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Solving nonlinear Fredholm integro-differential equations via modifications of some numerical methods

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Abstract

This paper presents the modifications of the variational iteration method (MVIM), the Laplace Adomian decomposition method (MLADM), and the homotopy perturbation method (MHPM) for solving the nonlinear Fredholm integro-differential equation of the second kind. In these techniques, a series is established, the summation of which gives the solution of the discussed equation. The conditions ensuring convergence of this series are presented. Some examples to illustrate the investigated methods are presented as well, and the results reveal that the proposed methods are very effective. Moreover, we present the comparison between our proposed methods with the exact solution and some traditional methods during numerical examples. The results show that (MHPM) and (MLADM) lead to an exact solution, whereas (MVIM) leads to limited solutions. Finally, the uniqueness of solutions and the convergence of the proposed methods are also proved.

Keywords: Fredholm integro-differential equation, MVIM, MHPM, MLADM.
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1. Introduction

Mathematical modeling of many physical systems leads to integrodifferential equations in various fields of engineering and physics. There are some methods to obtain approximate solutions to this kind of equations.

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From these methods are the homotopy perturbation method, Laplace Adomian decomposition method, and variational iteration method, which have undergone many modifications in the recent period. Fredholm integro-differential equation has been solved by some other methods, such as weighted mean value theorem [9]. The approximate solution for solving the nonlinear Fredholm integro-differential equation of the second kind in the complex plane by using the properties of rationalized Haar wavelet has been obtained in [14]. A two-dimensional nonlinear Volterra-Fredholm integro-differential equation by using some iterative methods is presented [13]. In [16], the author modified the existing homotopy perturbation method to solve the high-order integro-differential equations through canonical polynomials basis function. MADM is applied to find the approximate solution for Fredholm integral equation and its system in [4, 5]. Solving linear and nonlinear Volterra integral equations and their system by using some numerical methods are discussed by [6, 8]. Analytical methods with Laplace transform are implemented in [7] to find an approximate solution for Volterra integral equation with a convolution kernel. Non-standard finite difference methods to find the numerical solution of linear Fredholm integro-differential equation have been introduced by Pandey [17]. Al-Mdallal [2] presented the monotone iterative sequences for solving nonlinear integro-differential equations of the second-order. Whereas Atabakan et al. [12] used the spectral homotopy analysis method to solve nonlinear Fredholm integro-differential equations. Also, the same authors in [11] have applied the composite Chebyshev finite difference method to find the solution of Fredholm integro-differential equations. In this regard, Aloko et al. [3] discussed the new variational iteration method to find the numerical solutions of the second kind of nonlinear Fredholm integro-differential equations. Recently, Safavi and Khajehnasiri [18] have proposed two-dimensional block-pulse functions for solving nonlinear mixed Volterra-Fredholm integro-differential.

On the other hand, the Fredholm integral equation is solved in [1] using the homotopy analysis method. Legendre multi-wavelets collocation method for the numerical solution of linear and nonlinear integral equations was discussed lately by Asif et al. [10]. Variational iteration method and homotopy perturbation method to find the approximate solution of Volterra integral equations were achieved by Mirzaei [15]. Syam et al. [19] employed an efficient numerical algorithm for solving fractional higher-order nonlinear integro-differential equations.

Until recently, the applications of the HPM, LADM, and VIM have been developed by scientists and engineers in nonlinear problems, because these methods are the most convenient and powerful. Motivated by the above works, in this paper, we consider a nonlinear Fredholm integro-differential of the form

$$\sum_{n=0}^{k} \xi_n(x) y^{(n)}(x) = f(x) + \lambda \int_a^b k(x,t) G(y(t)) dt, \quad y^b(0) = c_b, 0 \leq b \leq (k-1), \tag{1}$$

where $f(x)$ and $\xi_n(x)$ are given real-valued functions and analytic functions, $k(x,t)$ is the kernel of the equation, $y^{(n)}(x)$ present the n-th derivative of $y(x)$ and $G(y(x))$ is a nonlinear function of $y(x)$.

The main motive for this research is to develop the applications of the modifications of HPM, LADM, and VIM in nonlinear problems because these methods are the most convenient for solving such types of equations, especially the nonlinear Fredholm integro-differential equations. Consequently, we apply the modifications of HPM, LADM, and VIM for solving some equations of type (1). Numerical examples are given to demonstrate the exact and approximate solutions. Also, we use the absolute error table and comparisons with current approaches to show the precision and effectiveness of these methods. Besides, we prove the uniqueness of the solution and the convergence of the proposed methods.

The remainder of the paper is displayed as follows. In Section 2, we give the formulation of MHPM, MLADM, and MVIM. Section 3 proves the uniqueness of the solution of Eq. (1) and the convergence of the proposed methods. Numerical examples to solve the nonlinear Fredholm integro-differential equations of the second kind are provided in Section 4. The comparison between the analytical and approximate solution obtained by the proposed methods and the other methods is discussed in Section 5. In the last section, we close this work with a conclusion.

2. Formulation of methods

Some effective methods have centered on the development of more advanced and efficient methods for solving nonlinear Fredholm integro-differential equations, such as the modified variational iteration method (MVIM), the modified homotopy perturbation method (MHPM), and the modified Laplace Adomian decomposition method (MLADM).

2.1. Modified variational iteration method

To illustrate the fundamental principles of MVIM, we consider the differential equation as follows:

$$Ly(x) + Ny(x) = g(x), \tag{2}$$

where L, N are linear, nonlinear terms respectively and $g(x)$ is an inhomogeneous term. The correction function for Eq. (2) using variational iteration method are presented as the form:

$$y_{n+1} = y_n(x) + \int_0^1 \lambda(\xi)\Big(Ly_n(\xi) + N\tilde{y}(\xi) - g(\xi)\Big)d(\xi), \tag{3}$$

where λ is a general Lagrange multiplier that can be optimally defined by variational theory, that is by part integration and by the use of restricted variation.
Putting $Ly_n(\xi) = y^{'}(\xi)$ we get

$$\begin{aligned}
\int_0^1 \lambda(\xi)\Big(y_n^{'}(\xi)\Big)d\xi &= \lambda(\xi)y_n(\xi) - \int_0^1 \lambda^{'}(\xi)(y_n(\xi))d\xi,\\
\int_0^1 \lambda(\xi)\Big(y_n^{''}(\xi)\Big)d\xi &= \lambda(\xi)y_n^{'}(\xi) - \lambda^{'}(\xi)y_n(\xi),\\
&+ \int_0^1 \lambda^{''}(\xi)(y_n(\xi))d\xi,\\
\int_0^1 \lambda(\xi)\Big(y_n^{'''}(\xi)\Big)d\xi &= \lambda(\xi)y_n^{''}(\xi) - \lambda^{'}(\xi)(y_n(\xi),\\
&+\lambda^{''}y_n(\xi)) - \int_0^1 \lambda^{'''}(\xi)(y_n(\xi))d\xi.
\end{aligned} \tag{4}$$

Generalized integration of the parts is

$$\begin{aligned}
\int_0^1 \lambda(\xi)\Big(y_n^{n}(\xi)\Big)d\xi &= \lambda(\xi)y_n^{n-1}(\xi) - \lambda^{'}(\xi)(y_n^{n-2}(\xi) + \lambda^{'}y_n^{n-3}(\xi))\\
&- \cdots - (-1)\int_0^1 \lambda^{n}(\xi)(y_n(\xi))d\xi,
\end{aligned}$$

we can note that there may be a constant or a function in this method, and δ is the restricted value behaves as a constant, $\tilde{y}_n(\xi)$ is considered as $\delta\tilde{y}_n(\xi) = 0$ restricted variation and so, the extreme condition demands that the stationary conditions should be met:

$$1 + \lambda\mid_{\xi=x} = 0, \qquad \lambda^{'}\mid_{\xi=x} = 0. \tag{5}$$

Thus, the general multiplier of Lagrange easily can be identified as:

$$\lambda = -1.$$

Successive approximations of $y_n(x)$, $\quad n \geq 0$, of solution $y(x)$ can be easily obtained by using selective function $y_0(x)$.
We will consider a particular case of Eq. (1) of the form:

$$y^{(n)}(x) + f(x) + \lambda \int_a^b k(x,t) y^k(t) y^m(t) dt = z(x), \tag{6}$$

with the initial condition $y^b = c_b, \quad b = 0, 1, \ldots, (n-1)$, where k, m are integers with $k \geq m \geq n$ and y^b are real constant.

Now we consider Eq. (2), where $g(x)$ is a known analytical function and the nonlinear operator $N(y_n)$ can be decomposed as

$$N\Big[\sum_{i=0}^{\infty} y_i\Big] = N(y_0) + \sum_{i=0}^{\infty}\Big(N(\sum_{j=0}^{i} y_j) - N(\sum_{j=0}^{i} y_j)\Big), \tag{7}$$

where y_j are the polynomials of x and the relationship of recurrence is determined as

$$\begin{aligned}
&y_0 = f(x), \\
&y_1 = N(y_0), \\
&y_2 = N(y_0 + y_1) - N(y_0), \\
&y_3 = N(y_0 + y_1 + y_2) - N(y_0 + y_1), \\
&\vdots \\
&y_{n+1} = N(y_0 + y_1 + \cdots + y_n) - N(y_0 + y_1 + \cdots + y_{n-1}), n = 1, 2, \ldots.
\end{aligned} \tag{8}$$

The nonlinear term in Eq. (3) can be written as $N\tilde{y}(\xi) = Ny_n(\xi)$ and the n th term approximate solution in Eq. (8) is

$$y_0 + y_1 + \cdots + y_{n+1} = N(y_0 + y_1 + \cdots + y_n),$$

so

$$y(x) = \sum_{n=0}^{n-1} y_n(x).$$

Appling $\mathcal{L}^{-1}$ to the recurrence relation for the finding of the components, the $(n+1)$th approximation of the exact solution for the unknown function $y(x)$ is determined as

$$y_{n+1}(x) = \mathcal{L}^{-1}(N(y_0 + y_1 + \cdots + y_n)) - \mathcal{L}^{-1}(N(y_0 + y_1 + \cdots + y_{n-1})), \tag{9}$$

we construct the solution as

$$y(x) = \mathcal{L}^{-1} \sum_{n=0}^{n-1} y_n(x), \quad n \geq 0. \tag{10}$$

The modification for Eq. (3) has formulated as

$$y_{n+1}(x) = y_n(x) + \int_a^b \lambda(\xi)\Big(Ly_n(\xi) - g(\xi) + \mathcal{L}^{-1}\sum_{n=0}^{n-1} y_n(\xi),\Big) d(\xi), \tag{11}$$

$$\begin{aligned}
y_{n+1}(x) = y_n(x) + \int_a^b (-1)^n \frac{1}{(n-1)!}(\xi - x)^{n-1}\Big(&Ly_n(\xi) - g(\xi) \\
&+\mathcal{L}^{-1}\int_a^{\xi} k(\xi, r) \sum_{n=0}^{n-1} y_n(\xi), dr\Big) d(\xi).
\end{aligned} \tag{12}$$

The zeroth approximation y can be any eclectic function. However, the initial values have been preferably using for the selective zeroth approximation y_0. Consequently, the solution is given by

$$y(x) = \lim_{n\to\infty} y_n(x). \tag{13}$$

2.2. Modified homotopy perturbation method

The method of homotopy perturbation first proposed by Ji-Huan He in (1997) [15, 16]. Consider the general form of nonlinear Fredholm integro-differential equations

$$y^{(n)}(x) = f(x) + \int_a^b k(x,t)[My(t) + Ny(t)]dt, \tag{14}$$

with initial conditions

$$y^b(x) = c_b, 0 \leq b \leq (n-1), n \geq 0,$$

where $M(y)$ and $N(y)$ are linear and nonlinear functions of y, respectively.
To explain the basic concept of this approach, we consider the following nonlinear differential equation:

$$A(y) = f(z), \quad z \in \Omega, \tag{15}$$

with boundary conditions

$$B(y, \frac{\partial y}{\partial n}) = 0, \quad z \in \Gamma, \tag{16}$$

where A, B are general differential operator and boundary operator respectively, Γ is the boundary of the domain Ω, and $f(r)$ is a known analytic function. Dividing the operator A into two parts: M and N. Therefore, Eq. (15) can be rewritten as follows:

$$M(y) + N(y) = f(z). \tag{17}$$

Using the homotopy technique, we construct a homotopy
$v(z,p) : \Omega \times [0,1] \to R$ which satisfies

$$H(v,p) = (1-p)[M(v) - M(y_0)] + p[A(v) - f(z)] = 0, p \in [0,1], \tag{18}$$

or

$$H(v,p) = M(v) - M(y_0) + pM(y_0) + p[N(v) - f(z)] = 0, \tag{19}$$

where p is an embedding parameter, and y_0 is an initial approximation of Eq. (15) which satisfies the boundary conditions. From Eqs. (18), (19), we have

$$\begin{aligned} H(v,0) &= M(v) - M(y_0) = 0, \\ H(v,1) &= A(v) - f(z) = 0. \end{aligned} \tag{20}$$

The changing in the process of p from zero to unity is just that of $v(z,p)$ from $y_0(z)$ to $y(z)$. In topology this is called deformation and $M(v) - M(y_0)$, and $A(v) - f(z)$ are called homotopic. Now, we assume that the solution of Eqs. (18), (19) can be expressed as

$$v = v_0 + pv_1 + p^2 v_2 + \cdots . \tag{21}$$

The approximate solution of Eq. (15) can be obtained by setting $p = 1$.

$$y = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{22}$$

To apply MHPM for solving the Fredholm integro-differential equations, we define a convex homotopy by

$$H(y,0) = M(y), \quad H(y,1) = N(y),$$

where $M(y)$ is a functional operator with known solution v_0, which has obtained as

$$v_0(x) = a + bx + cx^2 + dx^3,$$

which is dependent on the order of differentiation. In most cases, we may choose a convex homotopy by

$$H(y,p) = (1-p)M(y) + pN(y) = 0.$$

HPM reduces the disadvantages of the conventional perturbation method while retaining all its benefits. The series in Eq. (22) is convergent in most cases, and the convergence rate depends on $A(y)-f(z)=0$. Note that the components v_n for $n=1,2,\ldots$ must be determined in the HPM in order to achieve an approximate solution. Particularly for $n \geq 3$, large and sometimes complicated computations have needed. To avoid this problem, the MHPM is implemented in which v_0 is calculated in such a way that $v_1 = 0$ for $n \geq 1$. As a result, the number of calculations decreases relative to those in the HPM.

2.3. Modified Laplace Adomian decomposition method

Consider the general form of nonlinear Fredholm integro-differential equations in Eq.(14).

The general form of second-order nonlinear partial differential equations with initial conditions in the form

$$\begin{aligned} Ly(x,t) + My(x,t) + Ny(x,t) = z(x,t),\\ y(x,0) = f(x), \qquad y_t(x,0) = g(x), \end{aligned} \tag{23}$$

where $L = \dfrac{\partial^n}{\partial x^n}$ is the second order differential operator, M, N represent the remaining linear operator and the general non-linear differential operator respectively and $z(x,t)$ is the source term.
Using Laplace transform on both sides of Eq. (23), we have

$$\mathcal{L}[Ly(x,t)] + \mathcal{L}[My(x,t)] + \mathcal{L}[Ny(x,t)] = \mathcal{L}[z(x,t)],$$

applying Laplace Transform's differentiation property, we get:

$$s^2\mathcal{L}[Ly(x,t)] - sf(x) - g(x) + \mathcal{L}[My(x,t)] + \mathcal{L}[Ny(x,t)] = \mathcal{L}[z(x,t)],$$

and

$$\mathcal{L}[Ly(x,t)] = \frac{f(x)}{s} - \frac{g(x)}{s^2} - \frac{\mathcal{L}[My(x,t)]}{s^2} - \frac{\mathcal{L}[Ny(x,t)]}{s^2} + \frac{\mathcal{L}[z(x,t)]}{s^2}. \tag{24}$$

In the Laplace decomposition method, the next step is to represent the solution as an infinite series given below:

$$y(x,t) = \sum_{n=0}^{\infty} y_n(x,t), \tag{25}$$

decomposing the nonlinear operator as

$$Ny(x,t) = \sum_{n=0}^{\infty} A_n(x,t), \tag{26}$$

where A_n is the Adomian polynomial given below:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N(\sum_{i=0}^{\infty} \lambda^i y_i)\right]_{\lambda=0}, \quad \forall n \in N,$$

using (24), (25) and (26), we get

$$\sum_{n=0}^{\infty} \mathcal{L}[y_n(x,t)] = \frac{f(x)}{s} + \frac{g(x)}{s^2} - \frac{1}{s^2}\mathcal{L}[My(x,t)] - \frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty} A_n(x,t)] + \frac{1}{s^2}\mathcal{L}[z(x,t)], \tag{27}$$

we have to compare both sides of (27) as

$$\begin{aligned}
&\mathcal{L}[y_0(x,t)] = k_1(x,s),\\
&\mathcal{L}[y_1(x,t)] = k_2(x,s) - \frac{1}{s^2}\mathcal{L}[M_0 y(x,t)] - \frac{1}{s^2}\mathcal{L}[A_0(x,t)],\\
&\mathcal{L}[y_{n+1}(x,t)] = -\frac{1}{s^2}\mathcal{L}[M_n y(x,t)] - \frac{1}{s^2}\mathcal{L}[A_n(x,t)], \quad n \geq 1,
\end{aligned} \tag{28}$$

where $k_1(x,s)$ and $k_2(x,s)$ are Laplace transform of $k_1(x,t)$ and $k_2(x,t)$ respectively.

The application of the inverse Laplace transformation to Eq. (28) provides our requisite recursive relation as follows:

$$\begin{aligned}
&y_0(x,t) = k_1(x,t),\\
&y_1(x,t) = k_2(x,t) - \mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[M_0 y(x,t)] - \frac{1}{s^2}\mathcal{L}[A_0(x,t)]\right],\\
&y_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2}\mathcal{L}[M_n y(x,t)] - \frac{1}{s^2}\mathcal{L}[A_n(x,t)],\right] \quad n \geq 1.
\end{aligned} \tag{29}$$

The solution by using the modified Adomian decomposition method depends highly on the choice of $k_0(x,t)$ and $k_1(x,t)$, where $k_0(x,t)$ and $k_1(x,t)$ represent the terms resulting from the source term and the initial conditions prescribed.

3. MAIN RESULTS

In this section, we will prove the uniqueness of solution for Eq. (1) and the convergence of the proposed methods.

Theorem 3.1. *Assume that:*

(i) the nonlinear terms $G(y(x))$ and $D^j(y(x))$ are Lipschitz continuous, i.e.

$$|G(y_1) - G(y_2)| \leq \alpha |y_1 - y_2|, \quad \forall y_1, y_2 \in C[a,b],$$

and

$$|D^j(y_1) - D^j(y_2)| \leq \gamma_j |y_1 - y_2| \quad \forall y_1, y_2 \in C[a,b],$$

where α and $\gamma_j \geq 0, j = 0,1,2,\ldots,k$ are constants and $D^j(y(x)) = \dfrac{d^j}{dx^j}y(x) = \sum_{i=0}^{\infty}\gamma_i$.

(ii) $\psi(x)$ is bounded function for all $x \in J[a,b]$ with $0 < \psi = (\alpha\theta_1 + k\gamma\theta_3)(b-a) < 1$.
Then, there exists a unique solution $y(x) \in C$ to the problem (1).

Proof. Let y_1 and y_2 be two different solutions of the problem (1), then

$$\begin{aligned}
|y_1 - y_2| = |&\int_a^b \frac{\gamma(x-t)^k k(x,t)}{\xi_k k!}[G(y_1) - G(y_2)]dt\\
&-\sum_{j=0}^{k-1}\int_a^b \frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}[D^j(y_1) - D^j(y_2)]dt|\\
\leq &\int_a^b |\frac{\gamma(x-t)^k k(x,t)}{\xi_k k!}||G(y_1) - G(y_2)|dt\\
+&\sum_{j=0}^{k-1}\int_a^b |\frac{(x-t)^{k-1}\xi_j(t)}{\xi_k(t)(k-1)!}||D^j(y_1) - D^j(y_2)|dt,\\
\leq &(\alpha\theta_1 + k\gamma^*\theta*_3)(b-a)|y_1 - y_2|,
\end{aligned}$$

we get $(1-\psi)|y_1-y_2| \leq 0$. Since $0 \leq \psi \leq 1$, so $|y_1-y_2| = 0$. therefore, $y_1 = y_2$ and the proof is completed. □

Theorem 3.2. *If problem (1) has a unique solution, then the solution $y_n(x)$ obtained from the recursive relation (12) using VIM converges when*

$$0 < \phi = (\alpha\theta_2 + k\gamma^*\theta_4^*)(b-a) < 1.$$

Proof. We have the iteration formula:

$$y_0(x) = L^{-1}[\frac{f(x)}{\xi_k}(x)] + \sum_{b=0}^{k-1} \frac{(x-a)^b}{b!} c_b,$$

$$y_{n+1}(x) = y_n(x) - L^{-1}[\sum_{j=0}^{k} \xi_j(x) y_n^j(x) - f(x) - \gamma \int_a^b k(x,t)]G(y_n(t))dt], n \geq 0,$$

where L^{-1} is the multiple integration operator given as

$$L^{-1}(.) = \int_a^b \int_a^b \cdots \int_a^b (.)dxdx\ldots dx(k-times).$$

From the above equations, we get

$$y_{n+1}(x) - y(x) = y_n(x) - y(x) - (L^{-1}[\sum_{j=0}^{k} \xi_j(x)[y_n^j(x) - y_j(x)]])$$

$$-L^{-1}[\gamma \int_a^b k(x,t)[G(y_n(t)) - G(y(t))dt].$$

If we set, $\xi_k(x) = 1$, and $z_{n+1}(x) = y_{n+1}(x) - y_n$, $\quad z_n(x) = y_n(x) - y_n$, since $z_n(a) = 0$, then

$$z_{n+1}(x) = z_n(x) + \int_a^b \frac{\gamma(x-t)^k k(x,t)}{k!}[G(y_n(t)) - G(y(t))]dt$$

$$-\sum_{j=0}^{k-1} \int_a^b \frac{\lambda_1 \xi_j(t)(x-t)^{k-1}}{(k-1)!}[D^j(y_n(t)) - D^j(y)]dt - (z_n(x) - z_n(a)).$$

Therefore,

$$|z_{n+1}(x)| \leq \int_a^b |\frac{\gamma(x-t)^k k(x,t)}{k!}||z_n|\alpha dt$$

$$+\sum_{j=0}^{k-1} \int_a^b |\frac{\gamma\xi_j(t)(x-t)^{k-1}}{(k-1)!}|max|\gamma_j||z_n|dt.$$

$$\leq |z_n|[\int_a^b \alpha\theta_2 dt + \sum_{j=0}^{k-1} \int_a^b \theta_4^* max|\gamma_j|]$$

$$\leq |z_n|[\alpha\theta_2 + k\gamma^*\theta_4^*](b-a) = |z_n|\Phi.$$

Hence,

$$\| z_{n+1} \| = max_{\forall x \in J}|z_{n+1}(x)|$$

$$\leq \Phi max_{\forall x \in J}|z_n(x)|$$

$$\leq \Phi \| z_n \| .$$

□

Theorem 3.3. *Assume that*

$$\big(L(y)-L(z),y-z\big)\geq k\|y-z\|^2, k>0, \forall y,z\in H$$

and whatever may be $M>0$, there exist a constant $C(M)>0$ such that for $y,z\in H$ with $\|y\|\leq M, \|z\|\leq M$ we have:

$$\big(L(y)-L(z),y-z\big)\geq C(M)\|y-z\|\|v\|$$

for every $v\in H$, where H is the Hilbert space which may define by $H=L^2((\alpha,\beta)X[0,T])$.
Then the Laplace Adomian method applied to the nonlinear Fredholm integro-differential equation converges towards a particular solution.

Proof. We have to prove a special case from theorem 3.3 when $G(y(t))=y^2(t)$, firts we will start to verify the convergence of

$$\big(L(y)-L(z),y-z\big)\geq k\|y-z\|^2, k>0, \forall y,z\in H,$$

for the operator $L(y)$: i.e., $\exists k\geq 0, \forall y,z\in H$, we have

$$L(y)-L(z)=\int_a^x (y^2(t)-z^2(t))dt,$$

Then we get

$$(L(y)-L(z),y-z)=\Big(\int_a^x (y^2(t)-z^2(t))dt, y-z\Big).$$

According the Schwartz inequality, we get

$$\Big(\int_a^x (y^2(t)-z^2(t))dt, y-z\Big)\leq k\parallel y^2-z^2\parallel\parallel y-z\parallel.$$

Now we use the mean value theorem, then we have

$$\begin{aligned}&\Big(\int_a^x (y^2(t)-z^2(t))dt, y-z\Big)\leq k\parallel y^2-z^2\parallel\parallel y-z\parallel=\frac{1}{3}k_1\eta^3\parallel y-z\parallel^2\\&\leq\frac{1}{3}k_1M^3\parallel y-z\parallel^2,\\&\Big(-\int_a^x (y^2(t)-z^2(t))dt, y-z\Big)\geq\frac{1}{3}k_1M^3\parallel y-z\parallel^2,\end{aligned}$$

where $y\leq\eta\leq z$ and $\parallel y\parallel\leq M, \parallel z\parallel\leq M$. Therefore:

$$(L(y)-L(z),y-z)\geq k\parallel y-z\parallel^2$$

where $k=\frac{1}{3}k_1M^3$.
Now we verify the convergence of

$$\big(L(y)-L(z),y-z\big)\geq C(M)\|y-z\|\|z\|,$$

for the operator $L(y)$ which is for every $M\geq 0$, there exist a constant $C(M)\geq 0$ such that for $y,z\in H$ with $\parallel y\parallel\leq M, \parallel z\parallel\leq M$ we have $(L(y)-L(z),y-z)\leq C(M)\parallel y-z\parallel\parallel v\parallel$ for every $v\in H$. For that we have:

$$\begin{aligned}&\big(L(y)-L(z),v\big)=\Big(\int_0^x (y^2-z^2)dt, v\Big)\\&\leq M^3\parallel y-z\parallel\parallel v\parallel=C(M)\parallel y-z\parallel\parallel v\parallel,\end{aligned}$$

where $C(M)=M^3$ and therefore $\big(L(y)-L(z),y-z\big)\geq C(M)\|y-z\|\|z\|$ is hold. The proof is complete. □

4. Numerical Examples

This section contains numerical examples to illustrate the accuracy and effectiveness properties of the methods to solve the nonlinear Fredholm integro-differential equation of the second kind. The absolute errors used is defined as $|y(x)-y_n(x)|$, where $y(x), y_n(x)$ are the exact and approximate solutions respectively. The numerical solutions of our proposed methods will compare with the numerical solutions of other known methods.

Example 4.1. *Consider the following nonlinear Fredholm integro-differential equation of the second kind*

$$y^{'}(x) = xe^x + e^x - x + \int_0^1 xy(t)dt, \tag{30}$$

with the initial condition $y(0) = 0$ and the exact solution $y(x) = xe^x$.

- ***Using MVIM***

The correction functional for Eq. (30) is constructed as

$$y_{n+1}(x) = y_n(x) + \int_a^b \lambda(\xi)\left[Ly_n(\xi) - xe^x - e^x + x - \mathcal{L}^{-1}\sum_{j=0}^{i}\xi\tilde{y}_j(r)dr\right]d(\xi).$$

Making the functional stationary and noting that, $\tilde{y}_n$ is a restriction variation, $\delta\tilde{y}_n = 0$. To find the optimal $\lambda(\xi)$ and calculate variation with respect to y_n, we have the stationary conditions by applying Eq. (4):

$$\delta y_n : 1 + \lambda\ |_{\xi=x} = 0, \qquad \delta y_n : \lambda^{'}\ |_{\xi=x} = 0.$$

The Lagrange multiplier can be identified as $\lambda = -1$

$$y_{n+1}(x) = y_n(x) - \int_a^b \left[Ly_n(\xi) - xe^x - e^x + x - \mathcal{L}^{-1}\sum_{j=0}^{i}\xi\tilde{y}_j(r)dr\right]d(\xi).$$

- ***Using MHPM***

By letting $g_1(x) = xe^x + e^x$ and $g_2(x) = -x$ in order to obtain

$$\begin{aligned}
&p^0 : v_0^{'}(x) = g_1(x) \Longrightarrow v_0(x) = xe^x,\\
&p^1 : v_1^{'}(x) - g_2(x) - \int_0^1 k(x,t)v_0(t)dt = 0 \Longrightarrow v_1(x) = 0,\\
&p^2 : v_2^{'}(x) - \int_0^1 k(x,t)v_1(t)dt = 0 \Longrightarrow v_2(x) = 0.
\end{aligned}$$

Hence,

$$y(x) = \sum_{n=0}^{\infty} v_n(x) = xe^x,$$

which is the exact solution.

- ***Using MLADM***

Applying the Laplace transform and by using the initial condition, we have

$$sY(x) = \frac{1}{(s-1)^2} + \frac{1}{s-1} - \frac{1}{s^2} + \mathcal{L}\Big[\int_0^1 xy(t)dt\Big],$$

or

$$Y(x) = \frac{1}{s(s-1)^2} + \frac{1}{s(s-1)} - \frac{1}{s^3} + \frac{1}{s}\mathcal{L}\Big[\int_0^1 xy(t)dt\Big].$$

Using the inverse Laplace transform we get

$$y(x) = xe^x - \frac{x^{x^2}}{2} + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}(\int_0^1 xy(t))dt\Big]. \tag{31}$$

Decomposing the solution as an infinite sum given below

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{32}$$

Substituting (31) on (32) we get

$$\sum_{n=0}^{\infty} y_n(x) = xe^x - \frac{x^2}{2} + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}(\int_0^1 x\sum_{n=0}^{\infty} A_n(t))dt\Big],$$

in which $A_n = \sum_{j=0}^{n} y_j$. *The recursive relation is given below*

$$\begin{aligned} y_0(x) &= xe^x, \\ y_1(x) &= -\frac{x^2}{2} + \mathcal{L}^{-1}\Big(\frac{1}{s}\mathcal{L}(\int_0^1 xy_0(t))dt\Big) = 0, \\ y_{n+1}(x) &= \mathcal{L}^{-1}\Big(\frac{1}{s}\mathcal{L}(\int_0^1 xy_n(t))dt\Big) = 0, \qquad n \geq 0, \end{aligned}$$

in which of $A_n = \sum_{j=0}^{n} y_j$, *where for every* $n \geq 1, A_n = 0$. *Hence, the exact solution is*

$$y(x) = y_0(x) = xe^x.$$

Table 1: Numerical results of Example 4.1

X	$MVIM$	$MHPM$	$MLADM$	VIM	ADM
0.1	0.1105170888	0.1105170918	0.1105170918	0.1105170888	0.1105170888
0.2	0.2442805506	0.2442805516	0.2442805516	0.2442805397	0.2442805397
0.3	0.4049576457	0.4049576424	0.4049576424	0.4049576156	0.4049576156
0.4	0.5967298865	0.5967298792	0.5967298792	0.5967298316	0.5967298316
0.5	0.8243606264	0.8243606355	0.8243606355	0.8243605611	0.8243605611
0.6	1.0932712700	1.0932712800	1.0932712800	1.0932711730	1.0932711730
0.7	1.4096267890	21.4096268950	21.4096268950	1.4096267490	1.4096267490
0.8	1.7804325720	1.7804327420	1.7804327420	1.7804325510	1.7804325510
0.9	2.2136429590	2.2136428000	2.2136428000	2.2136425590	2.2136425590
1.0	2.7182818230	2.7182818280	2.7182818280	2.7182815300	2.7182815300

Table 2: Absolute errors of Example 4.1

X	E_{MVIM}	E_{VIM}	E_{ADM}
0.1	3×10^{-9}	3×10^{-9}	3×10^{-9}
0.2	1×10^{-9}	1.19×10^{-8}	1.19×10^{-8}
0.3	3.3×10^{-9}	2.68×10^{-8}	2.68×10^{-8}
0.4	7.3×10^{-9}	4.76×10^{-8}	4.76×10^{-8}
0.5	9.1×10^{-9}	7.44×10^{-8}	7.44×10^{-8}
0.6	10×10^{-9}	1.07×10^{-7}	1.07×10^{-7}
0.7	106×10^{-9}	$1.46\ \times10^{-7}$	$1.46\ \times10^{-7}$
0.8	170×10^{-9}	1.91×10^{-9}	1.91×10^{-9}
0.9	159×10^{-9}	2.41×10^{-7}	2.41×10^{-7}
1.0	5×10^{-9}	2.98×10^{-7}	2.98×10^{-7}

Example 4.2. *Consider the following nonlinear Fredholm integro-differential equation of the second kind*

$$y'(x) = \frac{5}{4} - \frac{x^2}{3} + \int_0^1 (x^2 - t)y^2(t)dt, \tag{33}$$

with the initial condition $y(0) = 0$ and exact solution $y(x) = x$.

- ***Using MVIM***

The correction functional for Eq. (33) is constructed as

$$y_{n+1}(x) = y_n(x) + \int_a^b \lambda(\xi)\left[Ly_n(\xi) - \frac{5}{4} + \frac{x^2}{3} - \mathcal{L}^{-1}\sum_{j=0}^{i}(\xi^2 - r)\tilde{y}_j^2(r)dr\right]d(\xi),$$

making the functional stationary and noting that, $\tilde{y}_n$ is a restriction variation, $\delta\tilde{y}_n = 0$. To find the optimal $\lambda(\xi)$ and calculate variation with respect to y_n, we have the stationary conditions by applying Eq. (4):

$$\delta y_n : 1 + \lambda\,|_{\xi=x} = 0, \qquad \delta y_n : \lambda'\,|_{\xi=x} = 0.$$

The Lagrange multiplier can be identified as $\lambda = -1$

$$y_{n+1}(x) = y_n(x) - \int_a^b \left[Ly_n(\xi) - \frac{5}{4} + \frac{x^2}{3} - \mathcal{L}^{-1}\sum_{j=0}^{i}(\xi^2 - r)\tilde{y}_j^2(r)dr\right]d(\xi),$$

- ***Using MHPM***

By letting $g_1(x) = 1$ and $g_2(x) = \frac{1}{4} - \frac{1}{3}x^2$ in order to obtain

$$p^0 : v_0'(x) = g_1(x) \Longrightarrow v_0(x) = x,$$

$$p^1 : v_1'(x) - g_2(x) + \int_0^1 k(x,t)v_0^2(t)dt = 0 \Longrightarrow v_1(x) = 0,$$

$$p^2 : v_2'(x) + \int_0^1 k(x,t)2v_0(t)v_1(t)dt = 0 \Longrightarrow v_2(x) = 0.$$

Hence,

$$y(x) = \sum_{n=0}^{\infty} v_n(x) = x,$$

which is the exact solution.

- ***Using MLADM***

Applying the Laplace transform and by using the initial condition, we have

$$sY(x) = \frac{5}{4s} - \frac{2}{3s^3} + \mathcal{L}\Big[\int_0^1 (x^2 - t)y^2(t)dt\Big],$$

or

$$Y(x) = \frac{5}{4s^2} - \frac{2}{3s^4} + \frac{1}{s}\mathcal{L}\Big[\int_0^1 (x^2 - t)y^2(t)dt\Big].$$

Using the inverse Laplace transform we get

$$y(x) = \frac{5}{4}x - \frac{2}{9}x^3 + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}(\int_0^1 (x^2 - t)y^2(t))dt\Big]. \tag{34}$$

Decomposing the solution as an infinite sum given below

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{35}$$

Substituting (34) on (35) we get

$$\sum_{n=0}^{\infty} y_n(x) = \frac{5}{4s^2} - \frac{2}{3s^4} + \mathcal{L}^{-1}\Big[\frac{1}{s}\mathcal{L}(\int_0^1 (x^2 - t)\sum_{n=0}^{\infty} A_n(t))dt\Big],$$

in which $A_n = \sum_{j=0}^{n} y_j^2$. *The recursive relation is given below*

$$\begin{aligned}
&y_0(x) = x,\\
&y_1(x) = \frac{1}{4} - \frac{1}{3}x^2 + \mathcal{L}^{-1}\Big(\frac{1}{s}\mathcal{L}(\int_0^1 (x^2 - t)y_0^2(t))dt\Big) = 0,\\
&y_{n+1}(x) = \mathcal{L}^{-1}\Big(\frac{1}{s}\mathcal{L}(\int_0^1 (x^2 - t)y_n^2(t))dt\Big) = 0, \qquad n \geq 0,
\end{aligned}$$

in which of $A_n = \sum_{j=0}^{n} y_j^2$, *where for every* $n \geq 1, A_n = 0$.
Hence, the exact solution is

$$y(x) = y_0(x) = x.$$

Table 3: Numerical results of Example 4.2

X	$MVIM$	$MHPM$	$MLADM$	BPM	$Er_{(MVIM)}$	$Er_{(BPM)}$
0.1	0.09987	0.10000	0.10000	0.08810	1.32×10^{-4}	1.19×10^{-2}
0.2	0.19974	0.20000	0.20000	0.17802	2.6×10^{-4}	2.20×10^{-2}
0.3	0.29962	0.30000	0.30000	0.26784	3.81×10^{-4}	3.22×10^{-2}
0.4	0.39951	0.40000	0.40000	0.35861	4.91×10^{-4}	4.14×10^{-2}
0.5	0.49941	0.50000	0.50000	0.45067	5.86×10^{-4}	4.93×10^{-2}
0.6	0.59934	0.60000	0.60000	0.54433	6.63×10^{-4}	5.57×10^{-2}
0.7	0.69928	0.70000	0.70000	0.63991	7.17×10^{-4}	5.57×10^{-2}
0.8	0.79925	0.80000	0.80000	0.73773	7.46×10^{-4}	6.23×10^{-2}
0.9	0.89926	0.90000	0.90000	0.83811	7.45×10^{-4}	6.19×10^{-2}
1.0	0.99929	1.00000	1.00000	0.99935	7.11×10^{-4}	6.50×10^{-4}

Example 4.3. *Consider the third-order nonlinear Fredholm integro-differential equation of the second kind*

$$y^{'''}(x) = \sin x - x - \int_0^{\frac{\pi}{2}} xty^{'}(t)dt, \tag{36}$$

wth the initial conditions $y(0) = 1, \quad y^{'}(0) = 0, \quad y^{''}(0) = -1$ *for* $x \in [0, \frac{\pi}{2}]$ *and exact solution* $y(x) = \cos x$.

- ***Using MVIM***

The correction functional for Eq. (36) is constructed as

$$y_{n+1}(x) = y_n(x) + \int_a^b \lambda(\xi)\left[Ly_n(\xi) - \sin\xi + \xi - \mathcal{L}^{-1}\sum_{j=0}^{i}(\xi r)\tilde{y}_j^{'}(r)dr\right]d(\xi).$$

Making the functional stationary and noting that, $\tilde{y}_n$ *is a restriction variation,* $\delta\tilde{y}_n = 0$. *To find the optimal* $\lambda(\xi)$ *and calculate variation with respect to* y_n*, we have the stationary conditions:*

$$\delta y_n : \lambda^{'''}\mid_{\xi=x}= 0, \qquad \delta y_n : \lambda^{''}\mid_{\xi=x}= 0, \qquad 1+\lambda^{'}\mid_{\xi=x}= 0, \qquad \lambda\mid_{\xi=x}= 0.$$

The Lagrange multiplier can be identified as $\lambda = -\frac{1}{2!}(\xi - x)^2$,

$$y_{n+1}(x) = y_n(x) - \int_a^b \frac{1}{2!}(\xi - x)^2\left[Ly_n(\xi) - \sin\xi + \xi - \mathcal{L}^{-1}\sum_{j=0}^{i}(\xi r)\tilde{y}^{'}j(r)dr\right]d(\xi).$$

Consequently, we have the following approximations:

$$y_0(x) = 1 - \frac{x^2}{2},$$
$$y_1(x) = \cos x - 0.04166666667x^4 + 0.008971723581x^7$$
$$y_2(x) = \cos x - 0.04166666667x^4 + 0.007137527979x^7$$
$$y_3(x) = \cos x - 0.04166666667x^4 + 0.007550623256x^7$$

- ***Using MHPM***

By letting $g_1(x) = \sin x$ *and* $g_2(x) = -x$ *in order to obtain*

$$p^0 : v_0^{'''}(x) = g_1(x) \Longrightarrow v_0(x) = \cos x,$$
$$p^1 : v_1^{'''}(x) - g_2(x) - \int_0^{\frac{\pi}{2}} k(x,t)v_0^{'}(t)dt = 0 \Longrightarrow v_1(x) = 0,$$
$$p^2 : v_2^{'''}(x) - \int_0^{\frac{\pi}{2}} k(x,t)v_1^{'}(t)dt = 0 \Longrightarrow v_2(x) = 0.$$

Hence

$$y(x) = \sum_{n=0}^{\infty} v_n(x) = \cos x,$$

which is the exact solution.

- ***Using MLADM***

Applying the Laplace transform and by using the initial condition, we have

$$s^3Y(x) - s^2 + 1 = \frac{1}{(s^2+1)} - \frac{1}{s^2} - \mathcal{L}\Big[\int_0^{\frac{\pi}{2}} xty'(t)dt\Big],$$

or

$$Y(x) = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^3(s^2+1)} - \frac{1}{s^5} - \frac{1}{s^3}\mathcal{L}\Big[\int_0^{\frac{\pi}{2}} xty'(t)dt\Big].$$

Using the inverse Laplace transform we get

$$y(x) = \cos x - \frac{x^4}{24} - \mathcal{L}^{-1}\Big[\frac{1}{s^3}\mathcal{L}(\int_0^{\frac{\pi}{2}} xty'(t))dt\Big]. \tag{37}$$

Decomposing the solution as an infinite sum given below

$$y(x) = \sum_{n=0}^{\infty} y_n(x). \tag{38}$$

Substituting (37) on (38) we get

$$\sum_{n=0}^{\infty} y_n(x) = \cos x - \frac{x^4}{24} - \mathcal{L}^{-1}\Big[\frac{1}{s^3}\mathcal{L}(\int_0^{\frac{\pi}{2}} xt\sum_{n=0}^{\infty} A_n(t))dt\Big],$$

in which $A_n = \sum_{j=0}^{n} y_j'$. *The recursive relation is given below:*

$$\begin{aligned}
y_0(x) &= \cos x,\\
y_1(x) &= -\frac{x^4}{24} - \mathcal{L}^{-1}\Big(\frac{1}{s^3}\mathcal{L}(\int_0^{\frac{\pi}{2}} xty_0'(t))dt)\Big) = 0,\\
y_{n+1}(x) &= -\mathcal{L}^{-1}\Big(\frac{1}{s^3}\mathcal{L}(\int_0^{\frac{\pi}{2}} xty_n'(t))dt)\Big) = 0, \qquad n \geq 0,
\end{aligned}$$

in which of $A_n = \sum_{j=0}^{n} y_j'$, *where for every* $n \geq 1, A_n = 0$. *Hence, the exact solution is*

$$y(x) = y_0(x) = \cos x.$$

Table 4: Numerical results of Example 4.3

X	$MVIM$	$MHPM$	$MLADM$	ADM	$Er_{(MVIM)}$	$Er_{(ADM)}$
5.0	-25.71633781	0.283662185	0.283662185	1.127197302	2.60×10^{1}	8.44×10^{-1}
10.0	-417.8390715	-0.839071529	-0.839071529	12.65749033	4.17×10^{2}	1.35×10^{1}
15.0	-2110.759688	-0.759687912	-0.759687912	67.5666565	2.11×10^{3}	6.83×10^{1}

5. Results and discussions

Tables 1, 2, 3, and 4 show the comparison between the analytical and approximate solution obtained by the proposed methods and the other methods viz VIM, ADM, and Bernstein Polynomials Method (BPM). The simplicity and accuracy of the proposed methods illustrate by computing the absolute error. The accuracy of the result can improve by introducing more terms of the approximate solutions. There is good agreement between the exact and approximate solution obtained by MVIM. MHPM and MLADM converge more easily than MVIM, and both techniques give us the same exact solutions as the examples.
Our results show that the proposed methods are efficient and powerful techniques that provide higher accuracy and closed-form solution approximations. And the approximate solution error is obtained by considering only the partial sum of the series.

6. Conclusion

In this paper, the modifications of the MVIM, MHPM, and MLADM have successfully been applied to find the solution of nonlinear Fredholm integro-differential equations of the second kind. The methods can be concluded that is very powerful and efficient techniques in finding exact solutions or approximate solutions for wide classes of problems.
The numerical results show that our proposed methods provide a sequence of functions that converges to the exact solution of the problem and reduce the computational difficulty for solving nonlinear Fredholm integro-differential equations of the second kind when compared to other traditional methods. The effectiveness of our methods examine in some examples and the results show that the techniques are easier than many other numerical techniques. These modifications are promising and readily implemented, which makes it a more efficient tool and more practical for solving linear and nonlinear integro-differential equations as well as give us an analytical solution for these type of equations.

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