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On The Convergence Properties of Kantorovich-Szász Type Operators Involving Tangent Polynomials

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Abstract

In this paper, we construct a variant of the Kantorovich Szász type operators involving the Tangent polynomials and estimate the convergence properties of these operators by using the Korovkin theorem. Also, we obtain the rate of convergence by using the modulus of continuity and Peetre's K-functional.

Keywords: Tangent polynomials; Kantorovich-Szász type operators; Rate of convergence; Generating functions; Korovkin theorems.

Tanjant Polinomlarını İçeren Kantorovich-Szász Tipli Operatörlerin Yakınsama Özellikleri Üzerine

Öz

Bu çalışmada, Tanjant polinomlarını içeren Kantorovich-Szász tipli operatörlerin bir varyantı oluşturulup, Korovkin teoremi kullanılarak bu operatörün yakınsaklık özellikleri tahmin edilmiştir. Ayrıca Peetre's K- fonksiyoneli ve süreklilik modülü kullanılarak yakınsaklık hızı elde edilmiştir.

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Anahtar Kelimeler: Tanjant polinomları; Kantorovich-Szász tipli operatörler; Yakınsaklık hızı; Üreteç fonksiyonu; Korovkin Teoremi.

1. Introduction

It is well known the Tangent numbers are the coefficients in Taylor expansion for $tant$ [1]:

$$\tan(t) = \sum_{n=0}^{\infty} (-1)^{n+1} T_{2n+1} \frac{t^{2n+1}}{(2n+1)!}, \ \ T_{2n} = 0 \tag{1}$$

C. S. Ryoo defined the Tangent polynomials with the aid of generating function by using fermionic p- adic invariant integral in [2]:

$$\sum_{k=0}^{\infty} T_k(x) \frac{t^k}{k!} = \left(\frac{2}{e^{2t}+1}\right) e^{xt}, |t| < \frac{\pi}{2} \tag{2}$$

In the case $x = 0, T_k(0) = T_k$ is called k th Tangent number. Expressions of these numbers through generating functions:

$$\sum_{k=0}^{\infty} T_k \frac{t^k}{k!} = \left(\frac{2}{e^{2t}+1}\right), \ \ |t| < \frac{\pi}{2} \tag{3}$$

Because of obtaining generating function for a set of functions, we derive differential recurrence relation, pure recurrence relation or calculate certain integral. Using generating functions, many fundamental properties and identities for special polynomials and numbers can be obtained [3, 4]. For $k \in \mathbb{Z}^+$, Tangent polynomials are expressed by using generating functions at the following identity:

$$T_k(x) = \sum_{l=0}^{k} \binom{k}{l} x^{k-l} T_l. \tag{4}$$

Nowadays, many researchers study the central moment and test function notations which are useful for examining the convergence of sequences of linear positive operators. Moreover, central moments of Kantorovich type operators are investigated with the aid of moment-generating functions [5-13].

The organization of this paper is as follows:

In section 2, we give a variant of the Kantorovich-Szász type operators involving the Tangent polynomials and estimate the convergence properties of these operators by using

Korovkin's theorem. In section 3, we evaluate the rate of convergence by using the modulus of continuity and Peetre's K-functional.

2. Convergence Properties

Firstly, we define our new operator involving Tangent polynomials at the following equation:

$$K_n^*(f,x) = n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)d_t, \tag{5}$$

where $T_{2k+1} > 0$ and $k \in \mathbb{N}$.

Now, we construct the moments and test functions for $r = \{0,1,2\}$ in the following lemmas:

$e_r = t^r$ and $\vartheta^r(t,x) = (t-x)^r$

We give the values of moments under $K_n^*(f,x)$ in Lemma 1 as follows:

Lemma 1. For $\forall x \in [0,\infty)$, Eqn. (1) has the following properties:

$K_n^*(e_0,x) = 1,$

$K_n^*(e_1,x) = x + \frac{1-3e^2}{2n(1+e^2)},$

$K_n^*(e_2,x) = x^2 + \frac{x(1-e^2)}{n}.$

Proof. Take $f = e_0$ in Eqn. (5). Then we have

$$\begin{aligned} K_n^*(1,x) &= n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} d_t \\ &= n\frac{e^2+1}{2}e^{-nx}\frac{2}{e^2+1}e^{nx}\left(\frac{k+1}{n}-\frac{k}{n}\right) \\ &= 1. \end{aligned}$$

Taking the derivative of both sides of Eqn. (5) with respect to t, we get

$$\sum_{k=0}^{\infty} kT_k(x)\frac{t^{k-1}}{k!} = \frac{-4e^{2t}}{(e^{2t}+1)^2}e^{xt} + x\frac{2}{e^{2t}+1}e^{xt}. \tag{6}$$

Replacing $t = 1$ and $x = nx$ in Eqn. (6), we have

$$\sum_{k=0}^{\infty}\frac{kT_k(x)}{k!}=\frac{-4e^2}{(e^2+1)^2}e^{xt}+x\frac{2}{e^2+1}e^{nx}. \tag{7}$$

In the same manner, we obtain

$$\sum_{k=0}^{\infty}\frac{k^2T_k(x)}{k!}=(nx)^2\frac{2}{e^2+1}e^{nx}+\left(\frac{2-6e^2}{(e^2+1)^2}\right)nxe^{nx}+\frac{2e^6-8e^4-12e^2}{(e^2+1)^4}e^{nx}. \tag{8}$$

For $r=1$, we have

$$\begin{aligned}K_n^*(t,x)&=n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}td_t\\
&=n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\left[\frac{t^2}{2}\right]_{\frac{k}{n}}^{\frac{k+1}{n}}\\
&=n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\frac{(2k+1)}{n^2}\\
&=2n\frac{e^2+1}{2n^2}e^{-nx}\sum_{k=0}^{\infty}k\frac{T_k(nx)}{k!}+n\frac{e^2+1}{2n^2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\\
&=x+\frac{1-3e^2}{2n(1+e^2)}.\end{aligned}$$

For $r=2$, we obtain

$$\begin{aligned}K_n^*(t^2,x)&=n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}t^2d_t\\
&=n\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\left[\frac{t^3}{3}\right]_{\frac{k}{n}}^{\frac{k+1}{n}}\\
&=\frac{n}{3}\frac{e^2+1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\frac{(3k^2+3k+1)}{n^3}\\
&=\frac{e^2+1}{2n^2}e^{-nx}\sum_{k=0}^{\infty}k^2\frac{T_k(nx)}{k!}+\frac{e^2+1}{2n^2}e^{-nx}\sum_{k=0}^{\infty}k\frac{T_k(nx)}{k!}\\
&\quad+\frac{1}{3}\frac{e^2+1}{2n^2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\\
&=x^2+\frac{x(1-e^2)}{n}.\end{aligned}$$

This completes the proof.

Lemma 2. For $\forall x \in [0,\infty)$, we have

$$K_n^*((t-x),x) = \frac{1-3e^2}{2n(1+e^2)}, \tag{9}$$

$$K_n^*((t-x)^2,x) = \frac{-e^4+3e^2}{n(1+e^2)}\,x. \tag{10}$$

Proof. By using the linearity of (K_n^*), we have

$$K_n^*((t-x),x) = K_n^*(t,x) - xK_n^*(1,x) = \frac{1-3e^2}{2n(1+e^2)}$$

$$\begin{aligned} K_n^*((t-x)^2,x) &= K_n^*(t^2,x) - 2xK_n^*(t,x) + x^2K_n^*(1,x) \\ &= x^2 + \frac{x(1-e^2)}{n} - 2x\left(x + \frac{1-3e^2}{2n(1+e^2)}\right) + x^2 \\ &= \frac{x(1-e^2)}{n} - \frac{x(1-3e^2)}{n(1+e^2)} \\ &= \frac{-e^4+3e^2}{n(1+e^2)}\,x. \end{aligned}$$

So the desired results are obtained.

Now we give the main theorem by using the Lemma 1 and Korovkin's theorem in the following:

Theorem 3. Let $f \in C_E[0,\infty] = C[0,\infty] \cap E$, where

$$E := \left\{f: x \in [0,\infty], \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty\right\}$$

and

$$\lim_{y\to\infty} \frac{B'(y)}{B(y)} = 1 \text{ and } \lim_{y\to\infty} \frac{B''(y)}{B(y)} = 1.$$

Then the sequence (K_n^*) converges uniformly to f on $[0,\infty]$ as $n \to \infty$.

Proof. By applying the well-known Korovkin's first theorem and Lemma 1, we obtain

$$\lim_{n\to\infty} K_n^*(e_i,x) = x^i\,, i = 0,1,2\,. \tag{11}$$

and the operator (K_n^*) converges uniformly in each compact subset of $[0,\infty]$.The proof is completed by using the property (vii) of Theorem 4.1.4 in [6].

Estimating the degree of approximation by positive linear operators involving generating functions of special polynomials has several methods. One of them is the modulus of continuity given by

$$w(f,\delta) = sup\{f(t) - f(x), x, t \in [a,b], |t - x| \leq \delta\}, f \in C[0,\infty], \qquad \delta > 0, \tag{12}$$

where $C[0,\infty]$ is called as space of uniformly continuous functions on $[0,\infty]$. Also,

$$|f(t) - f(x)| \leq w(f,\delta)\left(\frac{|t - x|}{\delta} + 1\right)$$

is satisfied for any $\delta > 0$ and each $x \in [0,\infty)$

We give an estimate for degree of approximation of $K_n^*(f,x)$ with the help of definition of modulus of continuity in the following theorem:

Theorem 4. If $\in C[0,\infty] \cap E$, then

$$|K_n^*(f,x) - f(x)| \leq 2w\left(f; \sqrt{\frac{-e^4 + 3e^2}{n(1 + e^2)}\, x}\right). \tag{13}$$

Proof. We have

$$|K_n^*(f,x) - f(x)| \leq n\frac{e^2 + 1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t) - f(x)|d_t \tag{14}$$

$$\leq n\frac{e^2 + 1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\left(\frac{|t - x|}{\delta} + 1\right)w(f;\delta)d_t$$

$$\leq \left(1 + \frac{n}{\delta}\frac{e^2 + 1}{2}e^{-nx}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t - x|\, d_t\right)w(f;\delta)\,.$$

By applying Cauchy-Schwarz inequality, we get

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t - x|\, d_t \leq \frac{1}{\sqrt{n}}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t - x|^2\, d_t\right)^{\frac{1}{2}}, \tag{15}$$

that satisfies

$$\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\,d_t \leq \frac{1}{\sqrt{n}}\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\left(\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|^2\,d_t\right)^{\frac{1}{2}}. \tag{16}$$

By using the Cauchy-Schwarz inequality for the sum on the right side of Eqn. (16), we have

$$\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\,d_t \leq \frac{\sqrt{\frac{2}{e^2+1}e^{nx}}}{\sqrt{n}}\left(\frac{\frac{2}{e^2+1}e^{nx}}{n}K_n^*((t-x)^2;x)\right)^{\frac{1}{2}} \tag{17}$$

$$=\frac{e^{nx}}{n}\left(\frac{2}{e^2+1}\right)\left(K_n^*((t-x)^2;x)\right)^{\frac{1}{2}}.$$

From Lemma 2, using the second-order test function of the operator (K_n^*), we have the final form of Eqn. (17) as follows:

$$\sum_{k=0}^{\infty}\frac{T_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|\,d_t \leq \frac{e^{nx}}{n}\frac{2}{e^2+1}\left(\frac{-e^4+3e^2}{n(1+e^2)}\right)^{\frac{1}{2}}.$$

Hence, the proof is completed.

Another method for estimate of rate of approximation is the second order modulus of continuity given by

$$w_2(f,h) = sup\left\{\left|f(t)-2f\left(\frac{t+x}{2}\right)+f(x)\right|; x,t\in[a,b], |t-x|\leq 2h\right\}, \tag{18}$$

where $f\in C[a,b], h>0$.

The classical Peetre's K- functional is defined by

$$K(f,\delta) = \inf_{g\in C_B^2[0,\infty)}\{\|f-g\|+\delta\|g''\|\}, \tag{19}$$

where $f\in C_B[0,\infty)$ and $C_B^2[0,\infty)=\{g\in C_B[0,\infty): g', g''\in C_B[0,\infty)\}$.

Theorem 5. For the $K_n^*(f,x)$ operators, we have

$$|K_n^*(f,x)-f(x)| \leq \psi\|f\|_{C_B^2}, \tag{20}$$

where $\psi := \frac{-e^4+3e^2}{n(1+e^2)}\,x+\frac{1-3e^2}{2n(1+e^2)}$.

Proof. By applying linearity properties of $K_n^*(f,x)$ and Taylor expansion of f, we have

$$K_n^*(f,x)-f(x)=f(x)'K_n^*(s-x,x)+\frac{1}{2}f(\rho)''K_n^*((s-x)^2,x),\rho\in(x,s). \tag{21}$$

From Lemma 2, we get

$$|K_n^*(f,x)-f(x)|\leq\frac{-e^4+3e^2}{n(1+e^2)}x\|f'\|_{C_B}+\frac{1}{2}\frac{1-3e^2}{2n(1+e^2)}\|f''\|_{C_B}. \tag{22}$$

Therefore, the desired result is obtained.

Theorem 6. Let $f\in C_B[0,\infty)$. Then

$$|K_n^*(f,x)-f(x)|\leq 2M\left\{w_2\left(f,\sqrt{\frac{1}{2}\psi}\right)+min\left(1,\frac{1}{2}\psi\right)\right\}\|f\|_{C_B}, \tag{23}$$

where $M>0$.

Proof. We rewrite $f(t)-f(x)$ as

$$f(t)-f(x)=f(t)-g(t)+g(t)-g(x)+g(x)-f(x). \tag{24}$$

From the linearity properties of $K_n^*(f,x)$, we have

$$|K_n^*(f,x)-f(x)|\leq|K_n^*(f-g,x)-f(x)|+|K_n^*(g,x)-g(x)|+|f(x)-g(x)|. \tag{25}$$

where we assume that the function $g\in C^2[0,\infty)$. By using Theorem 5, we get

$$|K_n^*(f,x)-f(x)|\leq 2\|f-g\|_{C_B}+\psi\|g\|_{C_B^2}=2\left[\|f-g\|_{C_B}+\psi\|g\|_{C_B^2}\right]. \tag{26}$$

From Eqn. (26), we obtain

$$|K_n^*(f,x)-f(x)|\leq 2K(f,\psi), \tag{27}$$

where $K(f,\psi)$ is Peetre's K- functional defined by Eq.(19). We obtain the desired result.

4. Conclusion

The generating function has many useful applications in several fields. For example, generating function of Bernstein polynomials provides important results to constructing Bezier curves. In literature, many properties and relations are obtained by using the generating function of Tangent polynomials.

Recently, the approximation theory is one of the important application fields of generating function of special polynomials which have useful properties for constructing linear positive operators.

In this paper, we constructed a Kantorovich-Szász type operator, $K_n^*(f,x)$, involving the generating function of Tangent polynomials. Then, we investigated some properties such as modulus of continuity, second-order modulus of continuity, and Peetre's K functional for $K_n^*(f,x)$.

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References

[1] Qi, F., *Derivaties of tangent function and tangent numbers,* Applied Mathematics and Computations, 268, 844-858, 2015.

[2] Ryoo, C.S., *A note on the tangent numbers and polynomials*, Advanced Studies in Theoretical Physics, 7(9), 447-454, 2013.

[3] Ryoo, C.S., *On poly-tangent numbers and polynomials and the distribution of their zeros*, Journal of Applied Mathematics and Informatics, 34, 487-494, 2016.

[4] Ryoo, C.S., *Differential equations assosiated with Tangent numbers*, Journal of Applied Mathematics and Informatics, 34, 487-494, 2016.

[5] Paltanea, R., *Approximation theory using positive linear operators*, Birkhauser Boston, 2004.

[6] Altomare, F., Campiti M. *Korovkin-type approximation theory and its applications*, Walter de Gruyter, 17, 2011.

[7] Aktaş, R., Söylemez, D., Taşdelen, F., *Stancu type generalization of Szász-Durrmeyer operators involving Brenke-type polynomials*, Filomat, 33(3), 855-868, 2019.

[8] Taşdelen, F., Aktaş, R., Altın, A., *A Kantorovich type of Szász operators including Brenke-type polynomials,* Abstract and Applied Analysis, 2012, 2012.

[9] Varma, S., Taşdelen, F., *Szász type operators involving Charlier polynomials*, Mathematical and Computer Modelling, 56(5-6), 118-122, 2012.

[10] Cekim, B., Aktaş, R., İçöz, G., *Kantorovich-Stancu type operators including Boas-Buck type polynomials*, Hacettepe Journal of Mathematics and Statistics, 48(2), 460-471, 2019.

[11] İçöz, G., Varma, S., Sucu, S., *Approximation by operators including generalized Appell polynomials*. Filomat, 30(2), 429-440, 2016.

[12] Atakut, Ç., Büyükyazıcı, İ., *Approximation by Kantorovich-Szász type operators based on Brenke type polynomials*, Numerical Functional Analysis and Optimization, 37(12), 1488-1502, 2016.

[13] Tasdelen, F., Söylemez, D., Aktaş, R. *Dunkl Gamma Type Operators ıncluding Appell polynomials*, Complex Analysis and Operator Theory, 13, 3359-3371, 2019.