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Some results on soft topological notions

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Abstract — Recently, the generalizations of soft open sets have become a popular subject. These generalizations define based on the concepts of the soft interior and soft closure. Therefore, the properties related to these concepts play an essential role in propositions concerning the generalizations. To this end, we consider the soft interior and soft closure through the concept of the soft element, and thus we clarify the relationships between a soft topological space and its soft subspace topologies. Afterwards, we mention soft α-open sets, soft α-closed sets, and soft α-T_0 space via soft elements. Finally, we discuss soft α-separation axioms for further research.

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1. Introduction

The topological notions of the soft sets [1] first were introduced in two different studies in 2011. In the first of these studies, Çağman et al. [2] have defined soft topology on a soft set. They have proposed basic topological concepts, such as soft open set, soft interior, soft closure, soft limit point, and soft boundary with elements of a soft set and investigated some of their basic properties. On the other, Shabir and Naz [3] have described the concept of soft topology on a universal set. Moreover, they have suggested the basic definitions and properties of the concept regarding a universal set's elements. Afterwards, Enginoğlu et al. [4] have updated the definition of the soft closed set provided in [2] and several theorems related to it to eliminate inconsistencies between definitions and theorems. So far, many researchers have conducted studies [5-14] on various topological concepts ranging from soft separation axioms to soft compactness.

Recently, the researchers have focused on soft α-open sets [15], soft pre-open sets [16], soft semi-open sets [17], and soft β-open sets [16]. Akdağ and Özkan [15] have defined soft α-continuous and soft α-open functions and investigated their relationships among the other soft continuous structures. After that, Akdağ and Özkan [18] have introduced soft α-separation axioms by using the elements of a universal set. Moreover, they have studied some of their fundamental properties and compared the soft α-separation axioms with the soft separation axioms. Khattak et al. [19] have propounded soft α-separation axioms in terms of soft points. Saleh and Sur [20] have proposed novel separation axioms called soft α-R_0, soft α-symmetric, and soft α-R_1 using soft points.

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The present study's primary goal is to study some relationships concerning the soft interior and soft closure of a soft set in a soft topological space and its soft subspace to avail from them in the next research. The second one is to define soft α-T_0 space by using the soft element's concept [5] and the soft topological notions provided in [2,4] and analyse some of its fundamental properties.

In Section 2 of the present paper, we present the soft topological notions, such as soft element, soft open set, soft interior, and soft closure, and some of their basic properties to be utilised in the following sections. In Section 3, we study the relationships related to soft interior and soft closure concepts between soft topological space and its soft subspace. In Section 4, as different from the literature, we propose some properties related to soft α-open sets and soft α-closed sets via the concept of the soft element. Furthermore, we describe soft α-T_0 space via the soft element concept and investigate several of its basic properties. In Section 5, we discuss soft α-separation axioms for further research.

2. Preliminaries

In this section, firstly, we present the concepts of soft sets [1,21] and soft element [5] and some of their basic properties [4,8,21] to be employed in the following sections. Throughout this study, let U be a universal set, E be a parameter set, and $P(U)$ be the power set of U.

Definition 2.1. [1] Let f be a function from E to $P(U)$. Then, the set $F := \{(x, f(x)) : x \in E\}$ is called a soft set parameterized via E over U (or briefly over U).

Definition 2.2. [21] Let $A \subseteq E$ and f_A be a function from E to $P(U)$ such that $f_A(x) = \emptyset$ if $x \notin A$. Then, the set $F_A := \{(x, f_A(x)) : x \in E\}$ is called a soft set parameterized via E over U (or briefly over U).

In the present paper, the set of all the soft sets over U is denoted by $S(U)$. Besides, as long as it causes no confusion, we do not display the elements $(x, \emptyset)$ in a soft set.

Definition 2.3. [21] Let $F_A \in S(U)$. For all $x \in A$, if $f_A(x) = \emptyset$, then F_A is called empty soft set and is denoted by F_Φ, and if $f_A(x) = U$, then F_A is called A-universal soft set and is denoted by $F_{\tilde{A}}$. Moreover, E-universal soft set is called universal soft set and is denoted by $F_{\tilde{E}}$.

Definition 2.4. [21] Let $F_A, F_B \in S(U)$. For all $x \in E$, if $f_A(x) \subseteq f_B(x)$ then, F_A is called a soft subset of F_B and is denoted by $F_A \tilde{\subseteq} F_B$, and if $f_A(x) = f_B(x)$, then F_A and F_B are called equal soft sets and is denoted by $F_A = F_B$.

Proposition 2.5. [21] Let $F_A, F_B, F_C \in S(U)$. Then, $F_A \tilde{\subseteq} F_{\tilde{E}}$, $F_\Phi \tilde{\subseteq} F_A$, $F_A \tilde{\subseteq} F_A$, $(F_A \tilde{\subseteq} F_B \wedge F_B \tilde{\subseteq} F_C) \Rightarrow F_A \tilde{\subseteq} F_C$, $(F_A = F_B \wedge F_B = F_C) \Rightarrow F_A = F_C$, and $(F_A \tilde{\subseteq} F_B \wedge F_B \tilde{\subseteq} F_A) \Leftrightarrow F_A = F_B$.

Definition 2.6. [21] Let $F_A, F_B, F_C \in S(U)$. For all $x \in E$, if $f_C(x) = f_A(x) \cup f_B(x)$, then F_C is called soft union of F_A and F_B and is denoted by $F_A \tilde{\cup} F_B$, and if $f_C(x) = f_A(x) \cap f_B(x)$, then F_C is called soft intersection of F_A and F_B and is denoted by $F_A \tilde{\cap} F_B$.

Definition 2.7. [21] Let $F_A, F_B, F_C \in S(U)$. For all $x \in E$, if $f_C(x) = f_A(x) \setminus f_B(x)$, then F_C is called soft difference between F_A and F_B and is denoted by $F_A \tilde{\setminus} F_B$.

Definition 2.8. [21] Let $F_A, F_B \in S(U)$. For all $x \in E$, if $f_B(x) = f_A^c(x)$, then F_B is called soft complement of F_A and is denoted by $F_A^{\tilde{c}}$. Here, $f_A^c(x) := f_{\tilde{E}}(x) \setminus f_A(x)$.

Proposition 2.9. [21] Let $F_A, F_B, F_C \in S(U)$. Then,

i. $F_A \tilde{\cup} F_A = F_A$ and $F_A \tilde{\cap} F_A = F_A$

ii. $F_A \tilde{\cup} F_\Phi = F_A$ and $F_A \tilde{\cap} F_{\tilde{E}} = F_A$

iii. $F_A \tilde{\cup} F_{\tilde{E}} = F_{\tilde{E}}$ and $F_A \tilde{\cap} F_\Phi = F_\Phi$

iv. $F_A \tilde{\cup} F_B = F_B \tilde{\cup} F_A$ and $F_A \tilde{\cap} F_B = F_B \tilde{\cap} F_A$

v. $(F_A \tilde{\cup} F_B) \tilde{\cup} F_C = F_A \tilde{\cup} (F_B \tilde{\cup} F_C)$ and $(F_A \tilde{\cap} F_B) \tilde{\cap} F_C = F_A \tilde{\cap} (F_B \tilde{\cap} F_C)$

vi. $F_A \tilde{\cup} (F_B \tilde{\cap} F_C) = (F_A \tilde{\cup} F_B) \tilde{\cap} (F_A \tilde{\cup} F_C)$ and $F_A \tilde{\cap} (F_B \tilde{\cup} F_C) = (F_A \tilde{\cap} F_B) \tilde{\cup} (F_A \tilde{\cap} F_C)$

vii. $F_A \tilde{\cup} F_A^{\tilde{c}} = F_{\tilde{E}}$ and $F_A \tilde{\cap} F_A^{\tilde{c}} = F_\Phi$

viii. $(F_A \tilde{\cup} F_B)^{\tilde{c}} = F_A^{\tilde{c}} \tilde{\cap} F_B^{\tilde{c}}$ and $(F_A \tilde{\cap} F_B)^{\tilde{c}} = F_A^{\tilde{c}} \tilde{\cup} F_B^{\tilde{c}}$

ix. $\left(F_A^{\tilde{c}}\right)^{\tilde{c}} = F_A$

x. $F_A \tilde{\setminus} F_B = F_A \tilde{\cap} F_B^{\tilde{c}}$

Proposition 2.10. [4,8] Let $F_A, F_B, F_C, F_D \in S(U)$. Then,

i. $(F_A \tilde{\subseteq} F_B \ \wedge \ F_C \tilde{\subseteq} F_D) \Rightarrow F_A \tilde{\cap} F_C \tilde{\subseteq} F_B \tilde{\cap} F_D$

ii. $(F_A \tilde{\subseteq} F_B \ \wedge \ F_C \tilde{\subseteq} F_D) \Rightarrow F_A \tilde{\cup} F_C \tilde{\subseteq} F_B \tilde{\cup} F_D$

iii. $(F_B, F_C \tilde{\subseteq} F_A \wedge F_B \tilde{\cap} F_C = F_\Phi) \Leftrightarrow F_B \tilde{\subseteq} \left(F_A \tilde{\setminus} F_C\right)$

Example 2.11. [2] Let $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$ such that $U = \{u_1, u_2, u_3\}$, $E = \{x_1, x_2, x_3\}$, and $A = \{x_1, x_2\}$. Then, all the soft subsets of F_A are as follows:

$F_{A_1} = \{(x_1, \{u_1\})\}$
$F_{A_2} = \{(x_1, \{u_2\})\}$
$F_{A_3} = \{(x_1, \{u_1, u_2\})\}$
$F_{A_4} = \{(x_2, \{u_2\})\}$
$F_{A_5} = \{(x_2, \{u_3\})\}$
$F_{A_6} = \{(x_2, \{u_2, u_3\})\}$
$F_{A_7} = \{(x_1, \{u_1\}), (x_2, \{u_2\})\}$
$F_{A_8} = \{(x_1, \{u_1\}), (x_2, \{u_3\})\}$
$F_{A_9} = \{(x_1, \{u_1\}), (x_2, \{u_2, u_3\})\}$
$F_{A_{10}} = \{(x_1, \{u_2\}), (x_2, \{u_2\})\}$
$F_{A_{11}} = \{(x_1, \{u_2\}), (x_2, \{u_3\})\}$
$F_{A_{12}} = \{(x_1, \{u_2\}), (x_2, \{u_2, u_3\})\}$
$F_{A_{13}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2\})\}$
$F_{A_{14}} = \{(x_1, \{u_1, u_2\}), (x_2, \{u_3\})\}$
$F_{A_{15}} = F_A$
$F_{A_{16}} = F_\Phi$

Definition 2.12. [22] Let $F_A \in S(U)$ and $F_B \tilde{\subseteq} F_A$. If, for $x_0 \in B$, $f_B(x_0) \neq \emptyset$, and for all $y \in B \setminus \{x_0\}$, $f_B(y) = \emptyset$, then F_B is called a soft point of F_A and is denoted by $F_B \tilde{\in} F_A$.

Definition 2.13. [5] Let $F_A \in S(U)$ and $F_B \tilde{\subseteq} F_A$. If, for $x_0 \in B$, $f_B(x_0)$ is a single point set and for all $y \in B \setminus \{x_0\}$, $f_B(y) = \emptyset$, then F_B is called a soft element of F_A and is denoted by $F_B \tilde{\in} F_A$.

This study exploits Definition 2.13. Throughout this paper, the set of all the soft elements of F_A is denoted by $\mathcal{S}(F_A)$ or briefly $\mathcal{S}$. It is clear that $F_B \tilde{\in} F_A$ and $F_B \in \mathcal{S}(F_A)$ are the same.

Example 2.14. For Example 2.11, $\{(x_1, \{u_1\})\}, \{(x_1, \{u_2\})\}, \{(x_2, \{u_2\})\}, \{(x_2, \{u_3\})\} \tilde{\in} F_A$ and $\{(x_1, \{u_1\})\}$, $\{(x_2, \{u_2\})\}, \{(x_2, \{u_3\})\} \tilde{\in} F_{A_9}$.

Proposition 2.15. [8] Let $F_A, F_B \in S(U)$. Then, $F_A \tilde{\subseteq} F_B$ iff $\varepsilon \tilde{\in} F_B$, for all $\varepsilon \tilde{\in} F_A$. Besides, $\mathcal{S}(F_A \tilde{\cup} F_B) = \{\varepsilon : \varepsilon \tilde{\in} F_A \ \vee \ \varepsilon \tilde{\in} F_B\}$. Namely, $\varepsilon \tilde{\in} F_A \tilde{\cup} F_B \Leftrightarrow \varepsilon \tilde{\in} F_A \ \vee \ \varepsilon \tilde{\in} F_B$. Similarly, $\varepsilon \tilde{\in} F_A \tilde{\cap} F_B \Leftrightarrow \varepsilon \tilde{\in} F_A \ \wedge \ \varepsilon \tilde{\in} F_B$, $\varepsilon \tilde{\in} F_A \tilde{\setminus} F_B \Leftrightarrow \varepsilon \tilde{\in} F_A \ \wedge \ \varepsilon \tilde{\notin} F_B$, and $\varepsilon \tilde{\in} F_A^{\tilde{c}} \Leftrightarrow \varepsilon \tilde{\notin} F_A \ \wedge \ \varepsilon \tilde{\in} F_{\tilde{E}}$.

Secondly, we provide some of the soft topological notions [2,4,5,7,15,23], such as soft open sets, soft interior, soft closure, and soft α-open sets and some of their basic properties to be used in the next sections.

Definition 2.16. [2] Let $F_A \in S(U)$ and $\tilde{\tau}$ be a collection of soft subsets of F_A. If

i. $F_\Phi, F_A \in \tilde{\tau}$

ii. Countable soft unions of the soft subsets of $\tilde{\tau}$ belong to $\tilde{\tau}$.

iii. Finite soft intersections of the soft subsets of $\tilde{\tau}$ belong to $\tilde{\tau}$.

then $\tilde{\tau}$ is called a soft topology on F_A. Moreover, the ordered pair $(F_A, \tilde{\tau})$ is referred to as a soft topological space.

Here, this study utilizes the arbitrary soft unions instead of the countable soft unions provided in Definition 2.16.

Definition 2.17. [2,4] Let $(F_A, \tilde{\tau})$ be a soft topological space. Then, every element of $\tilde{\tau}$ is called a $\tilde{\tau}$-soft open set (or briefly soft open). Furthermore, if $F_B \in \tilde{\tau}$, then $F_A \tilde{\setminus} F_B$ is referred to as a $\tilde{\tau}$-soft closed set (or briefly soft closed).

From now on, set of all the soft closed sets of $(F_A, \tilde{\tau})$ is denoted by $\tilde{\tau}^k$.

Proposition 2.18. [2,4] In $(F_A, \tilde{\tau})$, F_Φ and F_A are both soft open and soft closed.

Example 2.19. For Example 2.11, $\tilde{\tau} = \{F_\Phi, F_A, F_{A_1}, F_{A_7}, F_{A_{14}}\}$ is a soft topology on F_A and so $(F_A, \tilde{\tau})$ is a soft topological space.

Definition 2.20. [2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, the collection $\{F_{A_i} \tilde{\cap} F_B : F_{A_i} \in \tilde{\tau},\ i \in I\}$ is called a soft subspace topology on F_B and is denoted by $\tilde{\tau}_{F_B}$. Moreover, $(F_B, \tilde{\tau}_{F_B})$ is referred to as a soft topological subspace of $(F_A, \tilde{\tau})$.

Proposition 2.21. [2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, a soft subspace topology on F_B is a soft topology on F_B.

Example 2.22. For Example 2.19, $\tilde{\tau}_{F_{A_7}} = \{F_\Phi, F_{A_7}, F_{A_1}\}$ and $\tilde{\tau}_{F_{A_{12}}} = \{F_\Phi, F_{A_{12}}, F_{A_4}, F_{A_{11}}\}$ are soft subspace topologies on F_{A_7} and $F_{A_{12}}$, respectively. Therefore, $(F_{A_7}, \tilde{\tau}_{F_{A_7}})$ and $(F_{A_{12}}, \tilde{\tau}_{F_{A_{12}}})$ are soft topological subspaces of $(F_A, \tilde{\tau})$.

Theorem 2.23. [4] Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$. If $F_C \in \tilde{\tau}_{F_B}$, then there exists at least one $F_D \in \tilde{\tau}$ such that $F_C \tilde{\subseteq} F_D$.

Theorem 2.24. [4] Let $(F_A, \tilde{\tau})$ be a soft topological space. Then, $\tilde{\tau}^k$ provides the following conditions:

i. F_Φ and F_A are soft closed.

ii. Arbitrary soft intersections of the soft closed sets are soft closed.

iii. Finite soft unions of the soft closed sets are soft closed.

Definition 2.25. [5] Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \tilde{\subseteq} F_A$, and $\varepsilon \tilde{\in} F_B$. If there exists at least one $F_C \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_C$ and $F_C \tilde{\subseteq} F_B$, then ε is called a $\tilde{\tau}$-soft interior point (or briefly soft interior point) of F_B. Moreover, the soft union of all the soft interior points of F_B is called $\tilde{\tau}$-soft interior (or briefly soft interior) of F_B and is denoted by ${F_B}^\circ$.

Definition 2.26. [2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, the soft intersection of all the soft closed sets containing F_B is called $\tilde{\tau}$-soft closure (or briefly soft closure) of F_B and is denoted by $\overline{F_B}$.

Proposition 2.27. [2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, the soft interior of F_B is soft union all the soft open subsets of F_B. In other words, ${F_B}^\circ$ is the biggest soft open set contained by F_B. Moreover, $\overline{F_B}$ is the smallest soft closed set containing F_B.

Proposition 2.28. [2] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then,

i. $F_B \in \tilde{\tau} \Leftrightarrow F_B = {F_B}^\circ$

ii. $F_B \in \tilde{\tau}^k \Leftrightarrow F_B = \overline{F_B}$

iii. $({F_B}^\circ)^\circ = {F_B}^\circ$ and $\overline{(\overline{F_B})} = \overline{F_B}$

iv. $(F_B \tilde{\subseteq} F_C \Rightarrow {F_B}^\circ \tilde{\subseteq} {F_C}^\circ)$ and $(F_B \tilde{\subseteq} F_C \Rightarrow \overline{F_B} \tilde{\subseteq} \overline{F_C})$

v. ${F_B}^\circ \tilde{\cap} {F_C}^\circ = (F_B \tilde{\cap} F_C)^\circ$ and $\overline{(F_B \tilde{\cap} F_C)} \tilde{\subseteq} \overline{F_B} \tilde{\cap} \overline{F_C}$

vi. $F_B^{\circ} \tilde{\cup} F_C^{\circ} \tilde{\subseteq} (F_B \tilde{\cup} F_C)^{\circ}$ and $\overline{F_B} \tilde{\cup} \overline{F_C} = \overline{(F_B \tilde{\cup} F_C)}$

vii. $F_B^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F_B}$

Theorem 2.29. [2,4] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then, $\varepsilon \tilde{\in} \overline{F_B}$ iff, for all $F_C \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_C$, $F_B \tilde{\cap} F_C \neq F_\Phi$.

Proposition 2.30. [7] Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$.

i. $F_C \in \tilde{\tau}_{F_B}{}^k$ iff there exists at least one $F_D \in \tilde{\tau}^k$ such that $F_C = F_D \tilde{\cap} F_B$.

ii. If $F_C \in \tilde{\tau}$, then $F_C \in \tilde{\tau}_{F_B}$.

iii. If $F_C \in \tilde{\tau}^k$, then $F_C \in \tilde{\tau}_{F_B}{}^k$.

Proposition 2.31. [23] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. If F_B is soft open, then $F_B \tilde{\cap} \overline{F_C} \tilde{\subseteq} \overline{F_B \tilde{\cap} F_C}$.

Definition 2.32. [15] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. If $F_B \tilde{\subseteq} \left(\overline{F_B^{\circ}}\right)^{\circ}$, then F_B is called a soft α-open set in $\tilde{\tau}$. If F_B is a soft α-open set, then $F_A \tilde{\setminus} F_B$ is called a soft α-closed set in $\tilde{\tau}$.

Hereinafter, the set of all the soft α-open sets and all the soft α-closed sets in $(F_A, \tilde{\tau})$ are denoted by $S\alpha O(\tilde{\tau})$ and $S\alpha C(\tilde{\tau})$, respectively.

Proposition 2.33. [15] In a soft topological space, the arbitrary soft unions of soft α-open sets are soft α-open set. Moreover, the arbitrary soft intersections of soft α-closed sets are soft α-closed set.

Definition 2.34. [15] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, $\tilde{\tau}$-soft α-interior (or briefly soft α-interior) of F_B is defined by $\tilde{\bigcup}\{F_C \in S\alpha O(\tilde{\tau}) : F_C \tilde{\subseteq} F_B\}$ and is denoted by $(F_B)^{\circ}_{\alpha}$, and $\tilde{\tau}$-soft α-closure (or briefly soft α-closure) of F_B is defined by $\tilde{\bigcap}\{F_C \in S\alpha C(\tilde{\tau}) : F_B \tilde{\subseteq} F_C\}$ and is denoted by $\overline{(F_B)}_{\alpha}$.

Proposition 2.35. [15] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then,

i. $F_B \in S\alpha C(\tilde{\tau}) \Leftrightarrow \overline{(F_B)}_{\alpha} = F_B$

ii. $F_B \tilde{\subseteq} F_C \Rightarrow \overline{(F_B)}_{\alpha} \tilde{\subseteq} \overline{(F_C)}_{\alpha}$

Theorem 2.36. [15] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, $F_B \in S\alpha C(\tilde{\tau})$ iff $\overline{(\overline{F_B})^{\circ}} \tilde{\subseteq} F_B$.

Theorem 2.37. [23] Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then, $F_B \in S\alpha O(\tilde{\tau})$ iff there exists at least one $F_C \in \tilde{\tau}$ such that $F_C \tilde{\subseteq} F_B \tilde{\subseteq} (\overline{F_C})^{\circ}$.

3. Relationships between Soft Interior and Soft Closure in Soft Topological Spaces and Their Soft Subspaces

This section studies several properties containing relationships between the soft interior and soft closure of a soft set according to a soft topological space and its soft subspace.

Theorem 3.1. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

i. If F_B is soft open, then $F_B \tilde{\subseteq} \overline{F_B^{\circ}}$.

ii. If F_B is soft closed, then $(\overline{F_B})^{\circ} \tilde{\subseteq} F_B$.

Proof. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

i. If F_B is soft open, then $F_B = F_B^{\circ}$. Moreover, from Proposition 2.28 (vii), $F_B \tilde{\subseteq} \overline{F_B}$. Thus, $F_B \tilde{\subseteq} \overline{F_B^{\circ}}$.

ii. If F_B is soft closed, then $F_B = \overline{F_B}$. Moreover, from Proposition 2.28 (vii), $F_B^{\circ} \tilde{\subseteq} F_B$. Thus, $(\overline{F_B})^{\circ} \tilde{\subseteq} F_B$.

The converse of the propositions provided in Theorem 3.1 is not always correct. This situation is proved with the following example.

Example 3.2. For Example 2.19, $\overline{F_{A_9}}^{\circ} = \overline{F_{A_7}} = F_A$. Thus, $F_{A_9} \tilde{\subseteq} \overline{F_{A_9}}^{\circ}$, but F_{A_9} is not soft open. Similarly, $\left(\overline{F_{A_5}}\right)^{\circ} = {F_{A_{11}}}^{\circ} = F_{\Phi}$. Thus, $\left(\overline{F_{A_5}}\right)^{\circ} \tilde{\subseteq} F_{A_5}$, but F_{A_5} is not soft closed.

Theorem 3.3. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. If F_B is soft open, then $\overline{F_B \tilde{\cap} \overline{F_C}} = \overline{F_B \tilde{\cap} F_C}$.

Proof. Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B, F_C \tilde{\subseteq} F_A$, and F_B be a soft open. Then, from Proposition 2.31, $F_B \tilde{\cap} \overline{F_C} \tilde{\subseteq} \overline{F_B \tilde{\cap} F_C}$ and from Proposition 2.28 (iv), $\overline{F_B \tilde{\cap} \overline{F_C}} \tilde{\subseteq} \overline{\left(\overline{F_B \tilde{\cap} F_C}\right)}$. Therefore, $\overline{F_B \tilde{\cap} \overline{F_C}} \tilde{\subseteq} \overline{F_B \tilde{\cap} F_C}$. Moreover, since F_B is soft open, then $F_B \tilde{\subseteq} {F_B}^{\circ}$. From Proposition 2.28 (vii), $F_C \tilde{\subseteq} \overline{F_C}$. Therefore, $F_B \tilde{\cap} F_C \tilde{\subseteq} {F_B}^{\circ} \tilde{\cap} \overline{F_C}$ and so $\overline{F_B \tilde{\cap} F_C} \tilde{\subseteq} \overline{F_B \tilde{\cap} \overline{F_C}}$. Consequently, $\overline{F_B \tilde{\cap} \overline{F_C}} = \overline{F_B \tilde{\cap} F_C}$.

Theorem 3.4. Let $\left(F_B, \tilde{\tau}_{F_B}\right)$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$.

i. If $F_B \in \tilde{\tau}$ and $F_C \in \tilde{\tau}_{F_B}$, then $F_C \in \tilde{\tau}$.

ii. If $F_B \in \tilde{\tau}^k$ and $F_C \in {\tilde{\tau}_{F_B}}^{k}$, then $F_C \in \tilde{\tau}^k$.

Proof. Let $\left(F_B, \tilde{\tau}_{F_B}\right)$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$.

i. Let $F_B \in \tilde{\tau}$ and $F_C \in \tilde{\tau}_{F_B}$. Then, there exists at least one $F_D \in \tilde{\tau}$ such that $F_C = F_D \tilde{\cap} F_B$. Moreover, since $F_B, F_D \in \tilde{\tau}$, then $F_D \tilde{\cap} F_B \in \tilde{\tau}$. Therefore, $F_C \in \tilde{\tau}$.

ii. Let $F_B \in \tilde{\tau}^k$ and $F_C \in {\tilde{\tau}_{F_B}}^{k}$. Then, from Proposition 2.30 (i), there exists at least one $F_D \in \tilde{\tau}^k$ such that $F_C = F_D \tilde{\cap} F_B$. Moreover, since $F_B, F_D \in \tilde{\tau}^k$, then $F_D \tilde{\cap} F_B \in \tilde{\tau}^k$. Therefore, $F_C \in \tilde{\tau}^k$.

Henceforth, ${F_B}^{\circ\tilde{\tau}_1}$ and $\overline{F_B}^{\tilde{\tau}_1}$ indicate $\tilde{\tau}_1$-soft interior and $\tilde{\tau}_1$-soft closure of F_B, respectively.

Theorem 3.5. Let $(F_A, \tilde{\tau}_1)$ and $(F_A, \tilde{\tau}_2)$ be two soft topological spaces and $F_B \tilde{\subseteq} F_A$. Then,

i. $\tilde{\tau}_1 \subseteq \tilde{\tau}_2 \Leftrightarrow {F_B}^{\circ\tilde{\tau}_1} \tilde{\subseteq} {F_B}^{\circ\tilde{\tau}_2}$

ii. $\tilde{\tau}_1 \subseteq \tilde{\tau}_2 \Leftrightarrow \overline{F_B}^{\tilde{\tau}_2} \tilde{\subseteq} \overline{F_B}^{\tilde{\tau}_1}$

Proof. Let $(F_A, \tilde{\tau}_1)$ and $(F_A, \tilde{\tau}_2)$ be two soft topological spaces and $F_B \tilde{\subseteq} F_A$.

i. ($\Rightarrow$): Let $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$ and $\varepsilon \tilde{\in} {F_B}^{\circ\tilde{\tau}_1}$. Then, there exists at least one $F_C \in \tilde{\tau}_1$ such that $\varepsilon \tilde{\in} F_C$ and $F_C \tilde{\subseteq} F_B$. Since $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$, then $F_C \in \tilde{\tau}_2$. That is, there exists at least one $F_C \in \tilde{\tau}_2$ such that $\varepsilon \tilde{\in} F_C$ and $F_C \tilde{\subseteq} F_B$. In other words, $\varepsilon \tilde{\in} {F_B}^{\circ\tilde{\tau}_1} \Rightarrow \varepsilon \tilde{\in} {F_B}^{\circ\tilde{\tau}_2}$. Hence, ${F_B}^{\circ\tilde{\tau}_1} \tilde{\subseteq} {F_B}^{\circ\tilde{\tau}_2}$.

($\Leftarrow$): Let ${F_B}^{\circ\tilde{\tau}_1} \tilde{\subseteq} {F_B}^{\circ\tilde{\tau}_2}$, for all $F_B \tilde{\subseteq} F_A$. Then, for all $F_C \in \tilde{\tau}_1$, the hypothesis is valid. Since $F_C = {F_C}^{\circ\tilde{\tau}_1}$, then $F_C \tilde{\subseteq} {F_C}^{\circ\tilde{\tau}_2}$. Besides, ${F_C}^{\circ\tilde{\tau}_2} \tilde{\subseteq} F_C$. That is, ${F_C}^{\circ\tilde{\tau}_2} = F_C$. Therefore, $F_C \in \tilde{\tau}_1 \Rightarrow F_C \in \tilde{\tau}_2$. Hence, $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$.

ii. ($\Rightarrow$): Let $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$ and $\varepsilon \tilde{\in} \overline{F_B}^{\tilde{\tau}_2}$. Then, from Theorem 2.29, for all $F_C \in \tilde{\tau}_2$ such that $\varepsilon \tilde{\in} F_C$, $F_B \tilde{\cap} F_C \neq F_{\Phi}$. Since $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$, then for all $F_C \in \tilde{\tau}_1$ such that $\varepsilon \tilde{\in} F_C$, $F_B \tilde{\cap} F_C \neq F_{\Phi}$. Therefore, $\varepsilon \tilde{\in} \overline{F_B}^{\tilde{\tau}_1}$. Hence, $\overline{F_B}^{\tilde{\tau}_2} \tilde{\subseteq} \overline{F_B}^{\tilde{\tau}_1}$.

($\Leftarrow$): Let $\overline{F_B}^{\tilde{\tau}_2} \tilde{\subseteq} \overline{F_B}^{\tilde{\tau}_1}$, for all $F_B \tilde{\subseteq} F_A$. Then, for all $F_A \tilde{\setminus} F_C \in \tilde{\tau}_1$, the hypothesis is valid. Since $F_C = \overline{F_C}^{\tilde{\tau}_1}$, then $\overline{F_C}^{\tilde{\tau}_2} \tilde{\subseteq} F_C$. Besides, $F_C \tilde{\subseteq} \overline{F_C}^{\tilde{\tau}_2}$. That is, $\overline{F_C}^{\tilde{\tau}_2} = F_C$. Therefore, $F_A \tilde{\setminus} F_C \in \tilde{\tau}_2$. Hence, $\tilde{\tau}_1 \subseteq \tilde{\tau}_2$.

Theorem 3.6. Let $\left(F_B, \tilde{\tau}_{F_B}\right)$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B$. Then,

i. ${F_C}^{\circ} \tilde{\subseteq} {F_C}^{\circ\tilde{\tau}_{F_B}}$

ii. $\overline{F_C}^{\tilde{\tau}_{F_B}} \tilde{\subseteq} \overline{F_C}$

Proof. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B$.

i. Let $\varepsilon \tilde{\in} F_C^{\circ}$. Then, there exists at least one $F_D \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_D$ and $F_D \tilde{\subseteq} F_C$. Therefore, from Proposition 2.30 (ii), there exists at least one $F_D \in \tilde{\tau}_{F_B}$ such that $\varepsilon \tilde{\in} F_D$ and $F_D \tilde{\subseteq} F_C$. Therefore, $\varepsilon \tilde{\in} F_C^{\circ\tilde{\tau}_{F_B}}$. Hence, $F_C^{\circ} \tilde{\subseteq} F_C^{\circ\tilde{\tau}_{F_B}}$.

ii. Let $\varepsilon \tilde{\in} \overline{F_C}^{\tilde{\tau}_{F_B}}$. Then, from Theorem 2.29, for all $F_D \in \tilde{\tau}_{F_B}$ such that $\varepsilon \tilde{\in} F_D$, $F_C \tilde{\cap} F_D \neq F_\Phi$. Besides, for all $F_D \in \tilde{\tau}_{F_B}$, there exists at least one $F_K \in \tilde{\tau}$ such that $F_D = F_K \tilde{\cap} F_B$. Since $F_K \tilde{\cap} F_B \tilde{\subseteq} F_K$ and $F_K \in \tilde{\tau}$, then for all $F_K \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_K$, $F_C \tilde{\cap} F_K \neq F_\Phi$. Therefore, $\varepsilon \tilde{\in} \overline{F_C}$. Hence, $\overline{F_C}^{\tilde{\tau}_{F_B}} \tilde{\subseteq} \overline{F_C}$.

Theorem 3.7. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B$. Then,

i. If $F_B \in \tilde{\tau}$, then $F_C^{\circ\tilde{\tau}_{F_B}} \tilde{\subseteq} F_C^{\circ}$.

ii. If $F_B \in \tilde{\tau}$, then $\overline{F_C} \tilde{\subseteq} \overline{F_C}^{\tilde{\tau}_{F_B}}$.

Proof. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B$.

i. Let $F_B \in \tilde{\tau}$ and $\varepsilon \tilde{\in} F_C^{\circ\tilde{\tau}_{F_B}}$. Then, there exists at least one $F_D \in \tilde{\tau}_{F_B}$ such that $\varepsilon \tilde{\in} F_D$ and $F_D \tilde{\subseteq} F_C$. Therefore, from Theorem 3.4 (i), there exists at least one $F_D \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_D$ and $F_D \tilde{\subseteq} F_C$. Thus, $\varepsilon \tilde{\in} F_C^{\circ}$. Hence, $F_C^{\circ\tilde{\tau}_{F_B}} \tilde{\subseteq} F_C^{\circ}$.

ii. Let $F_B \in \tilde{\tau}$ and $\varepsilon \tilde{\in} \overline{F_C}$. Then, from Theorem 2.29, for all $F_D \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_D$, $F_C \tilde{\cap} F_D \neq F_\Phi$. Moreover, for all $F_D \tilde{\cap} F_B \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_D \tilde{\cap} F_B$, $F_C \tilde{\cap} (F_D \tilde{\cap} F_B) \neq F_\Phi$. Therefore, for all $F_D \tilde{\cap} F_B \in \tilde{\tau}_{F_B}$ such that $\varepsilon \tilde{\in} F_D \tilde{\cap} F_B$, $F_C \tilde{\cap} (F_D \tilde{\cap} F_B) \neq F_\Phi$. Thus, $\varepsilon \tilde{\in} \overline{F_C}^{\tilde{\tau}_{F_B}}$. Hence, $\overline{F_C} \tilde{\subseteq} \overline{F_C}^{\tilde{\tau}_{F_B}}$.

Corollary 3.8. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$, $F_B \in \tilde{\tau}$, and $F_C \tilde{\subseteq} F_B$. Then,

i. $F_C^{\circ} = F_C^{\circ\tilde{\tau}_{F_B}}$

ii. $\overline{F_C} = \overline{F_C}^{\tilde{\tau}_{F_B}}$

Corollary 3.9. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$. Then,

i. $F_C^{\circ} \tilde{\cap} F_B \tilde{\subseteq} F_C^{\circ\tilde{\tau}_{F_B}}$

ii. $\overline{F_C}^{\tilde{\tau}_{F_B}} = \overline{F_C} \tilde{\cap} F_B$

Proof. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$. Then,

i. From Theorem 3.6 (i), $F_C^{\circ} \tilde{\cap} F_B \tilde{\subseteq} F_C^{\circ} \tilde{\subseteq} F_C^{\circ\tilde{\tau}_{F_B}}$.

ii. From Theorem 3.6 (ii), $\overline{F_C}^{\tilde{\tau}_{F_B}} \tilde{\subseteq} \overline{F_C} \tilde{\cap} F_B$. On the other hand, let $\varepsilon \tilde{\in} \overline{F_C} \tilde{\cap} F_B$. Then, $\varepsilon \tilde{\in} \overline{F_C}$ and $\varepsilon \tilde{\in} F_B$. That is, for all $F_K \in \tilde{\tau}$ such that $\varepsilon \tilde{\in} F_K$, $F_C \tilde{\cap} F_K \neq F_\Phi$ and $\varepsilon \tilde{\in} F_B$. Therefore, for all $F_K \tilde{\cap} F_B \in \tilde{\tau}_{F_B}$ such that $\varepsilon \tilde{\in} F_K \tilde{\cap} F_B$, $F_C \tilde{\cap} (F_K \tilde{\cap} F_B) \neq F_\Phi$ and so $\varepsilon \tilde{\in} \overline{F_C}^{\tilde{\tau}_{F_B}}$. Hence, $\overline{F_C} \tilde{\cap} F_B \tilde{\subseteq} \overline{F_C}^{\tilde{\tau}_{F_B}}$. Consequently, $\overline{F_C}^{\tilde{\tau}_{F_B}} = \overline{F_C} \tilde{\cap} F_B$.

Theorem 3.10. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_B, F_C \tilde{\subseteq} F_A$. Then, $\overline{F_B \tilde{\cap} F_C}^{\tilde{\tau}_{F_B}} \tilde{\subseteq} F_B \tilde{\cap} \overline{F_C}$.

Proof. Let $(F_B, \tilde{\tau}_{F_B})$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_B, F_C \tilde{\subseteq} F_A$. We have $F_B \tilde{\cap} F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$. Because of Corollary 3.9 (ii) and $F_B \tilde{\subseteq} \overline{F_B}$,

$$\overline{F_B \tilde{\cap} F_C}^{\tilde{\tau}_{F_B}} = \overline{F_B \tilde{\cap} F_C} \tilde{\cap} F_B \tilde{\subseteq} \overline{F_B} \tilde{\cap} \overline{F_C} \tilde{\cap} F_B = (\overline{F_B} \tilde{\cap} F_B) \tilde{\cap} \overline{F_C} = F_B \tilde{\cap} \overline{F_C}$$

Hence, $\overline{F_B \tilde{\cap} F_C}^{\tilde{\tau}_{F_B}} \tilde{\subseteq} F_B \tilde{\cap} \overline{F_C}$.

4. Soft α-open Sets and Soft α-T_0 Spaces

In this section, we introduce some properties of soft α-open sets and soft α-closed sets. Moreover, we define soft α-T_0 space and study some of its basic properties.

Theorem 4.1. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. If $F_B \in \tilde{\tau}$ and $F_C \in S\alpha O(\tilde{\tau})$, then $F_B \tilde{\cap} F_C \in S\alpha O(\tilde{\tau})$.

Proof. Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B, F_C \tilde{\subseteq} F_A$, $F_B \in \tilde{\tau}$, and $F_C \in S\alpha O(\tilde{\tau})$. Since

$$
\begin{aligned}
F_B \in \tilde{\tau} \wedge F_C \in S\alpha O(\tilde{\tau}) \quad & \Rightarrow F_B = F_B^{\circ} \wedge F_C \tilde{\subseteq} \left(\overline{F_C^{\circ}}\right)^{\circ} \\
& \Rightarrow F_B \tilde{\cap} F_C \tilde{\subseteq} F_B^{\circ} \tilde{\cap} \left(\overline{F_C^{\circ}}\right)^{\circ} \\
& \Rightarrow F_B \tilde{\cap} F_C \tilde{\subseteq} \left(F_B \tilde{\cap} \overline{F_C^{\circ}}\right)^{\circ}, \text{ from Proposition 2.28 (v)} \\
& \Rightarrow F_B \tilde{\cap} F_C \tilde{\subseteq} \left(\overline{F_B \tilde{\cap} F_C^{\circ}}\right)^{\circ}, \text{ from Proposition 2.31} \\
& \Rightarrow F_B \tilde{\cap} F_C \tilde{\subseteq} \left(\overline{F_B^{\circ} \tilde{\cap} F_C^{\circ}}\right)^{\circ} \\
& \Rightarrow F_B \tilde{\cap} F_C \tilde{\subseteq} \left(\overline{(F_B \tilde{\cap} F_C)^{\circ}}\right)^{\circ}
\end{aligned}
$$

then $F_B \tilde{\cap} F_C \in S\alpha O(\tilde{\tau})$.

Theorem 4.2. Let $\left(F_B, \tilde{\tau}_{F_B}\right)$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$. If $F_C \in S\alpha O(\tilde{\tau})$, then $F_C \in S\alpha O\left(\tilde{\tau}_{F_B}\right)$.

Proof. Let $\left(F_B, \tilde{\tau}_{F_B}\right)$ be a soft topological subspace of $(F_A, \tilde{\tau})$ and $F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A$. Then,

$$
\begin{aligned}
F_C \in S\alpha O(\tilde{\tau}) \quad & \Rightarrow \exists F_D \in \tilde{\tau} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \left(\overline{F_D}\right)^{\circ}, \text{ from Theorem 2.37} \\
& \Rightarrow \exists F_D \tilde{\cap} F_B \in \tilde{\tau}_{F_B} \ni F_D \tilde{\cap} F_B \tilde{\subseteq} F_C \tilde{\cap} F_B \tilde{\subseteq} \left(\overline{F_D}\right)^{\circ} \tilde{\cap} F_B \\
& \Rightarrow \exists F_D \in \tilde{\tau}_{F_B} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \left(\overline{F_D}\right)^{\circ} \tilde{\cap} F_B \\
& (\text{Since } F_D \tilde{\subseteq} F_C \tilde{\subseteq} F_B, \text{ then } F_D \tilde{\cap} F_B = F_D \text{ and } F_C \tilde{\cap} F_B = F_C.) \\
& \Rightarrow \exists F_D \in \tilde{\tau}_{F_B} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \left(\overline{F_D}\right)^{\circ_{\tilde{\tau}_{F_B}}} \tilde{\cap} F_B, \text{ from Theorem 3.6 (i)} \\
& \Rightarrow \exists F_D \in \tilde{\tau}_{F_B} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \overline{F_D}^{\circ_{\tilde{\tau}_{F_B}}} \tilde{\cap} F_B^{\circ_{\tilde{\tau}_{F_B}}} \\
& (\text{Since } F_B \in \tilde{\tau}_{F_B}, \text{ then } F_B = F_B^{\circ_{\tilde{\tau}_{F_B}}}.) \\
& \Rightarrow \exists F_D \in \tilde{\tau}_{F_B} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \left(\overline{F_D} \tilde{\cap} F_B\right)^{\circ_{\tilde{\tau}_{F_B}}}, \text{ from Proposition 2.28 (v)} \\
& \Rightarrow \exists F_D \in \tilde{\tau}_{F_B} \ni F_D \tilde{\subseteq} F_C \tilde{\subseteq} \left(\overline{F_D}^{\tilde{\tau}_{F_B}}\right)^{\circ_{\tilde{\tau}_{F_B}}}, \text{ from Corollary 3.9 (ii)} \\
& (\text{Since } F_D \tilde{\subseteq} F_C \tilde{\subseteq} F_B \tilde{\subseteq} F_A, \text{ then } \overline{F_D}^{\tilde{\tau}_{F_B}} = \overline{F_D} \tilde{\cap} F_B) \\
& \Rightarrow F_C \in S\alpha O\left(\tilde{\tau}_{F_B}\right)
\end{aligned}
$$

The converse of the Theorem 4.2 is not always correct. In other words, a soft α-open set in a soft subspace of a soft topological space is may not soft α-open set in a soft topological space. This situation is shown in the following example.

Example 4.3. For $(F_A, \tilde{\tau})$ and $\left(F_{A_{12}}, \tilde{\tau}_{F_{A_{12}}}\right)$ provided in Example 2.19 and Example 2.22, $S\alpha O(\tilde{\tau}) = \{F_{\Phi}, F_{A_1}, F_{A_3}, F_{A_7}, F_{A_8}, F_{A_9}, F_{A_{13}}, F_{A_{14}}, F_A\}$ and $S\alpha O\left(\tilde{\tau}_{F_{A_{12}}}\right) = \{F_{\Phi}, F_{A_4}, F_{A_{11}}, F_{A_{12}}\}$. Hence, $F_{A_4} \in S\alpha O\left(\tilde{\tau}_{F_{A_{12}}}\right)$ but $F_{A_4} \notin S\alpha O(\tilde{\tau})$.

Theorem 4.4. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$. Then, $F_B \in S\alpha C(\tilde{\tau})$ if and only if there exists at least one $F_C \in \tilde{\tau}^k$ such that $\overline{F_C}^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} F_C$.

Proof. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B \tilde{\subseteq} F_A$.

($\Rightarrow$): Let $F_B \in S\alpha C(\tilde{\tau})$. From Theorem 2.36, $\overline{(\overline{F_B})^{\circ}} \tilde{\subseteq} F_B$. Since $F_B \tilde{\subseteq} \overline{F_B}$ and $\overline{F_B} \in \tilde{\tau}^k$, then $\overline{(\overline{F_B})^{\circ}} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F_B}$.

($\Leftarrow$): Let there exists at least one $F_C \in \tilde{\tau}^k$ such that $\overline{F_C}^{\circ} \tilde{\subseteq} F_B \tilde{\subseteq} F_C$. Since $F_C \in \tilde{\tau}^k$, then $\overline{F_C} = F_C$. Thus, $\overline{(\overline{F_C})^{\circ}} \tilde{\subseteq} F_B \tilde{\subseteq} \overline{F_C}$. Since $\overline{(\overline{F_C})^{\circ}} \tilde{\subseteq} F_B$, then $F_B \in S\alpha C(\tilde{\tau})$.

Definition 4.5. Let $(F_A, \tilde{\tau})$ be a soft topological space, $F_B \tilde{\subseteq} F_A$, and $\varepsilon \tilde{\in} F_B$. If there exists at least one $F_C \in S\alpha O(\tilde{\tau})$ such that $\varepsilon \tilde{\in} F_C$ and $F_C \tilde{\subseteq} F_B$, then ε is called a $\tilde{\tau}$-soft α-interior point (or briefly soft α-interior point) of F_B.

Theorem 4.6. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then, $F_A \tilde{\setminus} \overline{(F_B)}_\alpha = \left(F_A \tilde{\setminus} F_B\right)^{\circ}_{\alpha}$.

Proof. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then,

$$F_A \tilde{\setminus} \overline{(F_B)}_\alpha = F_A \tilde{\setminus} \left(\tilde{\bigcap_{\substack{F_B \tilde{\subseteq} F_{A_i} \\ F_A \tilde{\setminus} F_{A_i} \in S\alpha O(\tilde{\tau})}}} F_{A_i} \right) = \tilde{\bigcup_{\substack{F_A \tilde{\setminus} F_{A_i} \tilde{\subseteq} F_A \tilde{\setminus} F_B \\ F_A \tilde{\setminus} F_{A_i} \in S\alpha O(\tilde{\tau})}}} \left(F_A \tilde{\setminus} F_{A_i}\right) = \left(F_A \tilde{\setminus} F_B\right)^{\circ}_{\alpha}$$

Theorem 4.7. Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$. Then, $\varepsilon \tilde{\in} \overline{(F_B)}_\alpha$ iff, for all $F_C \in S\alpha O(\tilde{\tau})$ such that $\varepsilon \tilde{\in} F_C$, $F_B \tilde{\cap} F_C \neq F_\Phi$.

Proof: Let $(F_A, \tilde{\tau})$ be a soft topological space and $F_B, F_C \tilde{\subseteq} F_A$.

($\Rightarrow$): Let $\varepsilon \tilde{\notin} \overline{(F_B)}_\alpha$. Then, $\varepsilon \tilde{\in} F_A \tilde{\setminus} \overline{(F_B)}_\alpha$. From Theorem 4.6, $\varepsilon \tilde{\in} \left(F_A \tilde{\setminus} F_B\right)^{\circ}_{\alpha}$. Thus, there exists at least one $F_C \in S\alpha O(\tilde{\tau})$ such that $\varepsilon \tilde{\in} F_C \tilde{\subseteq} F_A \tilde{\setminus} F_B$. From Proposition 2.10 (iii), there exists at least one $F_C \in S\alpha O(\tilde{\tau})$ such that $\varepsilon \tilde{\in} F_C$ and $F_B \tilde{\cap} F_C = F_\Phi$.

($\Leftarrow$): Let there exists at least one $F_C \in S\alpha O(\tilde{\tau})$ such that $\varepsilon \tilde{\in} F_C$, $F_B \tilde{\cap} F_C = F_\Phi$. From Proposition 2.10 (iii), there exists $F_A \tilde{\setminus} F_C \in S\alpha C(\tilde{\tau})$ such that $F_B \tilde{\subseteq} F_A \tilde{\setminus} F_C$. In this case, from Proposition 2.35, $\overline{(F_B)}_\alpha \tilde{\subseteq} \overline{\left(F_A \tilde{\setminus} F_C\right)}_\alpha = F_A \tilde{\setminus} F_C$. Hence, $\varepsilon \tilde{\notin} \overline{(F_B)}_\alpha$.

Definition 4.8. Let $(F_A, \tilde{\tau})$ be a soft topological space. For all $\varepsilon_1, \varepsilon_2 \tilde{\in} F_A$ such that $\varepsilon_1 \neq \varepsilon_2$, if there exists at least one $F_B, F_C \in S\alpha O(\tilde{\tau})$ such that $\left(\varepsilon_1 \tilde{\in} F_B \wedge \varepsilon_2 \tilde{\notin} F_B\right)$ or $\left(\varepsilon_1 \tilde{\notin} F_C \wedge \varepsilon_2 \tilde{\in} F_C\right)$, then $(F_A, \tilde{\tau})$ is called a soft α-T_0 space.

Example 4.9. Let us consider Example 4.3. Since $F_A = \{(x_1, \{u_1, u_2\}), (x_2, \{u_2, u_3\})\}$ and $F_{A_1}, F_{A_3}, F_{A_8} \in S\alpha O(\tilde{\tau})$, then

$\{(x_1, \{u_1\})\} \tilde{\in} F_{A_1}$ and $\{(x_1, \{u_2\})\} \tilde{\notin} F_{A_1}$

$\{(x_1, \{u_1\})\} \tilde{\in} F_{A_1}$ and $\{(x_2, \{u_2\})\} \tilde{\notin} F_{A_1}$

$\{(x_1, \{u_1\})\} \tilde{\in} F_{A_1}$ and $\{(x_2, \{u_3\})\} \tilde{\notin} F_{A_1}$

$\{(x_1, \{u_2\})\} \tilde{\in} F_{A_3}$ and $\{(x_2, \{u_2\})\} \tilde{\notin} F_{A_3}$

$\{(x_1, \{u_2\})\} \tilde{\in} F_{A_3}$ and $\{(x_2, \{u_3\})\} \tilde{\notin} F_{A_3}$

$\{(x_2, \{u_2\})\} \tilde{\notin} F_{A_8}$ and $\{(x_2, \{u_3\})\} \tilde{\in} F_{A_8}$

Therefore, $(F_A, \tilde{\tau})$ is a soft α-T_0 space. On the other hand, for $\{(x_1, \{u_2\})\}, \{(x_2, \{u_3\})\} \tilde{\in} F_{A_{12}}$, there do not exist $F_B, F_C \in S\alpha O(\tilde{\tau})$ such that $\left(\{(x_1, \{u_2\})\} \tilde{\in} F_B \wedge \{(x_2, \{u_3\})\} \tilde{\notin} F_B\right)$ or $\left(\{(x_1, \{u_2\})\} \tilde{\notin} F_C \wedge \{(x_2, \{u_3\})\} \tilde{\in} F_C\right)$. Hence, $\left(F_{A_{12}}, \tilde{\tau}_{F_{A_{12}}}\right)$ is not a soft α-T_0 space.

Corollary 4.10. Every soft subspace of a α-T_0 space is not always a soft α-T_0 space. Therefore, being soft α-T_0 space is not a hereditary property.

Theorem 4.11. Let $(F_A, \tilde{\tau})$ be a soft topological space. Then, $(F_A, \tilde{\tau})$ is a soft α-T_0 space if and only if for all $\varepsilon_1, \varepsilon_2 \tilde{\in} F_A$ such that $\varepsilon_1 \neq \varepsilon_2$, $\overline{(\varepsilon_1)}_\alpha \neq \overline{(\varepsilon_2)}_\alpha$.

Proof.

($\Rightarrow$): Let $(F_A, \tilde{\tau})$ be a soft α-T_0 space. Then, for all $\varepsilon_1, \varepsilon_2 \tilde{\in} F_A$ such that $\varepsilon_1 \neq \varepsilon_2$, there exists at least one $F_B, F_C \in S\alpha O(\tilde{\tau})$ such that $\left(\varepsilon_1 \tilde{\in} F_B \wedge \varepsilon_2 \tilde{\notin} F_B\right)$ or $\left(\varepsilon_1 \tilde{\notin} F_C \wedge \varepsilon_2 \tilde{\in} F_C\right)$. Let $\varepsilon_1 \tilde{\in} F_B$ and $\varepsilon_2 \tilde{\notin} F_B$. Since $\varepsilon_2 \tilde{\notin} F_B$, then $\varepsilon_2 \tilde{\cap} F_B = F_\Phi$. Thus, there exists at least one $F_B \in S\alpha O(\tilde{\tau})$ such that $\varepsilon_1 \tilde{\in} F_B$ and $\varepsilon_2 \tilde{\cap} F_B = F_\Phi$. Because of Theorem 4.7, $\varepsilon_1 \tilde{\notin} \overline{(\varepsilon_2)}_\alpha$. Moreover, since $\varepsilon_1 \tilde{\in} \overline{(\varepsilon_1)}_\alpha$, then for all $\varepsilon_1, \varepsilon_2 \tilde{\in} F_A$ such that $\varepsilon_1 \neq \varepsilon_2$, $\overline{(\varepsilon_1)}_\alpha \neq \overline{(\varepsilon_2)}_\alpha$.

($\Leftarrow$): Let for all $\varepsilon_1, \varepsilon_2 \tilde{\in} F_A$ such that $\varepsilon_1 \neq \varepsilon_2$, $\overline{(\varepsilon_1)}_\alpha \neq \overline{(\varepsilon_2)}_\alpha$. Since $\varepsilon_1 \tilde{\in} \overline{(\varepsilon_1)}_\alpha$ and $\overline{(\varepsilon_1)}_\alpha \neq \overline{(\varepsilon_2)}_\alpha$, then $\varepsilon_1 \tilde{\notin} \overline{(\varepsilon_2)}_\alpha$. Because of Theorem 4.7, there exists at least one $F_B \in S\alpha O(\tilde{\tau})$ such that $\varepsilon_1 \tilde{\in} F_B$ and $\varepsilon_2 \tilde{\cap} F_B = F_\Phi$. Thus, there exists at least one $F_B \in S\alpha O(\tilde{\tau})$ such that $\varepsilon_1 \tilde{\in} F_B$ and $\varepsilon_2 \tilde{\notin} F_B$. Therefore, $(F_A, \tilde{\tau})$ is a soft α-T_0 space.

Example 4.12. For $\left(F_{A_7}, \tilde{\tau}_{F_{A_7}}\right)$ provided in Example 2.22, $S\alpha O\left(\tilde{\tau}_{F_{A_7}}\right) = \{F_\Phi, F_{A_1}, F_{A_7}\}$ and $S\alpha C\left(\tilde{\tau}_{F_{A_7}}\right) = \{F_\Phi, F_{A_4}, F_{A_7}\}$. For $\{(x_1, \{u_1\})\}, \{(x_2, \{u_2\})\} \tilde{\in} F_{A_7}$, since $\overline{\{(x_1, \{u_1\})\}}_\alpha = F_{A_7}$ and $\overline{\{(x_2, \{u_2\})\}}_\alpha = F_{A_4}$, then $\overline{\{(x_1, \{u_1\})\}}_\alpha \neq \overline{\{(x_2, \{u_2\})\}}_\alpha$. Therefore, $\left(F_{A_7}, \tilde{\tau}_{F_{A_7}}\right)$ is a soft α-T_0 subspace of $(F_A, \tilde{\tau})$.

5. Conclusion

This paper studied relationships between the soft interior and soft closure of a soft set in soft topological spaces and their soft subspaces through the concept of the soft element. Thus, the validity of many propositions on various generalizations of soft open sets can be proved easier. We then provided a few theorems concerning soft α-open sets and soft α-closed sets. Moreover, we defined soft α-T_0 space and investigated its basic properties. We showed that every soft subspace of a soft α-T_0 space is not always a soft α-T_0 space.

In the future, researchers can study that every soft subspace of a soft α-T_0 space is a soft α-T_0 space under which condition or conditions. Moreover, they can define through the concept of the soft element the other soft α-separation axioms, i.e., soft α-T_1 space, soft α-T_2 space (soft α-Hausdorff space), soft α-regular space, and soft α-normal space.

Author Contributions

All authors contributed equally to this work. They all read and approved the last version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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