}_{\alpha\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\phi}, \tag{3.9}$$

$$\tilde{B}_{\phi\cdot\gamma}=\widetilde{K}_{\phi\phi}+\tilde{J}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\phi}-\widetilde{K}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\phi}-\tilde{J}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\phi}, \tag{3.10}$$

$$\tilde{B}_{\phi\alpha\cdot\gamma}=\widetilde{K}_{\phi\alpha}-\tilde{J}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\alpha}-\widetilde{K}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\alpha}+\tilde{J}_{\phi\gamma}\tilde{J}_{\gamma\gamma}^{-1}\widetilde{K}_{\gamma\gamma}\tilde{J}_{\gamma\gamma}^{-1}\tilde{J}_{\gamma\alpha}. \tag{3.11}$$

To establish the asymptotic null distribution of our suggested test in (3.6), we need the following assumptions.

**Assumption 1**: *The disturbance terms $\{v_i\}_{i=1}^n$ are independent with means zero and variances $\sigma_i^2=\sigma_0^2h(z_i'\alpha_0)$. Furthermore, $E|v_i|^{4+\kappa}<\infty$ for some $\kappa>0$.*

**Assumption 2**: *The spatial weights matrices $W$ and $M$ are uniformly bounded in both row and column sums in absolute value.*

**Assumption 3**: *The exogenous variables in $X$ and $Z$ are non-stochastic and are uniformly bounded, and the limit $\lim_{n\to\infty}\frac{1}{n}X'X$ exists and is non-singular.*

Assumption 1 specifies the distribution of the disturbance terms. We need the moment condition $E|v_i|^{4+\kappa}<\infty$ for some $\kappa>0$ in order to use the central limit theorem in Kelejian & Prucha (2001, 2010) for the linear and quadratic forms of the disturbance terms. Assumptions 2 and 3 are standard assumptions in the spatial econometric literature, e.g., among others, see Kelejian & Prucha (2010) and Lee (2004). To show the asymptotic null distribution of their suggested tests, Baltagi et al. (2020) further assume that $S^{-1}(\lambda)$ and $R^{-1}(\rho)$ are uniformly bounded in both row and column sums in absolute value, uniformly in $\phi$ over a neighborhood of $\phi_0$. In our setting we do not need this restrictive assumption as our vector of score functions and our expressions on $J_{ab}(\theta_0)$ terms do not involve $S^{-1}(\lambda)$ and $R^{-1}(\rho)$.

The following theorem shows the asymptotic distribution of our test $T$ under $H_1^{\alpha}: \alpha_0 = \delta_\alpha/\sqrt{n}$, where $\delta_\alpha$ is a non-stochastic vector of bounded constants.

**Theorem 1**: *Assume that Assumptions 1-3 hold. Then, under $H_1^{\alpha}: \alpha_0 = \delta_\alpha/\sqrt{n}$ and irrespective of $H_0^{\phi}$ and $H_1^{\phi}$, it follows that*

$$T \overset{A}{\sim} \chi_p^2(\vartheta), \tag{3.12}$$

*where $\chi_p^2(\vartheta)$ is the chi-squared distribution with $p$ degrees of freedom and the non-centrality parameter*

$$\vartheta = \delta_\alpha'\left(J_{\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}\right)' D_{\alpha\cdot\gamma}^{-1}\left(J_{\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}\right)\delta_\alpha/n. \tag{3.13}$$

**Proof:** *See Appendix A.*

Theorem 1 shows that the asymptotic distribution of $T$ under $H_1^{\alpha}: \alpha_0 = \delta_\alpha/\sqrt{n}$ is $\chi_p^2(\vartheta)$ irrespective of whether $H_0^{\phi}$ or $H_1^{\phi}$ holds. Thus, the asymptotic null distribution of our test, i.e, when $\delta_\alpha = 0$ holds, is $\chi_p^2$. That is, we can use the critical values from the $\chi_p^2$ distribution to implement our test. Note that our test statistic simplifies significantly if $J_{\alpha\phi\cdot\gamma} = 0$ holds. In that case, the test statistic takes the following form

$$T = S'(\tilde{\theta})\tilde{B}_{\alpha\cdot\gamma}^{-1}S(\tilde{\theta}).$$

The asymptotic distribution of this simplified version under $H_1^{\alpha}: \alpha_0 = \delta_\alpha/\sqrt{n}$ will be $T \overset{A}{\sim} \chi_p^2(\vartheta)$, where $\vartheta = \delta_\alpha' J_{\alpha\cdot\gamma}' B_{\alpha\cdot\gamma}^{-1} J_{\alpha\cdot\gamma}\delta_\alpha$. However, this simplified version is invalid in our case since $J_{\alpha\phi\cdot\gamma} \neq 0$.

**Remark 1**: *Our suggested test statistic in (3.6) is formulated with $\tilde{\theta} = (\tilde{\beta}', \tilde{\sigma}^2, 0, 0, 0')'$, which is the constrained estimator obtained under $H_0^{\alpha}$ and $H_0^{\phi}$. Note that under $H_0^{\alpha}$ and $H_0^{\phi}$, our model reduces to $Y = X\beta_0 + V$. Thus, we can use the OLS estimator $\tilde{\beta} = (X'X)^{-1}X'Y$ and $\tilde{\sigma}^2 = \tilde{V}'\tilde{V}/(n-k)$ to calculate our test statistic. On the other hand, the test statistics suggested in Baltagi et al. (2020) require the estimation of $\phi_0$ by a non-linear optimization routine. Thus, it is clear that our approach has a computational advantage over the approach suggested in Baltagi et al. (2020).*

**4.Monte Carlo Simulations**

To investigate the finite sample size and power properties of our suggested test, we conduct Monte Carlo simulations in this section. We consider the following data generating process

$$Y = \lambda_0 WY + \beta_0 l_n + \beta_1 X + U, \quad U = \rho_0 MU + V,$$

where $l_n$ is the $n \times 1$ vector of ones. We set $\beta_0 = 5$, $\beta_1 = 1$ and $W = M$. The spatial autoregressive parameters take values from $\{0, 0.1, 0.2, 0.5\}$. We use two specifications for the spatial weights matrix: (i) the rook contiguity and (ii) the queen contiguity. To this end, the $n$ spatial units are randomly allocated into $\sqrt{n} \times \sqrt{n}$ square lattice graph. In the rook contiguity case, $w_{ij} = 1$ if the $j$'th observation is adjacent (left/right/above or below) to the $i$'th observation on the graph. In the queen contiguity case, $w_{ij} = 1$ if the $j$'th observation is adjacent to, or shares a border with the $i$'th observation. Both weights matrices are row normalized so that each row sums to unity. The sample size $n$ takes three values $\{169, 361, 529\}$. We generate $X$ according to $X \sim N(0_{n\times 1}, I_n)$. The heteroskedasticity is generated according to $\sigma_i^2 = \sigma_0^2 \exp(z_i\alpha_0)$, where $\sigma_0^2 = 1$, $z_i = x_i$ and $\alpha_0 \in \{0,1\}$. Thus, we will investigate the empirical size properties of our test under the case of $\alpha_0 = 0$, and the empirical power properties under the case of $\alpha_0 = 1$. The disturbance terms are drawn independently as $v_i \sim \sigma_i\epsilon_i$, where (i) $\epsilon_i \sim N(0,1)$ and (ii) $\epsilon_i \sim (\chi_3^2 - 3)/\sqrt{6}$. We set the number of repetitions to $1000$ and use the nominal sizes $\{10\%, 5\%, 1\%\}$ to calculate the empirical rejection rates in all cases.

The Monte Carlo results are summarized in Tables 1-4 for the rook contiguity case. To save space, we present the results for the queen contiguity case in Appendix B. The empirical size properties of the proposed test can be evaluated from the results in Tables 1 and 2. The first row in the first panel of Table 1 shows that when there are no spatial effects the empirical size of the proposed test statistic is close the nominal levels chosen. In this panel, we observe that the test still performs well when there is spatial dependence in the error terms. As expected though, as the value of the $\rho$ becomes quite large, the test becomes oversized. The first rows in panels 2, 3 and 4 can be used to evaluate the empirical size of the test when there is spatial dependence in the outcome variable. We observe that the empirical size of the test statistic is satisfactory but deteriorates as the value of $\lambda$ becomes large. The remaining rows in panels 2, 3 and 4 can be used to evaluate the size properties of the test when there is spatial dependence in the outcome variable and the error terms. We observe that the test statistic performs well. For example, when $\lambda = 0.2$, $\rho = 0.1$ and $n = 361$, at the 5% level, the empirical size of the test is about $6.1\%$. As expected though, when both $\lambda$ and $\rho$ become very large, the test becomes oversized. The overall findings from Table 2 are similar, but the rejection rates are slightly higher when the error terms are generated from the chi-squared distribution as opposed to the standard normal distribution. Although the proposed test statistic is asymptotically robust to non-normality of the error terms, in finite samples its behavior may be affected when the sample size is quite small. The empirical power properties of the test statistic can be assessed from the results in Tables 3 and 4. In both tables, we observe that the proposed test has the satisfactory power and rejects the false null hypothesis with probability close to one. The results seem robust to non-normality of the error terms as well. As expected, the empirical power of the test increases as the sample size increases.

**Table 1 Empirical size properties: Rook contiguity case and $\epsilon_i \sim N(\mathbf{0}, \mathbf{1})$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| 0 | 0 | 0.112 | 0.064 | 0.017 | 0.091 | 0.048 | 0.007 | 0.114 | 0.067 | 0.016 |
| 0 | 0.1 | 0.107 | 0.056 | 0.008 | 0.102 | 0.045 | 0.01 | 0.114 | 0.056 | 0.013 |
| 0 | 0.2 | 0.127 | 0.068 | 0.01 | 0.121 | 0.058 | 0.012 | 0.123 | 0.071 | 0.014 |
| 0 | 0.5 | 0.195 | 0.128 | 0.045 | 0.168 | 0.095 | 0.029 | 0.169 | 0.104 | 0.041 |
| | | | | | | | | | | |
| 0.1 | 0 | 0.111 | 0.061 | 0.017 | 0.094 | 0.048 | 0.018 | 0.114 | 0.06 | 0.015 |
| 0.1 | 0.1 | 0.115 | 0.063 | 0.014 | 0.113 | 0.058 | 0.017 | 0.106 | 0.065 | 0.015 |
| 0.1 | 0.2 | 0.146 | 0.083 | 0.022 | 0.127 | 0.077 | 0.02 | 0.131 | 0.066 | 0.015 |
| 0.1 | 0.5 | 0.193 | 0.12 | 0.044 | 0.197 | 0.133 | 0.049 | 0.191 | 0.128 | 0.031 |
| | | | | | | | | | | |
| 0.2 | 0 | 0.131 | 0.064 | 0.014 | 0.115 | 0.062 | 0.012 | 0.112 | 0.064 | 0.016 |
| 0.2 | 0.1 | 0.127 | 0.07 | 0.017 | 0.114 | 0.061 | 0.009 | 0.12 | 0.061 | 0.018 |
| 0.2 | 0.2 | 0.149 | 0.086 | 0.02 | 0.126 | 0.07 | 0.016 | 0.137 | 0.079 | 0.018 |
| 0.2 | 0.5 | 0.214 | 0.142 | 0.064 | 0.203 | 0.135 | 0.052 | 0.208 | 0.135 | 0.054 |
| | | | | | | | | | | |
| 0.5 | 0 | 0.205 | 0.134 | 0.049 | 0.194 | 0.113 | 0.035 | 0.193 | 0.117 | 0.039 |
| 0.5 | 0.1 | 0.215 | 0.135 | 0.052 | 0.205 | 0.134 | 0.048 | 0.215 | 0.134 | 0.05 |
| 0.5 | 0.2 | 0.245 | 0.185 | 0.092 | 0.241 | 0.169 | 0.068 | 0.271 | 0.19 | 0.092 |
| 0.5 | 0.5 | 0.495 | 0.427 | 0.303 | 0.489 | 0.406 | 0.271 | 0.473 | 0.4 | 0.266 |

**Table 2 Empirical size properties: Rook contiguity case and $\epsilon_i \sim (\chi_3^2 - 3)/6$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| 0 | 0 | 0.128 | 0.069 | 0.024 | 0.138 | 0.073 | 0.014 | 0.129 | 0.076 | 0.015 |
| 0 | 0.1 | 0.162 | 0.085 | 0.026 | 0.142 | 0.082 | 0.017 | 0.136 | 0.069 | 0.02 |
| 0 | 0.2 | 0.186 | 0.117 | 0.029 | 0.145 | 0.08 | 0.018 | 0.157 | 0.079 | 0.022 |
| 0 | 0.5 | 0.22 | 0.144 | 0.056 | 0.181 | 0.108 | 0.035 | 0.18 | 0.096 | 0.021 |
| | | | | | | | | | | |
| 0.1 | 0 | 0.157 | 0.091 | 0.03 | 0.15 | 0.089 | 0.031 | 0.118 | 0.069 | 0.024 |
| 0.1 | 0.1 | 0.139 | 0.082 | 0.025 | 0.14 | 0.082 | 0.022 | 0.135 | 0.072 | 0.019 |
| 0.1 | 0.2 | 0.225 | 0.145 | 0.046 | 0.159 | 0.093 | 0.023 | 0.17 | 0.091 | 0.028 |
| 0.1 | 0.5 | 0.243 | 0.157 | 0.058 | 0.242 | 0.158 | 0.068 | 0.225 | 0.148 | 0.064 |
| | | | | | | | | | | |
| 0.2 | 0 | 0.169 | 0.111 | 0.031 | 0.147 | 0.085 | 0.029 | 0.139 | 0.073 | 0.021 |
| 0.2 | 0.1 | 0.211 | 0.136 | 0.037 | 0.16 | 0.084 | 0.025 | 0.146 | 0.081 | 0.024 |
| 0.2 | 0.2 | 0.207 | 0.128 | 0.038 | 0.171 | 0.104 | 0.031 | 0.169 | 0.097 | 0.029 |
| 0.2 | 0.5 | 0.322 | 0.248 | 0.113 | 0.279 | 0.195 | 0.093 | 0.256 | 0.178 | 0.084 |
| | | | | | | | | | | |
| 0.5 | 0 | 0.248 | 0.167 | 0.065 | 0.259 | 0.175 | 0.077 | 0.265 | 0.172 | 0.06 |
| 0.5 | 0.1 | 0.307 | 0.213 | 0.115 | 0.278 | 0.191 | 0.082 | 0.283 | 0.185 | 0.076 |
| 0.5 | 0.2 | 0.328 | 0.252 | 0.132 | 0.333 | 0.248 | 0.126 | 0.341 | 0.269 | 0.129 |
| 0.5 | 0.5 | 0.586 | 0.52 | 0.391 | 0.542 | 0.459 | 0.344 | 0.535 | 0.448 | 0.316 |

**Table 3 Empirical power properties: Rook contiguity case and $\epsilon_i \sim N(0,1)$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| 0 | 0 | 0.999 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0 | 0.1 | 0.999 | 0.995 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0 | 0.2 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0 | 0.5 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| 0.1 | 0 | 0.997 | 0.996 | 0.995 | 0.998 | 0.998 | 0.998 | 1.000 | 1.000 | 1.000 |
| 0.1 | 0.1 | 0.995 | 0.994 | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.1 | 0.2 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.1 | 0.5 | 0.999 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| 0.2 | 0 | 0.999 | 0.999 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.2 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.2 | 0.5 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| 0.5 | 0 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.5 | 0.1 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.5 | 0.2 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| 0.5 | 0.5 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

**Table 4 Empirical size properties: Rook contiguity case and $\epsilon_i \sim (\chi_3^2 - 3)/6$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| **0** | **0** | 0.998 | 0.996 | 0.989 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |
| **0** | **0.1** | 0.998 | 0.998 | 0.993 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |
| **0** | **0.2** | 0.998 | 0.997 | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0** | **0.5** | 1.000 | 0.999 | 0.990 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| **0.1** | **0** | 0.997 | 0.997 | 0.988 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.1** | 0.999 | 0.998 | 0.986 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.2** | 0.999 | 0.998 | 0.994 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.5** | 1.000 | 1.000 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| **0.2** | **0** | 0.999 | 0.997 | 0.987 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.1** | 1.000 | 0.999 | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.2** | 0.999 | 0.998 | 0.990 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.5** | 0.999 | 0.999 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| | | | | | | | | | | |
| **0.5** | **0** | 1.000 | 0.999 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.1** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.2** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.5** | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

## 5.Conclusion

It is important to test the presence of homoskedasticity in spatial models because the conventional estimators suggested in the literature are usually formulated under the assumption of homoskedastic disturbance terms. In this paper, we suggested an outer-product-of-gradient (OPG) variant of Lagrange multiplier (LM) test statistic for testing homoskedasticity in a spatial model that has a spatial lag term in the dependent variable and the disturbance terms. Our suggested test is simple to compute since its computation only requires the OLS estimator of a linear regression model. More importantly, the implementation of our test does not require knowing whether or not spatial dependence is present in the dependent variable and/or the disturbance terms. That is, our test statistic is robust to the (local) presence of a spatial lag in the dependent variable and the disturbance terms. We designed a Monte Carlo simulation to assess the finite sample properties of our suggested test. Our simulation results attest that our test statistic has good finite sample size and power properties.

For a cross-sectional spatial autoregressive model, we first showed how to adjust the score function, and then how to determine the asymptotic variance of the adjusted score function to formulate a valid test statistic. The same approach can also be used to develop test statistics for some other popular spatial models, including (i) spatial models with endogenous weights matrices, (ii) higher order spatial autoregressive models, and (iii) matrix exponential spatial models. The same approach can also be used to develop test statistics for static and dynamic spatial panel data models with fixed or interactive effects. All of these extensions can be explored in future studies.

## BIBLIOGRAPHY

Anselin, L. (1988). *Spatial econometrics: Methods and Models*. Springer.

Baltagi, B. H., Chang, Y.-J., & Li, Q. (1992). Monte Carlo results on several new and existing tests for the error component model. *Journal of Econometrics*, *54*(1), 95–120. https://doi.org/https://doi.org/10.1016/0304-4076(92)90101-V

Baltagi, B. H., Pirotte, A., & Yang, Z. (2020). Diagnostic tests for homoskedasticity in spatial cross-sectional or panel models. *Journal of Econometrics*. https://doi.org/10.1016/j.jeconom.2020.10.002

Baltagi, B. H., & Yang, Z. (2013a). Standardized LM tests for spatial error dependence in linear or panel regressions. *The Econometrics Journal*, *16*(1), 103–134. https://doi.org/10.1111/j.1368-423X.2012.00385.x

Baltagi, B. H., & Yang, Z. (2013b). Heteroskedasticity and non-normality robust LM tests for spatial dependence. *Regional Science and Urban Economics*, *43*(5), 725–739. https://doi.org/10.1016/j.regsciurbeco.2013.05.001

Born, B., & Breitung, J. (2011). Simple regression-based tests for spatial dependence. *The Econometrics Journal*, *14*(2), 330–342.

Breusch, T. S., & Pagan, A. R. (1980). The Lagrange Multiplier Test and its Applications to Model Specification in Econometrics. *The Review of Economic Studies*, *47*(1).

Burridge, P. (1980). On the Cliff-Ord test for spatial correlation. *Journal of the Royal Statistical Society, Series B: Methodological*, *42*, 107–108.

Elhorst, J. P. (2014). *Spatial Econometrics: From Cross-Sectional Data to Spatial Panels*. Springer Berlin Heidelberg.

Jin, F., & Lee, L. (2018). Outer-product-of-gradients tests for spatial autoregressive models. *Regional Science and Urban Economics*, *72*, 35–57.

Jin, F., & Lee, L. (2019). GEL estimation and tests of spatial autoregressive models. *Journal of Econometrics*, *208*(2), 585–612.

Kelejian, H. H., & Prucha, I. R. (2001). On the Asymptotic Distribution of the Moran I Test Statistic with Applications. *Journal of Econometrics*, *104*(2), 219–257.

Kelejian, H. H., & Prucha, I. R. (2010). Specification and estimation of spatial autoregressive models with autoregressive and heteroskedastic disturbances. *Journal of Econometrics*, *157*, 53–67.

Koenker, R. (1981). A note on studentizing a test for heteroscedasticity. *Journal of Econometrics*, *17*(1), 107–112. https://doi.org/https://doi.org/10.1016/0304-4076(81)90062-2

Lee, L. (2004). Asymptotic Distributions of Quasi-Maximum Likelihood Estimators for Spatial Autoregressive Models. *Econometrica*, *72*(6), 1899–1925.

Lee, L. (2007). GMM and 2SLS estimation of mixed regressive, spatial autoregressive models. *Journal of Econometrics*, *137*(2), 489–514. https://doi.org/10.1016/j.jeconom.2005.10.004

LeSage, J., & Pace, R. K. (2009). Introduction to spatial econometrics. In *Introduction to Spatial Econometrics* (pp. 1–341). https://doi.org/10.1111/j.1467-985x.2010.00681_13.x

Lin, X., & Lee, L. (2010). GMM estimation of spatial autoregressive models with unknown heteroskedasticity. *Journal of Econometrics*, *157*(1), 34–52. https://doi.org/10.1016/j.jeconom.2009.10.035

Moulton, B. R., & Randolph, W. C. (1989). Alternative Tests of the Error Components Model. *Econometrica*, *57*(3), 685–693. http://www.jstor.org/stable/1911059

Neyman, J. (1959). Optimal asymptotic tests of composite statistical hypotheses. In U. Grenander (Ed.), *Probability and Statistics, the Harald Cramer Volume* (pp. 416–444). Wiley, New York.

Taşpınar, S., Doğan, O., & Bera, A. K. (2019). Heteroskedasticity-consistent covariance matrix estimators for spatial autoregressive models. *Spatial Economic Analysis*, *14*(2), 241–268. https://doi.org/10.1080/17421772.2019.1549366

White, H. (1994). *Estimation, Inference and Specification Analysis*. Cambridge University Press. https://doi.org/10.1017/CCOL0521252806

**Appendix**

## A. Proof of Theorem 1

We start with the mean value expansions of $\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta})$, $\frac{1}{\sqrt{n}}S_\phi(\tilde{\theta})$ and $\frac{1}{\sqrt{n}}S_\gamma(\tilde{\theta})$ around $\theta_0$ when both $H_1^\alpha: \alpha_0 = \delta_\alpha/\sqrt{n}$ and $H_1^\phi: \phi_0 = \delta_\phi/\sqrt{n}$ hold:

$$\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\alpha(\theta_0) - \frac{1}{n}\frac{\partial S_\alpha(\overline{\theta})}{\partial \alpha'}\delta_\alpha - \frac{1}{n}\frac{\partial S_\alpha(\overline{\theta})}{\partial \phi'}\delta_\phi + \frac{1}{n}\frac{\partial S_\alpha(\overline{\theta})}{\partial \gamma'}\sqrt{n}(\tilde{\gamma} - \gamma_0), \tag{A.1}$$

$$\frac{1}{\sqrt{n}}S_\phi(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\phi(\theta_0) - \frac{1}{n}\frac{\partial S_\phi(\overline{\theta})}{\partial \alpha'}\delta_\alpha - \frac{1}{n}\frac{\partial S_\phi(\overline{\theta})}{\partial \phi'}\delta_\phi + \frac{1}{n}\frac{\partial S_\phi(\overline{\theta})}{\partial \gamma'}\sqrt{n}(\tilde{\gamma} - \gamma_0), \tag{A.2}$$

$$\frac{1}{\sqrt{n}}S_\gamma(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\gamma(\theta_0) - \frac{1}{n}\frac{\partial S_\gamma(\overline{\theta})}{\partial \alpha'}\delta_\alpha - \frac{1}{n}\frac{\partial S_\gamma(\overline{\theta})}{\partial \phi'}\delta_\phi + \frac{1}{n}\frac{\partial S_\gamma(\overline{\theta})}{\partial \gamma'}\sqrt{n}(\tilde{\gamma} - \gamma_0), \tag{A.3}$$

where $\overline{\theta}$ lies between $\tilde{\theta}$ and $\theta_0$. Under Assumptions 1-3, it can be shown that $\frac{1}{n}\tilde{J}_{ab} = \frac{1}{n}J_{ab} + o_p(1)$ for $a, b \in \{\gamma, \phi, \alpha\}$, e.g., see the proof of Theorem 2.1 in Baltagi et al. (2020). Then, the results in (A.1)-(A.3) can be expressed as

$$\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\alpha(\theta_0) + \frac{1}{n}J_{\alpha\alpha}\delta_\alpha + \frac{1}{n}J_{\alpha\phi}\delta_\phi - \frac{1}{n}J_{\alpha\gamma}\sqrt{n}(\tilde{\gamma} - \gamma_0) + o_p(1), \tag{A.4}$$

$$\frac{1}{\sqrt{n}}S_\phi(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\phi(\theta_0) + \frac{1}{n}J_{\phi\alpha}\delta_\alpha + \frac{1}{n}J_{\phi\phi}\delta_\phi - \frac{1}{n}J_{\phi\gamma}\sqrt{n}(\tilde{\gamma} - \gamma_0) + o_p(1), \tag{A.5}$$

$$\frac{1}{\sqrt{n}}S_\gamma(\tilde{\theta}) = \frac{1}{\sqrt{n}}S_\gamma(\theta_0) + \frac{1}{n}J_{\gamma\alpha}\delta_\alpha + \frac{1}{n}J_{\gamma\phi}\delta_\phi - \frac{1}{n}J_{\gamma\gamma}\sqrt{n}(\tilde{\gamma} - \gamma_0) + o_p(1). \tag{A.6}$$

Note that $\frac{1}{\sqrt{n}}S_\gamma(\tilde{\theta}) = 0$ holds in (A.6) by definition. Then, solving (A.6) for $\sqrt{n}(\tilde{\gamma} - \gamma_0)$ and substituting the resulting equation into (A.4) and (A.5) yields

$$\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta}) = \left(I_p, -J_{\alpha\gamma}J_{\gamma\gamma}^{-1}\right)\begin{pmatrix}\frac{1}{\sqrt{n}}S_\alpha(\theta_0)\\ \frac{1}{\sqrt{n}}S_\gamma(\theta_0)\end{pmatrix} + \frac{1}{n}J_{\alpha\cdot\gamma}\delta_\alpha + \frac{1}{n}J_{\alpha\phi\cdot\gamma}\delta_\phi + o_p(1), \tag{A.7}$$

$$\frac{1}{\sqrt{n}}S_\phi(\tilde{\theta}) = \left(I_2, -J_{\phi\gamma}J_{\gamma\gamma}^{-1}\right)\begin{pmatrix}\frac{1}{\sqrt{n}}S_\phi(\theta_0)\\ \frac{1}{\sqrt{n}}S_\gamma(\theta_0)\end{pmatrix} + \frac{1}{n}J_{\phi\cdot\gamma}\delta_\phi + \frac{1}{n}J_{\phi\alpha\cdot\gamma}\delta_\alpha + o_p(1), \tag{A.8}$$

where $I_p$ is the $p \times p$ identity matrix. Under Assumptions 1-3, it can be shown that $\frac{1}{\sqrt{n}}S(\theta_0) \overset{A}{\sim} N(0, K/n)$, e.g., see the proof of Theorem 2.1 in Baltagi et al. (2020). Then, the asymptotic distribution of $\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta})$ can be determined from (A.7) by using the asymptotic normality of score functions. Thus, it follows that

$$\frac{1}{\sqrt{n}}S_\alpha(\tilde{\theta}) \overset{A}{\sim} N\left(\frac{1}{n}J_{\alpha\cdot\gamma}\delta_\alpha + \frac{1}{n}J_{\alpha\phi\cdot\gamma}\delta_\phi, B_{\alpha\cdot\gamma}/n\right), \tag{A.9}$$

where

$$B_{\alpha\cdot\gamma}=K_{\alpha\alpha}+J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\gamma}J_{\gamma\gamma}^{-1}J_{\alpha\gamma}'-K_{\alpha\gamma}J_{\gamma\gamma}^{-1}J_{\alpha\gamma}'-J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\alpha}. \tag{A.10}$$

Using (A.8) and similar steps, we obtain

$$\frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})\overset{A}{\sim}N\left(\frac{1}{n}J_{\phi\cdot\gamma}\delta_{\phi}+\frac{1}{n}J_{\phi\alpha\cdot\gamma}\delta_{\alpha},\,B_{\phi\cdot\gamma}/n\right), \tag{A.11}$$

where

$$B_{\phi\cdot\gamma}=K_{\phi\phi}+J_{\phi\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\gamma}J_{\gamma\gamma}^{-1}J_{\phi\gamma}'-K_{\phi\gamma}J_{\gamma\gamma}^{-1}J_{\phi\gamma}'-J_{\phi\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\phi}. \tag{A.12}$$

Under $H_0^{\alpha}$, the result in (A.11) shows that $nJ_{\phi\cdot\gamma}^{-1}\frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})\overset{A}{\sim}N\left(\delta_{\phi},\,B_{\phi\cdot\gamma}/n\right)$. Then, using (A.9) and this last result, an adjusted score function that has zero asymptotic mean in the local presence of $\phi_0$ can be derived as

$$\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta})=\frac{1}{\sqrt{n}}\left(S_{\alpha}(\tilde{\theta})-J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}S_{\phi}(\tilde{\theta})\right). \tag{A.13}$$

Next, we show how to determine the asymptotic distribution of $\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta})$. To that end, we combine (A.7) and (A.8), and obtain

$$\begin{pmatrix}\frac{1}{\sqrt{n}}S_{\alpha}(\tilde{\theta})\\ \frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})\end{pmatrix}=\begin{pmatrix}-J_{\alpha\gamma}J_{\gamma\gamma}^{-1} & I_p & 0_{p\times2}\\ -J_{\phi\gamma}J_{\gamma\gamma}^{-1} & 0_{2\times p} & I_2\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{n}}S_{\gamma}(\theta_0)\\ \frac{1}{\sqrt{n}}S_{\alpha}(\theta_0)\\ \frac{1}{\sqrt{n}}S_{\phi}(\theta_0)\end{pmatrix}+\begin{pmatrix}\frac{1}{n}J_{\alpha\cdot\gamma}\delta_{\alpha}+\frac{1}{n}J_{\alpha\phi\cdot\gamma}\delta_{\phi}\\ \frac{1}{n}J_{\phi\cdot\gamma}\delta_{\phi}+\frac{1}{n}J_{\phi\alpha\cdot\gamma}\delta_{\alpha}\end{pmatrix}+o_p(1), \tag{A.14}$$

where $0_{a\times b}$ is the $a\times b$ matrix of zeros. Using $\frac{1}{\sqrt{n}}S(\theta_0)\overset{A}{\sim}N(0,K/n)$, the joint asymptotic distribution of $\frac{1}{\sqrt{n}}S_{\alpha}(\tilde{\theta})$ and $\frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})$ can be determined as

$$\begin{pmatrix}\frac{1}{\sqrt{n}}S_{\alpha}(\tilde{\theta})\\ \frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})\end{pmatrix}\overset{A}{\sim}N\left(\begin{pmatrix}\frac{1}{n}J_{\alpha\cdot\gamma}\delta_{\alpha}+\frac{1}{n}J_{\alpha\phi\cdot\gamma}\delta_{\phi}\\ \frac{1}{n}J_{\phi\cdot\gamma}\delta_{\phi}+\frac{1}{n}J_{\phi\alpha\cdot\gamma}\delta_{\alpha}\end{pmatrix},\begin{pmatrix}B_{\alpha\cdot\gamma}/n & B_{\alpha\phi\cdot\gamma}/n\\ B_{\phi\alpha\cdot\gamma}/n & B_{\phi\cdot\gamma}/n\end{pmatrix}\right), \tag{A.15}$$

where

$$B_{\alpha\cdot\gamma}=K_{\alpha\alpha}+J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\gamma}J_{\gamma\gamma}^{-1}J_{\alpha\gamma}'-K_{\alpha\gamma}J_{\gamma\gamma}^{-1}J_{\alpha\gamma}'-J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\alpha}, \tag{A.16}$$

$$B_{\alpha\phi\cdot\gamma}=K_{\alpha\phi}-J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\phi}-K_{\alpha\gamma}J_{\gamma\gamma}^{-1}J_{\gamma\phi}+J_{\alpha\gamma}J_{\gamma\gamma}^{-1}K_{\gamma\gamma}J_{\gamma\gamma}^{-1}J_{\gamma\phi}, \tag{A.17}$$

$B_{\phi\cdot\gamma}$ and $B_{\phi\alpha\cdot\gamma}$ are defined similarly. Using $\frac{1}{n}\tilde{J}_{ab}=\frac{1}{n}J_{ab}+o_p(1)$, the adjusted score function can be written as

$$\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta})=\left(I_2,\quad -J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}\right)\begin{pmatrix}\frac{1}{\sqrt{n}}S_{\alpha}(\tilde{\theta})\\ \frac{1}{\sqrt{n}}S_{\phi}(\tilde{\theta})\end{pmatrix}+o_p(1). \tag{A.18}$$

Then, using (A.15) under $H_0^{\alpha}$ and $H_1^{\phi}$, we obtain

$$\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta}) \overset{A}{\sim} N\big(0,\, D_{\alpha\cdot\gamma}/n\big), \tag{A.19}$$

where

$$D_{\alpha\cdot\gamma} = B_{\alpha\cdot\gamma} + J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}B_{\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}B_{\phi\alpha\cdot\gamma} - B_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}. \tag{A.20}$$

Thus, the asymptotic distribution of $\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta})$ under $H_1^{\alpha}$ and $H_0^{\phi}$ can be derived as

$$\frac{1}{\sqrt{n}}S_{\alpha}^{*}(\tilde{\theta}) \overset{A}{\sim} N\left(\frac{1}{n}\big(J_{\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}\big)\delta_{\alpha},\, D_{\alpha\cdot\gamma}/n\right), \tag{A.21}$$

Then, using Theorem 8.6 of White (1994) on the asymptotic distribution of quadratic forms, we obtain $T \overset{A}{\sim} \chi_p^2(\vartheta)$, where $\vartheta = \delta_{\alpha}'\big(J_{\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}\big)' D_{\alpha\cdot\gamma}^{-1}\big(J_{\alpha\cdot\gamma} - J_{\alpha\phi\cdot\gamma}J_{\phi\cdot\gamma}^{-1}J_{\phi\alpha\cdot\gamma}\big)\delta_{\alpha}/n$ is the non-centrality parameter.

**B. Additional Simulation Results**

**Table 5 Empirical size properties: Queen contiguity case and $\epsilon_i \sim N(\mathbf{0},\mathbf{1})$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| **0** | **0** | 0.096 | 0.049 | 0.007 | 0.099 | 0.045 | 0.007 | 0.092 | 0.051 | 0.009 |
| **0** | **0.1** | 0.113 | 0.062 | 0.013 | 0.111 | 0.054 | 0.009 | 0.099 | 0.049 | 0.007 |
| **0** | **0.2** | 0.136 | 0.075 | 0.014 | 0.121 | 0.056 | 0.012 | 0.105 | 0.057 | 0.013 |
| **0** | **0.5** | 0.139 | 0.074 | 0.023 | 0.152 | 0.088 | 0.018 | 0.125 | 0.08 | 0.019 |
| **0.1** | **0** | 0.149 | 0.077 | 0.012 | 0.102 | 0.046 | 0.009 | 0.11 | 0.059 | 0.019 |
| **0.1** | **0.1** | 0.125 | 0.065 | 0.018 | 0.125 | 0.067 | 0.012 | 0.105 | 0.062 | 0.016 |
| **0.1** | **0.2** | 0.117 | 0.055 | 0.016 | 0.104 | 0.054 | 0.016 | 0.129 | 0.058 | 0.013 |
| **0.1** | **0.5** | 0.148 | 0.085 | 0.021 | 0.163 | 0.097 | 0.026 | 0.156 | 0.091 | 0.021 |
| **0.2** | **0** | 0.128 | 0.07 | 0.011 | 0.107 | 0.057 | 0.015 | 0.117 | 0.061 | 0.011 |
| **0.2** | **0.1** | 0.118 | 0.06 | 0.011 | 0.118 | 0.064 | 0.015 | 0.109 | 0.058 | 0.015 |
| **0.2** | **0.2** | 0.137 | 0.073 | 0.007 | 0.126 | 0.058 | 0.02 | 0.103 | 0.055 | 0.013 |
| **0.2** | **0.5** | 0.152 | 0.094 | 0.032 | 0.154 | 0.082 | 0.022 | 0.175 | 0.11 | 0.033 |
| **0.5** | **0** | 0.159 | 0.085 | 0.026 | 0.164 | 0.089 | 0.031 | 0.146 | 0.079 | 0.029 |
| **0.5** | **0.1** | 0.169 | 0.101 | 0.025 | 0.194 | 0.124 | 0.045 | 0.165 | 0.1 | 0.033 |
| **0.5** | **0.2** | 0.183 | 0.123 | 0.05 | 0.204 | 0.135 | 0.053 | 0.201 | 0.116 | 0.043 |
| **0.5** | **0.5** | 0.333 | 0.267 | 0.156 | 0.335 | 0.254 | 0.128 | 0.327 | 0.239 | 0.132 |

**Table 6 Empirical size properties: Queen contiguity case and $\epsilon_i \sim (\chi_3^2 - 3)/6$**

| | | n=169 | | | n=361 | | | n=529 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | 10% | 5% | 1% | 10% | 5% | 1% | 10% | 5% | 1% |
| **0** | **0** | 0.17 | 0.099 | 0.044 | 0.129 | 0.069 | 0.012 | 0.114 | 0.067 | 0.013 |
| **0** | **0.1** | 0.139 | 0.079 | 0.029 | 0.144 | 0.09 | 0.027 | 0.107 | 0.062 | 0.012 |
| **0** | **0.2** | 0.159 | 0.088 | 0.026 | 0.134 | 0.072 | 0.027 | 0.13 | 0.076 | 0.02 |
| **0** | **0.5** | 0.175 | 0.108 | 0.041 | 0.166 | 0.097 | 0.027 | 0.172 | 0.103 | 0.044 |
| **0.1** | **0** | 0.134 | 0.071 | 0.025 | 0.117 | 0.079 | 0.023 | 0.122 | 0.065 | 0.02 |
| **0.1** | **0.1** | 0.166 | 0.096 | 0.03 | 0.127 | 0.069 | 0.019 | 0.122 | 0.06 | 0.02 |
| **0.1** | **0.2** | 0.157 | 0.084 | 0.026 | 0.138 | 0.067 | 0.021 | 0.135 | 0.075 | 0.023 |
| **0.1** | **0.5** | 0.2 | 0.118 | 0.045 | 0.186 | 0.116 | 0.043 | 0.179 | 0.115 | 0.044 |
| **0.2** | **0** | 0.174 | 0.105 | 0.037 | 0.144 | 0.081 | 0.024 | 0.117 | 0.064 | 0.022 |
| **0.2** | **0.1** | 0.184 | 0.105 | 0.026 | 0.141 | 0.085 | 0.026 | 0.119 | 0.061 | 0.016 |
| **0.2** | **0.2** | 0.202 | 0.134 | 0.042 | 0.16 | 0.086 | 0.029 | 0.149 | 0.088 | 0.022 |
| **0.2** | **0.5** | 0.291 | 0.198 | 0.079 | 0.241 | 0.155 | 0.067 | 0.235 | 0.155 | 0.064 |
| **0.5** | **0** | 0.231 | 0.145 | 0.063 | 0.19 | 0.111 | 0.038 | 0.197 | 0.122 | 0.034 |
| **0.5** | **0.1** | 0.257 | 0.186 | 0.079 | 0.23 | 0.155 | 0.052 | 0.208 | 0.137 | 0.048 |
| **0.5** | **0.2** | 0.328 | 0.256 | 0.106 | 0.261 | 0.184 | 0.082 | 0.264 | 0.176 | 0.08 |
| **0.5** | **0.5** | 0.469 | 0.393 | 0.234 | 0.406 | 0.332 | 0.19 | 0.415 | 0.339 | 0.204 |

**Table 7 Empirical power properties: Queen contiguity case and $\epsilon_i \sim N(\mathbf{0}, \mathbf{1})$**

| | | **n=169** | | | **n=361** | | | **n=529** | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | **10%** | **5%** | **1%** | **10%** | **5%** | **1%** | **10%** | **5%** | **1%** |
| **0** | **0** | 0.996 | 0.995 | 0.994 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.999 |
| **0** | **0.1** | 0.990 | 0.987 | 0.980 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| **0** | **0.2** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0** | **0.5** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0** | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 | 0.999 | 0.999 |
| **0.1** | **0.1** | 0.999 | 0.999 | 0.997 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.2** | 0.999 | 0.998 | 0.995 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.5** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0** | 0.999 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.1** | 1.000 | 0.999 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.2** | 1.000 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.5** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.1** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.2** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.5** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

**Table 8 Empirical power properties: Queen contiguity case and $\epsilon_i \sim (\chi^2_3 - 3)/6$**

| | | **n=169** | | | **n=361** | | | **n=529** | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda$ | $\rho$ | **10%** | **5%** | **1%** | **10%** | **5%** | **1%** | **10%** | **5%** | **1%** |
| **0** | **0** | 0.997 | 0.997 | 0.979 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0** | **0.1** | 0.995 | 0.992 | 0.989 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 0.999 |
| **0** | **0.2** | 0.993 | 0.991 | 0.967 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0** | **0.5** | 1.000 | 0.999 | 0.988 | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0** | 1.000 | 0.999 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.1** | 0.994 | 0.992 | 0.985 | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.2** | 1.000 | 0.996 | 0.984 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.1** | **0.5** | 1.000 | 1.000 | 0.993 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0** | 0.998 | 0.998 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.1** | 0.999 | 0.996 | 0.980 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.2** | 1.000 | 0.998 | 0.992 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.2** | **0.5** | 1.000 | 1.000 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0** | 1.000 | 1.000 | 0.996 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.1** | 1.000 | 1.000 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.2** | 0.999 | 0.999 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |
| **0.5** | **0.5** | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

**Beyan ve Açıklamalar (Disclosure Statements)**

1. Bu çalışmanın yazarları, araştırma ve yayın etiği ilkelerine uyduklarını kabul etmektedirler (The authors of this article confirm that their work complies with the principles of research and publication ethics).

2. Yazarlar tarafından herhangi bir çıkar çatışması beyan edilmemiştir (No potential conflict of interest was reported by the authors).

3. Bu çalışma, intihal tarama programı kullanılarak intihal taramasından geçirilmiştir (This article was screened for potential plagiarism using a plagiarism screening program).

4. Bu makale için etik kurul izni gerekmemektedir (Ethics committee permission is not required for this article).