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Araştırma Makalesi - Research Article

Generalized Sub-Equation Method for the (1+1)-Dimensional Resonant Nonlinear Schrodinger's Equation

(1+1)-Boyutlu Rezonant Doğrusal Olmayan Schrödinger Denklemi İçin Genelleştirilmiş Alt Denklem Yöntemi

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ABSTRACT

Interest in studying nonlinear models has been increasing in recent years. Dynamical systems, in which the state of the system changes continuously over time, have nonlinear interactions. The use of unique nonlinear differential equations is inescapable in the evaluation of such systems. In mathematical point of view, for obtaining analytical solutions of nonlinear differential equations, it must be fully integrable. Consequently, the importance of fully integrable nonlinear differential equations for nonlinear science has become indisputable. Among these equations, one of the most studied by physicists and mathematicians is the nonlinear Schrödinger equation. This equation has undergone many modifications to evaluate different phenomena. In this study, the resonant nonlinear Schrödinger equation, which is the most important of these physical equations in terms of explaining many physical phenomena, is solved analytically with the generalized sub-equation method.

Keywords- Generalized Sub-Equation Method, (1+1)-Dimensional Resonant Nonlinear Schrodinger's Equation, Exact Solution

ÖZ

Doğrusal olmayan modelleri incelemeye olan ilgi son yıllarda artmaktadır. Sistemin durumunun zaman içinde sürekli olarak değiştiği dinamik sistemler doğrusal olmayan etkileşimlere sahiptir. Bu tür sistemlerin değerlendirilmesinde benzersiz doğrusal olmayan diferansiyel denklemlerin kullanılması kaçınılmazdır. Matematiksel bakış açısına göre, doğrusal olmayan diferansiyel denklemlerin analitik çözümlerini elde etmek için, tamamen integre edilebilir olmalıdır. Sonuç olarak, doğrusal olmayan bilim için tamamen integre edilebilir doğrusal olmayan diferansiyel denklemlerin önemi tartışılmaz hale gelmiştir. Bu denklemler arasında fizikçiler ve matematikçiler tarafından en çok çalışılanlardan biri doğrusal olmayan Schrödinger denklemidir. Bu denklem, farklı olayları değerlendirmek için birçok değişikliğe uğramıştır. Bu çalışmada birçok fiziksel olguyu açıklama açısından bu fiziksel denklemlerin en önemlisi olan rezonans doğrusal olmayan Schrödinger denklemi genelleştirilmiş alt denklem yöntemi ile analitik olarak çözülmüştür.

Anahtar Kelimeler- Genelleştirilmiş Alt Denklem Yöntemi, (1+1) Boyutlu Resonant Doğrusal Olmayan Schrödinger Denklemi, Tam Çözüm

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I. INTRODUCTION

The interest on nonlinear phenomena has been increasing in recent decades. The main reason behind this is the fact that many events in nature are based on nonlinear interactions. These types of systems where interactions are nonlinear are called dynamical systems. Modelling and examining of such dynamical systems are generally done with some specific nonlinear differential equations. For instance, Korteweg-de Vries (KdV) [1] equation is best fitting equation for examining surface waves of shallow waters. On the other hand, Ginzburg-Landau equation is very useful in evaluating many concepts, such as superfluidity [2], Bose-Einstein condensation [3], strings in eld theory [4] and lasers [5], etc. Although 150 years have passed since it was first observed by John Scott Russell (1808-1882), the concept of soliton is growing in importance. Many researchers, especially from the eld of physics and mathematics, concentrate heavily on solitons [6, 7]. While mathematicians are interested in solitons because of the aesthetic and compelling appeal of mathematics used to explain solitons [8]. Physicists are interested in solitons due to their particle grade stability which can be described as remaining unchanged after a collision. Due to these unique properties, solitons are used in many areas of physics, such as plasma physics [9], particle physics [10], condensed matter physics [11] and astrophysics [12], etc. The fully integrability of the differential equation used to explain a nonlinear phenomenon in terms of mathematics is vital for the obtaining the exact solutions. Among the fully integrable nonlinear differential equations, nonlinear Schrödinger equation (NLSE) is of great importance in both physics and mathematics. NLSE is very successful in describing the evolution of slowly varying packets of quasi-monochromatic waves in weakly nonlinear media that have dispersion. With the development of our understanding of explaining nonlinear phenomena, some modifications were made on the NLSE and some physical events could be better explained. The most generic example for modified NLSEs is the resonant nonlinear Schrödinger equation (rNLSE) which is used for describing intermediate cases (inter-modal dispersion) between focusing and defocusing [13].

The (1+1)-dimensional resonant nonlinear Schrödinger's equation is given by [14,15],

$$i\psi_t + a\psi_{xx} + \left(b|\psi|^2 + c|\psi|^4\right)\psi + d\left(\frac{|\psi|_{xx}}{|\psi|}\right)\psi = 0,\ i = \sqrt{-1}. \tag{1}$$

where $\psi(x,t)$ is the complex wave profile and x,t spatial and temporal variables. In the last term $\frac{|\psi|_{xx}}{|\psi|}$ the differentiation with respect to x describes the Broglie quantum potential. Its coefficient d plays an important role in the form of the rNLSE, as it determines solutions with different behavior. Many studies on Schrödinger and rNLSE are made by various scientists [19-24]. For instance, Williams et al. [17] argued the stability and dynamical properties of soliton waves in rLNSE. Lee and Pashaev [18] used the Hirota bilinear approach to consider physically relevant soliton solutions of the resonant nonlinear Schrödinger equation with nontrivial boundary conditions.

To the best of our knowledge sub equation method is used for the first time to obtain the solution exact solutions of rLNSe. By this method trigonometric and hyperbolic solutions are obtained. Using chain rule with this method gives us a chance to turn nonlinear partial differential equation into nonlinear ordinary differential equation without using any normalization or discretization.

II. DESCRIPTION OF GENEALIZED SUB-EQUATION METHOD

In this section a brief description of the considered method called sub-equation method [16] can be expressed step by step. Take into account following nonlinear partial differential equation (NLPDE),

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, \ldots\right) = 0. \tag{2}$$

Step 1. The wave transform

$$\xi = x + \lambda t, \tag{3}$$

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Here, λ describes the velocity of the wave. By the help of this transformation and chain rule the function $u(x,t)$ changes into $U(\xi)$ which includes only one independent variable, also NLPDe turns into a nonlinear ordinary differential equation (NODE),

$$G(U,U',U'',...)=0 \tag{4}$$

where prime indicates the Newtonian concept derivative with respect to ξ.

Step 2. Assume that the solution of Eq. (4) can be obtained in the following form

$$U(\xi)=a_{-1}\left[\left(\varphi(\xi)\right)\right]^{-n}+a_0+a_1\left[\left(\varphi(\xi)\right)\right]^{n}\ ,\ a_{-1}\neq 0\ or\ a_1\neq 0 \tag{5}$$

where a_{-1}, a_0, a_1 are constant coefficients and going to be achieved later. Also n can be obtained using balancing procedure in Eq. (4) and $\varphi(\xi)$ is the solution of the following ODE

$$\varphi'(\xi)=\sigma+\left(\varphi(\xi)\right)^2 \tag{6}$$

and σ is a constant. For the Eq. (6), some special solutions can be stated as follows

$$\varphi(\xi)=\begin{cases}-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right), & \sigma<0,\\ -\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right), & \sigma<0,\\ \sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right) & ,\sigma>0,\\ \sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right) & ,\sigma>0,\\ \dfrac{-1}{\xi+\varpi}, \varpi\ is\ cons. & ,\sigma=0.\end{cases} \tag{7}$$

Step 3. By replacing the Eqs. (5) and (6) into Eq. (4) and regulating the obtained equation due to powers of $\varphi^i(\xi)$ and also equating the coefficients of $\varphi^i(\xi)$ to zero, we deduce an algebraic equation system with respect to $a_i\ (i=0,...,n), k, w$ and σ.

Step 4. Finally, the determined values of $a_i\ (i=0,...,n), k, w$ and σ are put into Eq. (5) by the help of formulas given in (7). So, we get the exact solutions for Eq. (2).

III. ANALYTICAL SOLUTIONS OF THE (1+1) DIMENSIONAL RESONANT NONLINEAR SCHRÖDINGER'S EQUATION

Using the following transformations in Eq. (1):

$$\psi(x,t)=\Phi(\xi)e^{i\Omega},\ \xi=x+\lambda t,\ \Omega=-kx+\omega t+\theta \tag{8}$$

leads to

$$(a+d)\Phi''-\left(\omega+ak^2\right)\Phi+b\Phi^3+c\Phi^5=0 \tag{9}$$

and

$$\lambda=2ak.$$

Considering the terms Φ'' and Φ^5 for in Eq. (9) for balancing procedure, yields $n=\dfrac{1}{2}$. Here, balancing term is a non-integer value. So, this concludes the solution of the Eq. (9) as in the following form

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$$\Phi(\xi)=a_{-1}\left[\left(\varphi(\xi)\right)\right]^{-\frac{1}{2}}+a_0+a_1\left[\left(\varphi(\xi)\right)\right]^{\frac{1}{2}}. \tag{10}$$

Subrogating Eq. (10) into Eq. (9), yields following algebraic equation system.

$$12a_2\sigma^2(a+d)+4a_2^5c=0,$$
$$5a_0a_2^4c=0,$$

$$40a_0^2a_2^3c+20a_1a_2^4c+4a_2^3b-a_1\sigma^2(a+d)=0,$$
$$10a_0^3a_2^2c+20a_0a_1a_2^3c+3a_0a_2^2b=0,$$
$$a_0^5c+20a_0^3a_1a_2c+a_0^3b+30a_0a_1^2a_2^2c+6a_0a_1a_2b-a_0\left(ak^2+\omega\right)=0,$$
$$a_1\sigma(a+d)-2a_1\left(ak^2+\omega\right)+10a_0^4a_1c+60a_0^2a_1^2a_2c+6a_0^2a_1b+20a_1^3a_2^2c+6a_1^2a_2b=0,$$
$$10a_0^3a_1^2c+20a_0a_1^3a_2c+3a_0a_1^2b=0,$$
$$a_2(a+d)-40a_0^2a_1^3c-20a_1^4a_2c-4a_1^3b=0,$$
$$5a_0a_1^4c=0,$$
$$12a_1(a+d)+4a_1^5c=0,$$
$$a_2\sigma(a+d)+2a_2\left(-ak^2-\omega\right)+10a_0^4a_2c+60a_0^2a_1a_2^2c+6a_0^2a_2b+20a_1^2a_2^3c+6a_1a_2^2b=0.$$

Solving the above systems yields:

For $\sigma<0$,

$$c=-\frac{3b}{16a_1^2\sqrt{-\sigma}},a_0=0,d=\frac{-4a\sqrt{-\sigma}+a_1^2b}{4\sqrt{-\sigma}},a_2=a_1\sqrt{-\sigma},\omega=\left(-ak^2+a_1^2b\sqrt{-\sigma}\right), \tag{11}$$

$$\psi_1(x,t)=\left(a_1\sqrt{-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right)}+\frac{a_1\sqrt{-\sigma}}{\sqrt{-\sqrt{-\sigma}\tanh\left(\sqrt{-\sigma}\xi\right)}}\right)e^{i\Omega},$$

$$\psi_2(x,t)=\left(a_1\sqrt{-\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right)}+\frac{a_1\sqrt{-\sigma}}{\sqrt{-\sqrt{-\sigma}\coth\left(\sqrt{-\sigma}\xi\right)}}\right)e^{i\Omega}.$$

For $\sigma>0$,

$$c=\frac{3b}{16a_1^2\sqrt{\sigma}},a_0=0,d=\frac{-4a\sqrt{\sigma}-a_1^2b}{4\sqrt{\sigma}},a_2=-a_1\sqrt{\sigma},\omega=\frac{1}{4}\left(-4ak^2-5a_1^2b\sqrt{\sigma}\right), \tag{12}$$

$$\psi_3(x,t)=\left(a_1\sqrt{\sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right)}-\frac{a_1\sqrt{\sigma}}{\sqrt{\sqrt{\sigma}\tan\left(\sqrt{\sigma}\xi\right)}}\right)e^{i\Omega},$$

$$\psi_4(x,t)=\left(a_1\sqrt{\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right)}-\frac{a_1\sqrt{\sigma}}{\sqrt{\sqrt{\sigma}\cot\left(\sqrt{\sigma}\xi\right)}}\right)e^{i\Omega}.$$

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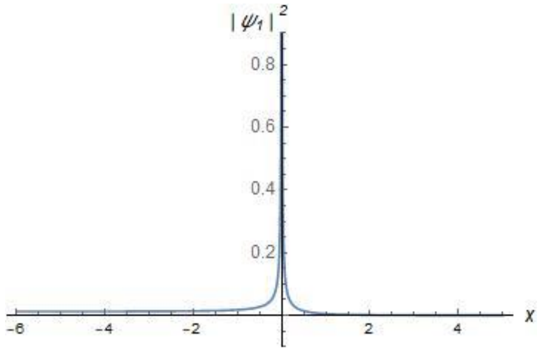
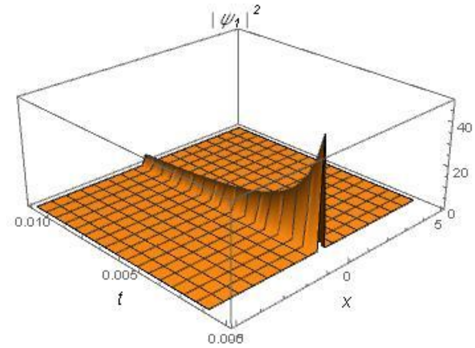
IV. GRAPHICAL REPRESENTATION OF PROBABILITY DISTRIBUTION FOR SOME SOLUTIONS

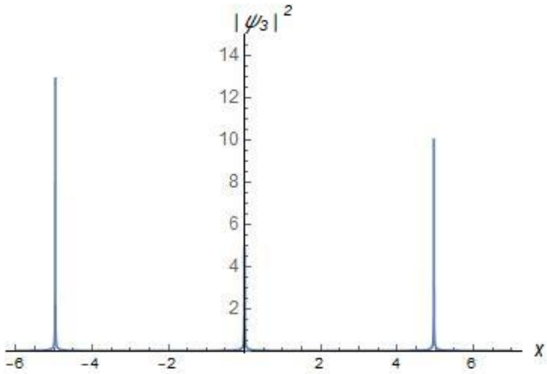
A single soliton propagated along x axis can be seen from the Figure 1(a). Moreover, the fading of the amplitude with time due to compulsion can be seen from the Figure 1(b). Three solitons having different amplitudes propagated along x axis can be seen from the Figure 1(c). Contrary to $\psi_1(x,t)$, three distinct solitons corresponding to $\psi_3(x,t)$ localized at different x positions without fading of the amplitude can be seen from the Figure 1(d).



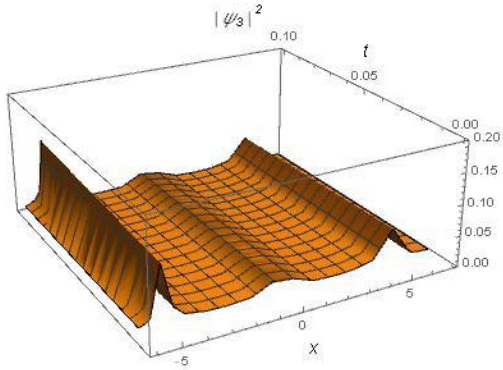
(a) $\sigma = -0.1, \theta = -5, t = 0.01$



(b) $\sigma = -0.1, \theta = -5$



(c) $\sigma = 0.1, \theta = 5, t = 0.01$



(d) $\sigma = 0.1, \theta = 5$

Figure 1. 3D and 2D graphical representations of $\psi_3(x,t)$ for different values

V. CONCLUSION

In this article authors find new trigonometric and hyperbolic solutions of the Schrödinger equation and interpret them graphically, and the generalized sub-equation method was used to achieve these goals. Also, the graphical illustrations and explanations of some solutions are given to express the physical nature of the solutions. All generated solutions are verified by utilizing symbolic computation. The results obtained here can be useful to understand the physics of various problems encountered in nature.

REFERENCES

[1] Tahir, M., & Awan, A. U. (2019). The study of complexitons and periodic solitary-wave solutions with fifth-order KdV equation in (2+ 1) dimensions. *Modern Physics Letters B, 33(33)*, 1950411.

[2] Berti, A., & Berti, V. (2013). A thermodynamically consistent Ginzburg–Landau model for superfluid transition in liquid helium. *Zeitschrift für angewandte Mathematik und Physik*, *64(4)*, 1387-1397.

[3] Kengne, E., Lakhssassi, A., Vaillancourt, R., & Liu, W. M. (2012). Exact solutions for generalized variable-coefficients Ginzburg-Landau equation: Application to Bose-Einstein condensates with multi-body interatomic interactions. *Journal of mathematical physics*, 53(12), 123703.

[4] Rivers, R. J. (2001). Zurek-Kibble causality bounds in time-dependent Ginzburg-Landau theory and quantum field theory. *Journal of low temperature physics*, 124(1), 41-83.

[5] Tasbozan, O., Kurt, A., & Tozar, A. (2019). New optical solutions of complex Ginzburg–Landau equation arising in semiconductor lasers. *Applied Physics B*, 125(6), 1-12.

[6] Khamrakulov, K. P. (2019). Two-soliton molecule bouncing in a dipolar Bose–Einstein condensates under the effect of gravity. *Modern Physics Letters B*, 33(36), 1950452.

[7] Seadawy, A. R., Iqbal, M., & Lu, D. (2019). Analytical methods via bright–dark solitons and solitary wave solutions of the higher-order nonlinear Schrödinger equation with fourth-order dispersion. *Modern Physics Letters B*, 33(35), 1950443.

[8] Yan, X. W. (2020). Generalized (3+ 1)-dimensional Boussinesq equation: Breathers, rogue waves and their dynamics. *Modern Physics Letters B*, 34(01), 2050003.

[9] Díaz, P., Laroze, D., Ávila, A., & Malomed, B. A. (2019). Two-dimensional composite solitons in a spin-orbit-coupled fermi gas in free space. *Communications in Nonlinear Science and Numerical Simulation*, 70, 372-383.

[10] Tian, Z., & Du, J. (2019). Analogue Hawking radiation and quantum soliton evaporation in a superconducting circuit. *The European Physical Journal C*, 79(12), 1-7.

[11] Arshad, M., Seadawy, A. R. & Lu, D. (2019). Study of bright–dark solitons of strain wave equation in micro-structured solids and its applications. *Modern Physics Letters B*, 33(33), 1950417.

[12] Marsh, D. J., & Pop, A. R. (2015). Axion dark matter, solitons and the cusp–core problem. *Monthly Notices of the Royal Astronomical Society*, 451(3), 2479-2492.

[13] Tala-Tebue, E., & Seadawy, A. R. (2018). Construction of dispersive optical solutions of the resonant nonlinear Schrödinger equation using two different methods. *Modern Physics Letters B*, 32(33), 1850407.

[14] Zayed, E. M. E., & Alurrfi, K. A. E. (2016). Extended auxiliary equation method and its applications for finding the exact solutions for a class of nonlinear Schrödinger-type equations. *Applied Mathematics and Computation*, 289, 111-131.

[15] Durur, H. (2020). Different types analytic solutions of the (1+ 1)-dimensional resonant nonlinear Schrödinger's equation using G'/G -expansion method. *Modern Physics Letters B*, 34(03), 2050036.

[16] Tasbozan, O., & Kurt, A. (2020). New Exact Solutions of the MkdV-Sine-Gordon Equation, *Sohag Journal of Mathematics*, 7, 1-4.

[17] Williams, F., Tsitoura, F., Horikis, T. P., & Kevrekidis, P. G. (2020). Solitary waves in the resonant onlinear Schrödinger equation: Stability and dynamical properties. Physics Letters A, 384(22), 126441.

[18] Lee, J. H., & Pashaev, O. K. (2007). Solitons of the resonant nonlinear Schrödinger equation with nontrivial boundary conditions: Hirota bilinear method. *Theoretical and Mathematical Physics*, 152(1), 991-1003.

[19] Başhan, A., Uçar, Y., Yağmurlu, N. M., & Esen, A. (2018). A new perspective for quintic B-spline based Crank-Nicolson-differential quadrature method algorithm for numerical solutions of the nonlinear Schrödinger equation. *The European Physical Journal Plus*, 133(1), 1-15.

[20] Başhan, A., & Esen, A. (2021). Single soliton and double soliton solutions of the quadratic-nonlinear Korteweg-de Vries equation for small and long-times. *Numerical Methods for Partial Differential Equations*, 37(2), 1561-1582.

[21] Başhan, A. (2019). A mixed methods approach to Schrödinger equation: Finite difference method and quartic B-spline based differential quadrature method. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 9(2), 223-235.

[22] Başhan, A. (2018). An effective application of differential quadrature method based on modified cubic B-splines to numerical solutions of the KdV equation. *Turkish Journal of Mathematics*, 42(1), 373-394.

[23] Başhan, A. (2021). Modification of quintic B-spline differential quadrature method to nonlinear Korteweg-de Vries equation and numerical experiments. *Applied Numerical Mathematics*, 167, 356-374.

[24] Bashan, A., Yagmurlu, N. M., Ucar, Y., & Esen, A. (2017). An effective approach to numerical soliton solutions for the Schrödinger equation via modified cubic B-spline differential quadrature method. *Chaos, Solitons & Fractals*, 100, 45-56.