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Compact and Matrix Operators on the Space $\left|\overline{N}_p^{\phi}\right|_k$

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Abstract

In this paper, determining the operator norm, we give certain characterizations of matrix transformations from the space $\left|\overline{N}_p^{\phi}\right|_k$, the space of all series summable by the absolute weighted mean summability method, to one of the classical sequence spaces c_0, c, l_∞. Also, we obtain the necessary and sufficient conditions for each matrix in these classes to be compact and establish a number of estimates or identities for the Hausdorff measures of noncompactness of the matrix operators in these classes.

1. Introduction

The summability theory has an important role in analysis, applied mathematics and engineering sciences, and has been studied by many authors for a long time. One of the main subjects in the summability theory is the theory of sequence spaces that concerns with the generalization of the notions of convergence for sequences and series. The main purpose is to assign a limit value for non-convergent series or sequences by using a transformation which is given by the most general linear mappings of infinite matrices. In this concept, the literature has still enlarged, concerned with characterizing completely all matrices which transform one given sequence space into another and also, many sequence spaces defined as domain of special matrices such as Euler, Nörlund, Hausdorff, Cesàro and weighted mean matrices and related matrix operators have been investigated by several authors (see, [**?**, **?**]). On the other hand, from a different point of view, using the concept of absolute summability, several new spaces of series summable by the absolute summability methods have taken place in the literature (see, for instance, [**?**]-[**?**]). In a recent paper, the sequence space $\left|\overline{N}_p^{\phi}\right|_k$ has introduced and studied by Sarıgöl [**?**, **?**], Mohapatra and Sarıgöl [**?**].

The present paper aims to characterize the infinite matrix classes $\left(\left|\overline{N}_p^{\phi}\right|_k, c\right)$, $\left(\left|\overline{N}_p^{\phi}\right|_k, c_0\right)$, $\left(\left|\overline{N}_p^{\phi}\right|_k, l_\infty\right)$ and to determine the operator norms for $1 \le k < \infty$. Further, the necessary and sufficient conditions for each matrix in these classes to be compact are obtained and certain identities or estimates for the Hausdorff measure of noncompactness are established.
A vector subspace of ω, the space of all sequences of real or complex numbers, is called a *sequence space*. The sequence spaces Φ, l_∞, c, c_0, b_s, c_s and l_k $(k \ge 1)$ stand for the sets of all finite, bounded, convergent and null sequences and the sets of all bounded, convergent and k-absolutely convergent series, respectively.
Let Λ and Γ be two arbitrary sequence spaces and $R = (r_{nv})$ be an infinite matrix of complex components. The transform sequence $R(\lambda)$ of the sequence $\lambda = (\lambda_v)$ is deduced by the usual matrix product and the components of $R(\lambda)$ are written as

$$R_n(\lambda) = \sum_{v=0}^{\infty} r_{nv}\lambda_v,$$

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provided that the series converges for all $n \in \mathbb{N}$. If the sequence $R(\lambda)$ exists and $R(\lambda) \in \Gamma$ for $\lambda \in \Lambda$, then, it is said that R is a matrix mapping from Λ into Γ. The collection of all such infinite matrices is denoted by (Λ, Γ).
The set

$$\Lambda_R = \{\lambda \in \omega : R(\lambda) \in \Lambda\}$$

is called domain of an infinite matrix R in the space Λ. Note that it is also a sequence space.
The β-dual of $\Lambda \subset \omega$ is the set

$$\Lambda^\beta = \left\{ a : \forall \lambda \in \Lambda, \sum_{v=0}^{\infty} a_v \lambda_v \text{ converges} \right\}.$$

Let Λ and Γ be Banach spaces. By $\mathscr{B}(\Lambda, \Gamma)$, we mean the set of all bounded (continuous) linear operators L from Λ to Γ. $\mathscr{B}(\Lambda, \Gamma)$ is also a Banach space with the operator norm given by

$$\|L\| = \sup_{\lambda \in S_\Lambda} \|L(\lambda)\|_\Gamma$$

for all $L \in \mathscr{B}(X, Y)$. Here, S_Λ represents the unit sphere in Λ, i.e.,

$$S_\Lambda = \{\lambda \in \Lambda : \|\lambda\| = 1\}.$$

If $a \in \omega$ and $\Lambda \supset \Phi$ is a BK-space, a Banach space on which all coordinate functional $p_n(\lambda) = \lambda_n$ are continuous for all n, then

$$\|a\|_\Lambda^* = \sup_{\lambda \in S_\Lambda} \left| \sum_{k=0}^{\infty} a_k \lambda_k \right|$$

provided the expression on the right side is defined and finite which is the case whenever $a \in \Lambda^\beta$.
If, for each $\lambda \in \Lambda$,

$$\left\| \lambda - \sum_{j=0}^{m} \lambda_j e^{(j)} \right\| \to 0 \text{ as } m \to \infty$$

then it is said that the BK-space Λ has AK property, and in this case we write $\lambda = \sum_{j=0}^{\infty} \lambda_j e^{(j)}$ where $e^{(j)}$ is a sequence whose only non-zero term is one in jth place for $j \in \mathbb{N}$.
Throughout the whole paper, assume that $\phi = (\phi_n)$ is a sequence of positive constants and $R = (r_{nv})$ is an infinite matrix of complex numbers for all $n, v \in \mathbb{N}$. Also, k^* is the conjugate of k, that is, $1/k + 1/k^* = 1$ for $k > 1$ and $1/k^* = 0$ for $k = 1$.
Let $\sum \lambda_n$ be an infinite series with its partial sum s_n. The series $\sum \lambda_v$ is said to be summable $|R, \phi_n|_k$, if (see[**?**]).

$$\sum_{n=1}^{\infty} \phi_n^{k-1} |\Delta R_n(s)|^k < \infty,$$

where $1 \leq k < \infty$ and $\Delta R_n(s) = R_n(s) - R_{n-1}(s)$. In the special case, when R is a weighted mean matrix, the summability method $|R, \phi_n|_k$ is reduced to $\left|\overline{N}, p_n, \phi\right|_k$ [**?**]. In recent paper, $\left|\overline{N}_p^\phi\right|_k$ has been generated from the space l_k as a set of all series summable by the absolute weighted mean method by Mohapatra and Sarıgöl [**?**] and Sarıgöl [**?**, **?**]. The space $\left|\overline{N}_p^\phi\right|_k$ can be expressed as

$$\left|\overline{N}_p^\phi\right|_k = \left\{ \lambda = (\lambda_v) : \sum_{n=1}^{\infty} \phi_n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v \right|^k < \infty \right\},$$

or equivalently, according to notation of domain, $\left|\overline{N}_p^\phi\right|_k = (l_k)_{T^{(p)}}$ where the matrix $T^{(p)}$ is given by

$$t_{nv}^{(p)} = \begin{cases} 1, \ n = 0, \ v = 0 \\ \phi_n^{1/k^*} \frac{p_n P_{v-1}}{P_n P_{n-1}}, \ 1 \leq v \leq n \\ 0, \qquad v > n, \end{cases}$$

whose inverse $S^{(p)}$ is

$$s_{nv}^{(p)} = \begin{cases} 1, \qquad n = 0, v = 0 \\ -\phi_{n-1}^{-1/k^*} \frac{P_{n-2}}{p_{n-1}}, \ v = n-1 \\ \phi_n^{-1/k^*} \frac{P_n}{p_n}, \qquad v = n \\ 0, \qquad v \neq n-1, n. \end{cases} \tag{1.1}$$

Besides, it is obvious that the space $\left|\overline{N}_p^\phi\right|_k$ is a BK-space with the norm $\|\lambda\|_{\left|\overline{N}_p^\phi\right|_k} = \left\|T^{(p)}(\lambda)\right\|_{l_k}$ and it is also linearly isomorphic to the space l_k for $1 \leq k < \infty$ [**?**].
We recall the following lemmas which are useful in proving our results:

Lemma 1.1. *[?] Let U be a triangle. Then,*

(i) For $\Lambda,\Gamma\subset\omega$, $R\in(\Lambda,\Gamma_U)$ iff $B=UR\in(\Lambda,\Gamma)$.

(ii) If Λ,Γ are BK-spaces and $R\in(\Lambda,\Gamma_U)$, then $\|L_R\|=\|L_B\|$.

Lemma 1.2. *[?] The following statements hold:*

1. $R\in(l,c)\Leftrightarrow$ *(i)* $\lim\limits_n r_{nv}$ *exists for all* $v\geq 0$, *(ii)* $\sup\limits_{n,v}|r_{nv}|<\infty$ *and* $R\in(l,l_\infty)\Leftrightarrow$ *(ii) holds.*
2. *If* $1<k<\infty$, *then,* $R\in(l_k,c)\Leftrightarrow(i)$ *holds,* $(iii)\sup\limits_n\sum\limits_{v=0}^{\infty}|r_{nv}|^{k^*}<\infty$ *and* $R\in(l_k,l_\infty)\Leftrightarrow(iii)$ *holds.*
3. $R\in(l,c_0)\Leftrightarrow$ *(ii) holds, (iv)* $\lim\limits_n r_{nv}=0$ *for all* $v\geq 0$.
4. *If* $1<k<\infty$, *then,* $R\in(l_k,c_0)\Leftrightarrow(iii)$ *and* (iv) *hold.*

Lemma 1.3. *[?] Let $1\leq k<\infty$. Then, $R\in(l,l_k)$ iff*

$$\|R\|_{(l,l_k)}=\sup_v\left\{\sum_{n=0}^{\infty}|r_{nv}|^k\right\}^{1/k}.$$

Lemma 1.4. *[?] Let $1<k<\infty$. Then, $R\in(l_k,l)$ iff*

$$\|R\|'_{(l_k,l)}=\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|r_{nv}|\right)^{k^*}\right\}^{1/k^*}<\infty.$$

Since

$$\|R\|_{(l_k,l)}\leq\|R\|'_{(l_k,l)}\leq 4\,\|R\|_{(l_k,l)},$$

there exists $1\leq\xi\leq 4$ such that $\|R\|'_{(l_k,l)}=\xi\,\|R\|_{(l_k,l)}$ where

$$\|R\|_{(l_k,l)}=\sup_{N\in\mathfrak{F}}\left\{\sum_{v=0}^{\infty}\left|\sum_{n\in N}r_{nv}\right|^{k^*}\right\}^{1/k^*}$$

and $\mathfrak{F}$ represents the collection of all finite subsets of $\mathbb{N}$.

Lemma 1.5. *[?] Let $1<k<\infty$ and k^* denote the conjugate of k. Then, we have $l_k^\beta=l_{k^*}$ and $l_\infty^\beta=c^\beta=c_0^\beta=l$, $l^\beta=l_\infty$. Also, if $\Lambda\in\{l_\infty,c,c_0,l,l_k\}$ then, we have*

$$\|a\|_\Lambda^*=\|a\|_{\Lambda^\beta}$$

for all $a\in\Lambda^\beta$, where $\|.\|_{\Lambda^\beta}$ is the natural norm on Λ^β.

Lemma 1.6. *[?] Let $\Lambda\supset\Phi$ be a BK-space and $\Gamma\in\{c,c_0,\ell_\infty\}$. If $R\in(\Lambda,\Gamma)$, then*

$$\|L_R\|=\|R\|_{(\Lambda,l_\infty)}=\sup_n\|R_n\|_\Lambda^*<\infty.$$

The Hausdorff measure of noncompactness χ was introduced by Goldenstein, Gohberg and Markus [?]. Using the Hausdorff measure of noncompactness, some compact operators on various sequence spaces are characterized by many authors. For example, Mursaleen and Noman in [?, ?], Malkowsky and Rakocevic in [?] have used the Hausdorff measure of noncompactness method to characterize the class of compact operators on some known spaces, (see also [?, ?, ?, ?], [?]-[?]).
Let (Λ,d) be a metric space and $H,M\subset\Lambda$. If there exists an $h\in H$ such that $d(h,m)<\varepsilon$ for every $m\in M$, then it is said that H is an ε-net of M; if H is finite, then the ε-net H of M is called a finite ε-net of M. Let Q be a bounded subset of the metric space Λ. Then, the Hausdorff measure of noncompactness of Q is defined by

$$\chi(Q)=\inf\{\varepsilon>0:Q\text{ has a finite }\varepsilon-\text{net in }\Lambda\},$$

and χ is called the Hausdorff measure of noncompactness.
Let Λ,Γ be Banach spaces. A linear operator L from Λ into Γ is called compact if its domain is all of Λ and, for every bounded sequence (λ_n) in Λ, $(L(\lambda_n))$ has a convergent subsequence in Γ. The class of all compact operators in $\mathscr{B}(\Lambda,\Gamma)$ is denoted by $\mathscr{C}(\Lambda,\Gamma)$.
The following lemmas give a calculation method for the Hausdorff measure of noncompactness of a bounded subset and the necessary and sufficient conditions a linear operator to be compact.

Lemma 1.7. *[?] Let Λ be one of the spaces c_0 or l_k for $1 \leq k < \infty$ and Q be a bounded subset of Λ. If $P_r : \Lambda \to \Lambda$ is the operator described by $P_r(\lambda) = (\lambda_0, \lambda_1, ...\lambda_r, 0, 0, ...)$ for all $\lambda \in \Lambda$, then*

$$\chi(Q) = \lim_{r\to\infty}\left(\sup_{\lambda\in Q}\|(I-P_r)(\lambda)\|\right).$$

Assume that χ_1, χ_2 are two Hausdorff measures on the spaces Λ, Γ and Q is a bounded subset of Λ. The linear operator $L : \Lambda \to \Gamma$ is said to be (χ_1, χ_2)- bounded if $L(Q)$ is a bounded subset of Γ and there exists a positive constant M such that $\chi_2(L(Q)) \leq M\chi_1(Q)$ for every Q. If an operator L is (χ_1, χ_2)- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded sets } Q \subset \Lambda\}$$

is called the (χ_1, χ_2)-measure noncompactness of L. In particular, for $\chi_1 = \chi_2 = \chi$, it is written by $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

Lemma 1.8. *[?] $L \in \mathscr{B}(\Lambda, \Gamma)$ and S_Λ be the unit sphere in X. Then,*

$$\|L\|_\chi = \chi(L(S_\Lambda))$$

and

$$L \text{ is compact } \Leftrightarrow \|L\|_\chi = 0.$$

Lemma 1.9. *[?] Let Λ be a normed sequence space, $U = (u_{nv})$ be an infinite triangle matrix, χ_U and χ denote the Hausdorff measures of noncompactness on M_{Λ_U} and M_Λ, the collections of all bounded sets in Λ_U and Λ, respectively. Then, $\chi_U(Q) = \chi(U(Q))$ for all $Q \in M_{\Lambda_U}$.*

Lemma 1.10. *[?] Let $\Lambda \supset \Phi$ be a BK-space with AK property or $\Lambda = l_\infty$. If $R \in (\Lambda, c)$, then we have*

$$\lim_{n\to\infty} r_{nk} = \alpha_k \text{ exists for all } k,$$

$$\alpha = (\alpha_k) \in \Lambda^\beta,$$

$$\sup_n \|R_n - \alpha\|^*_\Lambda < \infty,$$

$$\lim_{n\to\infty} R_n(\lambda) = \sum_{k=0}^{\infty} \alpha_k\lambda_k \text{ for every } \lambda = (\lambda_k) \in \Lambda.$$

Lemma 1.11. *[?] Let $X \supset \Phi$ be a BK-space. Then,*
(a) If $R \in (\Lambda, c_0)$, then

$$\|L_R\|_\chi = \lim_{r\to\infty}\left(\sup_{n>r}\|R_n\|^*_\Lambda\right).$$

(b) If Λ has AK property or $\Lambda = l_\infty$ and $R \in (\Lambda, c)$, then

$$\frac{1}{2}\lim_{r\to\infty}\left(\sup_{n>r}\|R_n-\alpha\|^*_\Lambda\right) \leq \|L_R\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n>r}\|R_n-\alpha\|^*_\Lambda\right)$$

where $\alpha = (\alpha_k)$ defined by $\alpha_k = \lim_{n\to\infty} r_{nk}$, for all $n \in \mathbb{N}$.
(c) If $R \in (\Lambda, l_\infty)$, then

$$0 \leq \|L_R\|_\chi \leq \lim_{r\to\infty}\left(\sup_{n>r}\|R_n\|^*_\Lambda\right).$$

2. Matrix and compact operators on space $\left|\overline{N}_p^\phi\right|_k$

In this section, by computing the operator norms we characterize infinite matrix classes $\left(\left|\overline{N}_p^\phi\right|_k, c\right)$, $\left(\left|\overline{N}_p^\phi\right|_k, c_0\right)$, $\left(\left|\overline{N}_p^\phi\right|_k, l_\infty\right)$ and also compact matrix classes $\mathscr{C}\left(\left|\overline{N}_p^\phi\right|_k, c\right)$, $\mathscr{C}\left(\left|\overline{N}_p^\phi\right|_k, c_0\right)$, $\mathscr{C}\left(\left|\overline{N}_p^\phi\right|_k, l_\infty\right)$. Moreover, we establish some identities or estimates for the Hausdorff measure of noncompactness.
For simplicity of notation, in what follows, we use

$$\sigma_{nv} = \Delta r_{nv}\frac{P_v}{p_v} + r_{n,v+1},$$

where $\Delta r_{nv} = r_{nv} - r_{n,v+1}$.

Lemma 2.1. *Let $1<k<\infty$. Then,*
(i) If $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$, then, $\tilde{a}^{(k)}=(\tilde{a}_v^{(k)})\in l_{k^*}$ for all $\lambda\in\left|\overline{N}_p^{\phi}\right|_k$
(ii) If $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$, then, $\tilde{a}^{(1)}=(\tilde{a}_v^{(1)})\in l_{\infty}$ for all $\lambda\in\left|\overline{N}_p^{\phi}\right|$
and the equality

$$\sum_{v=0}^{\infty}a_v\lambda_v=\sum_{v=0}^{\infty}\widetilde{a}_v^{(k)}y_v \tag{2.1}$$

holds, where $y=T^{(p)}(\lambda)$ and

$$\widetilde{a}_v^{(k)}=\phi_v^{-1/k^*}\left(\Delta a_v\frac{P_v}{p_v}+a_{v+1}\right) \text{ for } v>0, a_0=\widetilde{a}_0^{(k)}.$$

Proof. (i) Let $a=(a_v)\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$. By (**??**), the equation (**??**) is immediately obtained. Also, it follows from Lemma **??** that $\tilde{a}^{(k)}\in l_{k^*}$ whenever $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$, which completes the proof.
The proof of (ii) is left to reader. □

Lemma 2.2. *Let $1<k<\infty$. Then, we have $\|a\|^*_{\left|\overline{N}_p^{\phi}\right|_k}=\left\|\tilde{a}^{(k)}\right\|_{l_{k^*}}$ for all $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ and $\|a\|^*_{\left|\overline{N}_p^{\phi}\right|}=\left\|\tilde{a}^{(1)}\right\|_{\infty}$ for all $a\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$.*

Proof. Take $a\in\left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$. It is obvious from Lemma **??** that $\tilde{a}^{(k)}\in l_{k^*}$. Also, it follows from Lemma **??** and Lemma **??** that

$$\|a\|^*_{\left|\overline{N}_p^{\phi}\right|_k}=\sup_{\lambda\in S_{\left|\overline{N}_p^{\phi}\right|_k}}\left|\sum_{v=0}^{\infty}a_v\lambda_v\right|=\sup_{y\in S_{l_k}}\left|\sum_{v=0}^{\infty}\tilde{a}_v^{(k)}y_v\right|=\left\|\tilde{a}^{(k)}\right\|^*_{l_k}=\left\|\tilde{a}^{(k)}\right\|_{l_{k^*}}.$$

For $a\in\left\{\left|\overline{N}_p^{\phi}\right|\right\}^{\beta}$, the proof is similar, so it is left to reader. □

Theorem 2.3. *Let $1\le k<\infty$, Λ be arbitrary sequence space. Further, let $B=(b_{nj})$ be a matrix satisfying*

$$b_{nj}=\phi_n^{1/k^*}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}r_{vj}. \tag{2.2}$$

Then, $R\in\left(\Lambda,\left|\overline{N}_p^{\phi}\right|_k\right)$ iff $B\in(\Lambda,l_k)$.

Proof. Let $\lambda\in\Lambda$. Then, it follows from (**??**) that

$$\sum_{j=0}^{\infty}b_{nj}\lambda_j=\phi_n^{1/k^*}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}\sum_{j=0}^{\infty}\lambda_jr_{vj},$$

which implies that $B_n(\lambda)=T_n^{(p)}(R(\lambda))$. This gives that $R_n(\lambda)\in\left|\overline{N}_p^{\phi}\right|_k$ for all $\lambda\in\Lambda$ iff $B(\lambda)\in l_k$ for all $\lambda\in\Lambda$. So, the proof of the theorem is completed.
□

Let us define the matrix $\tilde{R}^{(k)}=\left(\tilde{r}_{nv}^{(k)}\right)$ with $\tilde{r}_{nv}^{(k)}=\frac{1}{\phi_v^{1/k^*}}\sigma_{nv}$ for $v>0, \tilde{r}_{n0}^{(k)}=r_{n0}$. It is clear that the matrices R and $\tilde{R}^{(k)}$ are connected by (**??**).

Theorem 2.4. (i) *Let $1<k<\infty$ and $\Lambda\in\{c_0,c,l_{\infty}\}$. Then,*

$$R\in\left(\left|\overline{N}_p^{\phi}\right|_k,\Lambda\right)\Rightarrow\|L_R\|=\sup_n\left\|\tilde{R}_n^{(k)}\right\|_{l_{k^*}}=\sup_n\left(\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*}\right)^{1/k^*},$$

$$R\in\left(\left|\overline{N}_p^{\phi}\right|,\Lambda\right)\Rightarrow\|L_R\|=\sup_n\left\|\tilde{R}_n^{(1)}\right\|_{l_{\infty}}=\sup_{n,v}\left|\tilde{r}_{nv}^{(1)}\right|.$$

(ii) *Let $1<k<\infty$. Then, there exists $1\le\xi\le4$ such that*

$$R \in \left(\left|\overline{N}_p^{\phi}\right|_k, l\right) \Rightarrow \|L_R\| = \frac{1}{\xi}\left\|\tilde{R}^{(k)}\right\|'_{(l_k,l)} = \frac{1}{\xi}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*}\right\}^{1/k^*},$$

$$R \in \left(\left|\overline{N}_p^{\phi}\right|, l_k\right) \Rightarrow \|L_R\| = \left\|\tilde{R}^{(1)}\right\|_{(l,l_k)} = \sup_v\left\{\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^k\right\}^{\frac{1}{k}},$$

$$R \in \left(\left|\overline{N}_p^{\phi}\right|, l\right) \Rightarrow \|L_R\| = \left\|\tilde{R}_n^{(1)}\right\|_{(l,l)} = \sup_v\sum_{n=0}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|.$$

Proof. The proof of the theorem is obtained from Lemma **??**, Lemma **??**, Lemma **??**, and Lemma **??**. □

Theorem 2.5. *Let $1 < k < \infty$. Then,*
a) $R \in (\left|\overline{N}_p^{\phi}\right|_k, c_0)$ iff

$$\lim_{n\to\infty}\tilde{r}_{nv}^{(k)} = 0 \text{ for all } v \tag{2.3}$$

$$\sup_n\sum_{v=0}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|^{k^*} < \infty \tag{2.4}$$

$$\sup_m\left\{\sum_{v=1}^{m-1}\frac{1}{\phi_v}\left|\sigma_{mv}\right|^{k^*} + \frac{1}{\phi_m}\left|r_{nm}\frac{P_m}{p_m}\right|^{k^*}\right\} < \infty \tag{2.5}$$

hold.
b) $R \in (\left|\overline{N}_p^{\phi}\right|_k, c)$ iff (**??**), (**??**) *and*

$$\lim_{n\to\infty}\tilde{r}_{nv}^{(k)} \text{ exists for all } v$$

hold.
c) $R \in (\left|\overline{N}_p^{\phi}\right|_k, l_\infty)$ iff (**??**), (**??**) *hold.*

Proof. Prove only the part (a) since the proofs of the other parts can be made the same way. $R \in (\left|\overline{N}_p^{\phi}\right|_k, c_0)$ if and only if $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ and $R(\lambda) \in c_0$ for every $\lambda \in \left|\overline{N}_p^{\phi}\right|_k$. It is seen immediately from Theorem 2.1 in [**?**] $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^{\phi}\right|_k\right\}^{\beta}$ if and only if (**??**) holds. Also, if any matrix $R \in (l_k, c_0)$, then the series $\sum_v r_{nv}\lambda_v$ converges uniformly in n and so

$$\lim_n\sum_v r_{nv}\lambda_v = \sum_v\lim_n r_{nv}\lambda_v. \tag{2.6}$$

On the other hand,

$$\lim_m\sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m\sum_{v=0}^{m} b_{mv}^{(n)}y_v$$

where $B^{(n)} = (b_{mv}^{(n)})$ is defined by

$$b_{mv}^{(n)} = \begin{cases} r_{n0}, & v = 0, \\ \frac{P_v}{\phi_v^{1/\mu_v^*}p_v}\left(r_{nv} - \frac{P_{v-1}}{P_v}r_{n,v+1}\right), & 1 \le v < m-1 \\ \frac{P_m r_{nm}}{\phi_m^{1/\mu_m^*}p_m}, & v = m, m \ge 1 \\ 0, & v > m. \end{cases}$$

So, it follows from (**??**)

$$R_n(\lambda) = \lim_m\sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m\sum_{v=0}^{m} b_{mv}^{(n)}y_v = \sum_{v=0}^{\infty}\tilde{r}_{nv}^{(k)}y_v = \tilde{R}_n^{(k)}(y).$$

It is clear that $R(\lambda) \in c_0$ for every $\lambda \in \left|\overline{N}_p^{\phi}\right|_k$ equals to $\tilde{R}^{(k)}(y) \in c_0$ for every $y \in l_k$ since $\left|\overline{N}_p^{\phi}\right|_k \cong l_k$. This means that $\tilde{R}^{(k)} \in (l_k, c_0)$. Applying Lemma **??** to the matrix $\tilde{R}^{(k)}$ the conditions (**??**) and (**??**) are obtained which completes the proof of the part (a).

□

Theorem 2.6. *The following statements hold:*
a) $R \in (\left|\overline{N}_p^\phi\right|, c_0)$ *iff*

$$\lim_{n\to\infty} \tilde{r}_{nv}^{(1)} = 0 \, for \; all \; v \tag{2.7}$$

$$\sup_{n,v} \left|\tilde{r}_{nv}^{(1)}\right| < \infty \tag{2.8}$$

$$\sup_{v} \left\{ |\sigma_{nv}| + \left| r_{nv} \frac{P_v}{p_v} \right| \right\} < \infty, \, for \; all \; n \tag{2.9}$$

hold.
b) $R \in (\left|\overline{N}_p^\phi\right|, c)$ *iff* (**??**), (**??**) *and*

$$\lim_{n\to\infty} \tilde{r}_{nv}^{(1)} \; exists \; for \; all \; v$$

hold.
c) $R \in (\left|\overline{N}_p^\phi\right|, l_\infty)$ *iff* (**??**), (**??**) *hold.*

Proof. (b) Let $R \in (\left|\overline{N}_p^\phi\right|, c)$. $R \in (\left|\overline{N}_p^\phi\right|, c)$ if and only if $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^\phi\right|\right\}^\beta$ and $R(\lambda) \in c$ for every $\lambda \in \left|\overline{N}_p^\phi\right|$. It follows from Theorem 2.1 in [**?**], $(r_{nv})_{v=0}^{\infty} \in \left\{\left|\overline{N}_p^\phi\right|\right\}^\beta$ if and only if (**??**) holds. Further, for any matrix $R \in (l, c)$, the series $\sum_v r_{nv}\lambda_v$ converges uniformly in n and so

$$\lim_n \sum_v r_{nv}\lambda_v = \sum_v \lim_n r_{nv}\lambda_v. \tag{2.10}$$

Also,

$$\lim_m \sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m \sum_{v=0}^{m} d_{mv}^{(n)} y_v$$

where the matrix $D^{(n)} = (d_{mv}^{(n)})$ is given by

$$d_{mv}^{(n)} = \begin{cases} \frac{P_v}{p_v}\left(r_{nv} - \frac{P_{v-1}}{P_v} r_{n,v+1}\right), 0 \le v < m-1 \\ \frac{P_m}{p_m} r_{nm}, \qquad v = m, m \ge 1 \\ 0, \qquad v > m. \end{cases}$$

So, it is deduced from (**??**)

$$R_n(\lambda) = \lim_m \sum_{v=0}^{m} r_{nv}\lambda_v = \lim_m \sum_{v=0}^{m} d_{mv}^{(n)} y_v = \sum_{v=0}^{\infty} \tilde{r}_{nv}^{(1)} y_v = \tilde{R}_n^{(1)}(y).$$

It is obvious that $R(\lambda) \in c$ for every $\lambda \in \left|\overline{N}_p^\phi\right|$ if and only if $\tilde{R}^{(1)}(\lambda) \in c$ for every $y \in l$, i.e., $\tilde{R}^{(1)} \in (l, c)$. Applying Lemma **??** to the matrix $\tilde{R}^{(1)}$ the conditions (**??**), (**??**) are obtained. This completes the proof of the part (b). The other parts can be proved by the similar way with Lemma **??**. □

Take the matrix $L = (l_{nj})$ defined by

$$l_{nj} = \begin{cases} 1, \; 0 \le j \le n \\ 0, \qquad j > n. \end{cases}$$

Then, since $b_s = \{l_\infty\}_L$ and $c_s = \{c\}_L$, the matrix classes $\left(\left|\bar{N}_p^\phi\right|_k, c_s\right)$ and $\left(\left|\bar{N}_p^\phi\right|_k, b_s\right)$ can be characterized as follows with Lemma **??**:

Corollary 2.7. *Let* $1 < k < \infty$. $R \in (\left|\overline{N}_p^\phi\right|_k, c_s)$ *iff*

$$\lim_{n\to\infty} \tilde{r}(n,v) exists \; for \; all \; v$$

$$\sup_n \sum_{v=0}^{\infty} |\tilde{r}(n,v)|^{k^*} < \infty \tag{2.11}$$

$$\sup_m \left\{ \sum_{v=1}^{m-1} |\tilde{r}(n,v)|^{k^*} + \frac{1}{\phi_m} \left| r(n,m) \frac{P_m}{p_m} \right|^{k^*} \right\} < \infty, \tag{2.12}$$

*$R \in (\left|\overline{N}_p^{\phi}\right|_k, b_s)$ if and only if (**??**) and (**??**) hold where $r(n,v) = \sum_{j=0}^{n} r_{jv}$, $R(n,v)$ and $\tilde{R}(n,v)$ are connected by (**??**).*

Theorem 2.8. *Suppose that $1 < k < \infty$. Then,*
a) $R \in (\left|\overline{N}_p^{\phi}\right|_k, c_0)$ iff

$$\|L_R\|_\chi = \limsup_{n\to\infty} \left(\sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)}\right|^{k^*} \right)^{1/k^*}$$

and $R \in \mathscr{C}(\left|\bar{N}_p^{\phi}\right|_k, c_0)$ iff $\limsup_{n\to\infty} \sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)}\right|^{k^*} = 0$.
b) $R \in (\left|\overline{N}_p^{\phi}\right|_k, c)$ iff

$$\frac{1}{2} \limsup_{n\to\infty} \left(\sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)} - \alpha_v\right|^{k^*} \right)^{1/k^*} \leq \|L_R\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)} - \alpha_v\right|^{k^*} \right)^{1/k^*}$$

and $R \in \mathscr{C}(\left|\overline{N}_p^{\phi}\right|_k, c_0)$ iff $\limsup_{n\to\infty} \sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)} - \alpha_v\right|^{k^*} = 0$ where $\alpha_v = \lim_{n\to\infty} \tilde{r}_{nv}^{(k)}$.
c) $R \in (\left|\overline{N}_p^{\phi}\right|_k, l_\infty)$ iff

$$0 \leq \|L_R\|_\chi \leq \limsup_{n\to\infty} \left(\sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)}\right|^{k^*} \right)^{1/k^*}$$

also, if $\limsup_{n\to\infty} \sum_{v=0}^{\infty} \left|\tilde{r}_{nv}^{(k)}\right|^{k^*} = 0$, then $R \in \mathscr{C}(\left|\overline{N}_p^{\phi}\right|_k, l_\infty)$.

Proof. To avoid repetition, only the proof of b is made and the proofs of (a) and (c) are left to the reader.
(b) Let $R \in \left(\left|\overline{N}_p^{\phi}\right|_k, c\right)$. To compute the Hausdorff measure of noncompactness of L_R, take the unit sphere $S_{\left|\overline{N}_p^{\phi}\right|_k}$ in the space $\left|\overline{N}_p^{\phi}\right|_k$. It is written from Lemma **??** that

$$\|L_R\|_\chi = \chi(RS_{\left|\overline{N}_p^{\phi}\right|_k}).$$

On the other hand, since $\left|\overline{N}_p^{\phi}\right|_k \cong l_k$, $R \in \left(\left|\overline{N}_p^{\phi}\right|_k, c\right)$ if and only if $\tilde{R}^{(k)} \in (l_k, c)$, and so

$$\|L_R\|_\chi = \chi(RS_{\left|\overline{N}_p^{\phi}\right|_k}) = \chi(\tilde{R}^{(k)} T^{(p)} S_{\left|\overline{N}_p^{\phi}\right|_k}) = \left\|L_{\tilde{R}^{(k)}}\right\|_\chi$$

which implies, by Lemma **??**,

$$\frac{1}{2} \lim_{r\to\infty} \left(\sup_{n\geq r} \left\|\tilde{R}_n^{(k)} - \alpha\right\|_{l_k}^{*} \right) \leq \|L_R\|_\chi \leq \lim_{r\to\infty} \left(\sup_{n\geq r} \left\|\tilde{R}_n^{(k)} - \alpha\right\|_{l_k}^{*} \right), \tag{2.13}$$

where $\alpha_v = \lim_{n\to\infty} \tilde{r}_{nv}^{(k)}$, for all $v \geq 0$.
By Lemma **??**, $\left\|\tilde{R}_n^{(k)} - \alpha\right\|_{l_k}^{*} = \left\|\tilde{R}_n^{(k)} - \alpha\right\|_{l_{k^*}}$. The last equality completes the first part of the proof of (b) with (**??**). Moreover, the compactness of L_R is immediately deduced from Lemma **??**. So, the proof of (b) is completed. □

We have the following theorems by following the above lines:

Theorem 2.9. (a) *If $R \in \left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$. Then*

$$\|L_R\|_\chi = \limsup_{n\to\infty} \left\|\tilde{R}_n^{(k)}\right\|_{l_\infty} = \limsup_{n\to\infty} \sup_v \left|\tilde{r}_{nv}^{(1)}\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$ *iff* $\lim\limits_{n\to\infty}\sup \sup\limits_{v}\left|\tilde{r}_{nv}^{(1)}\right| = 0$.

(b) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, c\right)$, *then*

$$\frac{1}{2}\lim_{n\to\infty}\sup \sup_{v}\left|\tilde{r}_{nv}^{(1)} - \alpha_v\right| \le \|L_R\|_{\chi} \le \lim_{n\to\infty}\sup \sup_{v}\left|\tilde{r}_{nv}^{(1)} - \alpha_v\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c\right)$ *iff* $\lim\limits_{n\to\infty}\sup \sup\limits_{v}\left|\tilde{r}_{nv}^{(1)} - \alpha_v\right| = 0$ *where* $\alpha_v = \lim\limits_{n\to\infty}\tilde{r}_{nv}^{(1)}$, *for all* $v \in \mathbb{N}$.

(c) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, l_\infty\right)$, *then*

$$0 \le \|L_R\|_{\chi} \le \lim_{n\to\infty}\sup \sup_{v}\left|\tilde{r}_{nv}^{(1)}\right|$$

and $R \in \mathscr{C}\left(\left|\overline{N}_p^{\phi}\right|, c_0\right)$ *if* $\lim\limits_{n\to\infty}\sup \sup\limits_{v}\left|\tilde{r}_{nv}^{(1)}\right| = 0$.

Theorem 2.10. (a) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|, l_k\right)$, $1 \le k < \infty$, *then*

$$\|L_R\|_{\chi} = \lim_{j\to\infty}\left\{\sup_{v}\left(\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^{k}\right)^{1/k}\right\},$$

and R is a compact operator iff $\lim\limits_{j\to\infty}\sup\limits_{v}\sum\limits_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^{k} = 0$.

(b) *If* $R \in \left(\left|\overline{N}_p^{\phi}\right|_k, l\right)$, $1 < k < \infty$, *then there exists* $1 \le \xi \le 4$ *such that*

$$\|L_R\|_{\chi} = \frac{1}{\xi}\lim_{j\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*}\right\}^{1/k^*},$$

and R is compact a compact operator iff $\lim\limits_{j\to\infty}\sum\limits_{v=0}^{\infty}\left(\sum\limits_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(k)}\right|\right)^{k^*} = 0$.

Proof. (a) Let $S_{\left|\overline{N}_p^{\phi}\right|}$ be a unit sphere in the space $\left|\overline{N}_p^{\phi}\right|$ and $R = \tilde{R}^{(1)} \circ T^{(p)}$. Since $\lambda \in S_{\left|\overline{N}_p^{\phi}\right|}$, $y = T^{(p)}(\lambda) \in S_l$. So, by Lemma **??**, Lemma **??** and Lemma **??**, it is written that

$$\begin{aligned}
\|R\|_{\chi} &= \chi\left(RS_{\left|\overline{N}_p^{\phi}\right|}\right) = \chi\left(\tilde{R}^{(1)} \circ T^{(p)}S_{\left|\overline{N}_p^{\phi}\right|_k}\right) \\
&= \lim_{j\to\infty}\left(\sup_{y\in T^{(p)}S_{\left|\overline{N}_p^{\phi}\right|}}\left\|(I-P_j)(\tilde{R}^{(1)}(y))\right\|\right) \\
&= \lim_{j\to\infty}\sup_{v}\left\{\sum_{n=j+1}^{\infty}\left|\tilde{r}_{nv}^{(1)}\right|^{k}\right\}^{1/k}
\end{aligned}$$

which completes the proof of the first part with Lemma **??**. The proof of (b) is similar, so it is omitted. □

3. Conclusion

The approach of constructing a lot of new sequence spaces by means of the matrix domain of some particular limitation methods have recently been employed by several authors in many research papers. Also, with a different point of view, using the concept of absolute summability method new sequence spaces have taken into the literature. For instance, in recent paper, $\left|\overline{N}_p^{\phi}\right|_k$ has been generated from the space l_k as a set of all series summable by the absolute weighted mean method by Mohapatra and Sarıgöl [**?**] and Sarıgöl [**?**, **?**]. In the present study, as a continuation of these papers, certain compact and matrix operatos from this space to one of the classical sequence spaces c, l_∞, c_0 are characterized and their norms and Hausdorff measures of noncompactness are determined. So, it has been brought a different perspective and studying field.

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References

[1] M. Ilkhan, *Matrix domain of a regular matrix derived by Euler Totient function in the spaces c_0 and c*, Mediterr. J. Math., **17**(1) (2020), 1-21.
[2] E. E. Kara, M. Başarır, *On compact operators and some Euler B(m)-difference sequence spaces*, J. Math. Anal. Appl., **379**(2) (2011), 499-511.
[3] T.M. FLett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc., **3**(1) (1957), 113-141.
[4] F. Gökçe, M.A. Sarıgöl, *Some matrix and compact operators of the absolute Fibonacci series spaces*, Kragujevac J. Math., **44**(2) (2020), 273–286.
[5] F. Gökçe, M.A. Sarıgöl, *Series spaces derived from absolute Fibonacci summability and matrix transformations*, Boll. Unione Mat. Ital., **13**(1) (2020), 29-38.
[6] F. Gökçe, M.A. Sarıgöl, *On absolute Euler spaces and related matrix operators*, Proc. Nat. Acad. Sci. India Sect. A, **90**(5) (2020), 769-775.
[7] F. Gökçe, M.A. Sarıgöl, *Generalization of the absolute Cesàro space and some matrix transformations*, Numer. Funct. Anal. Optim., **40**(9) (2019), 1039-1052.
[8] M.A. Sarıgöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar., **48**(3) (2011), 331-341.
[9] M.A. Sarıgöl, *Necessary and sufficient conditions for the equivalence of the summability methods $|N,p_n|_k$ and $|C,1|_k$*, Indian J. Pure Appl. Math., **22**(6) (1991), 483-489.
[10] R. N. Mohapatra, M.A. Sarıgöl, *On matrix operators on the series space $|\bar{N}_p^\theta|_k$*, Ukrainian Math. J., **69**(11) (2018), 1772-1783.
[11] M.A. Sarıgöl, *On the local properties of factored Fourier series*, Appl. Math. Comput., **216**(11) (2010), 3386-3390.
[12] W.T. Sulaiman, *On summability factors of infinite series*, Proc. Amer. Math. Soc., **115**(2) (1992), 313- 317.
[13] E. Malkowsky, V. Rakocevic, *An introduction into the theory of sequence space and measures of noncompactness*, Zb. Rad.(Beogr), **9**(17) (2000), 143-234.
[14] M. Stieglitz, H. Tietz, *Matrix transformationen von folgenraumen*, Eine Ergebnisübersicht. Math. Z. **154**(1) (1977), 1-16.
[15] I.J.Maddox, *Elements of functinal analysis*, Cambridge University Press, London, New York, 1970.
[16] M.A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci., **42**(3) (2015), 28-35.
[17] I. Djolovic, E. Malkowsky, *Matrix transformations and compact operators on some new mth-order difference sequences*, Appl. Math. Comput., **198**(2) (2008), 700-714.
[18] L.S. Goldenstein, I.T. Gohberg, A.S. Markus, *Investigations of some properties of bounded linear operators with their q-norms*, Uchen. Zap. Kishinev. Gos. Univ., **29** (1957), 29-36.
[19] M. Mursaleen, A.K. Noman, *Compactness of matrix operators on some new difference sequence spaces*, Linear Algebra Appl., **436**(1) (2012), 41-52.
[20] M. Mursaleen, A.K. Noman, *Compactness by the Hausdorff measure of noncompactness*, Nonlinear Anal., **73**(8) (2010), 2541-2557.
[21] E. Malkowsky, V. Rakocevic, *Measure of noncompactness of linear operators between spaces of sequences that are $(\bar{N},q)$ summable or bounded*, Czechoslovak Math. J., **51**(3) (2001), 505-522.
[22] G. C. Hazar Güleç, *Applications of measure of noncompactness in the series spaces of generalized absolute Cesàro means*, KFBD, **10**(1),(2020) 60-73.
[23] G. C. Hazar Güleç, *Compact matrix operators on absolute Cesàro spaces*, Numer. Funct. Anal. Optim., **41**(1) (2020), 1-15.
[24] E. Malkowsky, *Compact matrix operators between some BK− spaces, in: M. Mursaleen (Ed.), Modern Methods of Analysis and Its Applications*, Anamaya Publ., New Delhi, (2010), 86-120.
[25] M.A. Sarıgöl, *Norms and compactness of operators on absolute weighted mean summable series*, Kuwait J. Sci., **43**(4) (2016), 68-74.
[26] V. Rakocevic, *Measures of noncompactness and some applications*, Filomat, **12**(2) (1998), 87-120.
[27] A.M. Jarrah, E. Malkowsky, *Ordinary absolute and strong summability and matrix transformations*, Filomat **17** (2003), 59-78.
[28] E. Malkowsky, V. Rakocevic, *On matrix domains of triangles*, Appl. Math. Comput., **189**(2) (2007), 1146-1163.