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Conformable Fractional Calculus on Fuzzy Logic

Abdullah Akkurt[1]

[1]*Department of Mathematics, Faculty of Science and Arts, Kahramanmaraş Sütçü İmam University, 46100, Kahramanmaraş, Turkey.*

Abstract

In this article, we present a new general definition of fuzzy conformable fractional derivative and fractional integral, that depends on an unknown kernel. We will get some new applications with the help of this concept.

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1. Introduction

Since Zadeh's introduction of the concept of fuzzy sets [21], many scientists have explored fuzzy set theory. The fuzzy logic theory is part of mathematical analysis and has been extensively researched in recent years.
The fractional calculation, dealing with arbitrary - order integral and derivative operators, is a very popular subject with a history of over 300 years or so. The fractional analysis compared to traditional analysis is a very useful tool in modeling real-world problems. In the near future, applications of fractional derivatives and integrals can be seen in many areas.
Today, there are many real-valued fractional integrals and fractional derivatives such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard, Riesz. For these, please see [7], [14]. Here, all fractional derivatives do not certain some properties such as Product Rule, Quotient Rule, Chain Rule, Roll's Theorem and Mean Value Theorem.
To overcome such issues; Khalil et al. proposed the following concept [9]. This formula is the limit definition of the conformable derivative.

$$T_{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{1.1}$$

In [6], Almeida et al. gave the following limit definition formula for conformable fractional derivatives using kernels.

$$f^{(\alpha)}(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon k(t)^{1-\alpha}\right)-f(t)}{\varepsilon}. \tag{1.2}$$

For more information about conformable fractional derivatives and integrals, please refer to [1]-[5], [15]-[19].
In the field of derivatives, analytical approaches to the theory of fuzzy fractional analysis are based essentially on Riemann-Liouville or Caputo-Liouville versions. These versions are most often incorporated into scientific research in recent years. This fuzzy conformal fractional derivative seems to be a natural extension of the $H-$derivative [22] with its own mathematical details associated with the definition of its two-sided limits.
Omar and Mohammed [11] have given the following conformable fractional definition for fuzzy valued functions.

$$\begin{aligned} D^{\alpha}f(t) &= \lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon(t-t_0)^{1-\alpha}\right)\ominus f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\frac{f(t)\ominus f\left(t-\varepsilon(t-t_0)^{1-\alpha}\right)}{\varepsilon}, \end{aligned}$$

Email addresses: **abdullahmat@gmail.com (Abdullah Akkurt)**

and

$$D^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t) \ominus f\left(t + \varepsilon (t - t_0)^{1-\alpha}\right)}{-\varepsilon} = \lim_{\varepsilon \to 0} \frac{f\left(t - \varepsilon (t - t_0)^{1-\alpha}\right) \ominus f(t)}{-\varepsilon}.$$

2. Fuzzy-valued fractional calculus

Now, we give some preliminary information about the fuzzy numbers mentioned in [8], [20].

Definition 2.1. *A fuzzy number is a fuzzy set $\mathbb{R}_{\mathscr{F}} = \{u : \mathbb{R} \to [0,1]\}$ which satisfies the following conditions (i)-(iv):*
(i) u is normal, that is, there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex in $\mathbb{R}$, that is, for $0 \le t \le 1$,

$$u(ta + (1-t)b) \ge \min\{u(a), u(b)\}, \text{ for any } a, b \in \mathbb{R}, \tag{2.1}$$

(iii) u is upper semicontinuous,
(iv) $[u]^0 = \overline{\{x \in \mathbb{R} \mid u(x) > 0\}}$ is compact.

Here, space E is the space of all fuzzy numbers on $\mathbb{R}$.
If u is a fuzzy number on $\mathbb{R}$, we define $[u]^{\lambda} = \{x \in \mathbb{R} \mid u(x) \ge \lambda\}$ the $\lambda-$level of u, with $\lambda \in (0,1]$. From the conditions (i) to (iv), it follows that the λ-level set of $u \in E, [u]^{\lambda}$, is a nonempty compact interval, for any $\lambda \in [0,1]$. We denote by $[\underline{u}(r), \bar{u}(\lambda)]$ the λ-level of a fuzzy number u. For $u_1,\ u_2 \in E$, and $r \in \mathbb{R}$, the sum $u_1 + u_2$ and the product $r \cdot u_1$ are defined by

$$[u_1 + u_2]^{\lambda} = [u_1]^{\lambda} + [u_2]^{\lambda},\ [r \cdot u_1]^{\lambda} = r[u_1]^{\lambda},\ \forall \lambda \in [0,1], \tag{2.2}$$

where $[u_1]^{\lambda} + [u_2]^{\lambda}$ means the usual addition of two intervals of $\mathbb{R}$ and $r[u_1]^{\lambda}$ means the usual scalar product between r and an real interval. For $u \in E$, we define the diameter of the λ-level set of u as $diam[u]^{\lambda} = \bar{u}(\lambda) - \underline{u}(\lambda)$.

Definition 2.2. *([8]) The generalized Hukuhara difference of two fuzzy numbers $u, v \in E$ (gH-difference for short) is defined as follows:*

$$u \ominus_{gH} v = w \Leftrightarrow \begin{cases} (i)\ u = v + w \\ or\ (ii)\ v = u + (-1)w. \end{cases} \tag{2.3}$$

A function $u : [a,b] \to E$ is called d -increasing (d -decreasing) on $[a,b]$ if for every $r \in [0,1]$ the function $t \mapsto diam[u(t)]^r$ is nondecreasing (nonincreasing) on $[a,b]$. If u is d -increasing or d -decreasing on $[a,b]$, then we say that u is d -monotone on $[a,b]$.

Definition 2.3. *([8]) Let $f : (a,b) \to E$ and $x \in (a,b)$. The fuzzy function f is said to be generalized Hukuhara differentiable (gH-differentiable) at x_0, if there exists an element $f'(x_0) \in E$ such that*

$$f'_g(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) \ominus_g f(x_0)}{h}. \tag{2.4}$$

Denote by $C([a,b],E)$ the set of all continuous fuzzy functions and $AC([a,b],E)$ the set of all absolutely continuous fuzzy functions on the interval $[a,b]$ with values in E. Let $L([a,b],E)$ be the set of all fuzzy functions $f : [a,b] \to E$ such that the functions $x_0 \mapsto D_0[f(x_0), \hat{0}]$ belongs to $L^1[a,b]$.

Lemma 2.4. ([12]) *Let $u, v : \mathbb{R}_{\mathscr{F}} \longrightarrow [0,1]$ be the fuzzy sets. Then, $u = v$ if and only if $[u]^{\lambda} = [v]^{\lambda}$ for all $\lambda \in [0,1]$. The following arithmetic operations on fuzzy numbers are well known and frequently used below. If $u, v \in \mathbb{R}_{\mathscr{F}}$, then*

$$[u+v]^{\lambda} = \left[u_1^{\lambda} + v_1^{\lambda}, u_2^{\lambda} + v_2^{\lambda}\right] \tag{2.5}$$

$$[\lambda u]^{\lambda} = k[u]^{\lambda} = \begin{cases} \left[ku_1^{\lambda}, ku_2^{\lambda}\right], & \text{if } k \ge 0, \\ \left[ku_2^{\lambda}, ku_1^{\lambda}\right], & \text{if } k < 0. \end{cases} \tag{2.6}$$

Definition 2.5. *([10], [13]). Let $u, v \in \mathbb{R}_{\mathscr{F}}$. If there exists $w \in \mathbb{R}_{\mathscr{F}}$ such as $u = v + w$, then w is called the H-difference of u, v, and it is denoted as $u \ominus v$.*

Definition 2.6. *([23]) Let we denote*

$$\overline{0} = \begin{cases} 1, & t = 0 \\ 0, & t \ne 0. \end{cases} \tag{2.7}$$

Then, $\overline{0} \in \mathbb{R}_{\mathscr{F}}$ be a neutral element with respect to $+$, i.e., $u + \overline{0} = \overline{0} + u \quad u \in \mathbb{R}_{\mathscr{F}}$:

(i) With respect to $\overline{0}$, none of $u \in \mathbb{R}_{\mathscr{F}}/\mathbb{R}$ has opposite in $\mathbb{R}_{\mathscr{F}}$
(ii) For any $a,b \in \mathbb{R}$ with $a,b \geq 0$ or $a,b \leq 0$ and any $u \in \mathbb{R}_{\mathscr{F}}$, we have $(a+b)\cdot u = a\cdot u + b\cdot u$, for general $a,b \in \mathbb{R}$ the above property does not hold.
(iii) For any $k \in \mathbb{R}$ and any $u,v \in \mathbb{R}_{\mathscr{F}}$, we have $k\cdot(u+v) = k\cdot u + k\cdot v$ (iv) For any $k,v \in \mathbb{R}$ and any $u \in \mathbb{R}_{\mathscr{F}}$, we have $k\cdot(v\cdot u) = (k\cdot v)\cdot u$.
Define $d : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \longrightarrow \mathbb{R}_{+} \cup \{0\}$ by the equation

$$d(u,v) = \sup_{\beta\in[0,1]} d_H\left([u]^{\beta},[v]^{\beta}\right), \text{ for all } u,v \in \mathbb{R}_{\mathscr{F}} \tag{2.8}$$

where d_H is the Hausdorff metric.

$$d_H\left([u]^{\beta},[v]^{\beta}\right) = \max\left\{\left|u_1^{\beta} - v_1^{\beta}\right|, \left|u_2^{\beta} - v_2^{\beta}\right|\right\}. \tag{2.9}$$

It is well known that $(\mathbb{R}_{\mathscr{F}}, d)$ is a complete metric space. We list the following properties of $d(u,v)$:

$$\begin{aligned} d(u+w,v+w) &= d(u,v) \\ d(u,v) &= d(v,u) \\ d(ku,kv) &= |k|d(u,v) \\ d(u,v) &\leq d(u,w)+d(w,v) \end{aligned} \tag{2.10}$$

for all $u,v,w \in \mathbb{R}_{\mathscr{F}}$ and $\lambda \in \mathbb{R}$.
Now, let's give some notations that we will use in definitions as follows:
$\mathbb{A} := (t_0,T] \subset \mathbb{R}$ with $t_0 \geq 0, \mathbb{D} := [0,1]-\{0\}$, and $\mathscr{F}(\mathbb{R})$ denotes to set of fuzzy numbers on $\mathbb{R}$. As long as, $t \in \mathbb{A}$, $\alpha \in \mathbb{D}, t_0 \in \mathbb{R}$, $P \in \mathscr{F}(\mathbb{R})$, $f \in C(\mathbb{A}\times\mathscr{F}(\mathbb{R}),\mathscr{F}(\mathbb{R}))$, and $h \in C(\mathbb{A},\mathscr{F}(\mathbb{R}))$. So, $D^{\alpha}f(t)$ denotes to fuzzy conformable fractional derivative of t over $\mathbb{A}$.
The purpose of this article is to give the definition of the conformable fractional derivative and fractional integral of a fuzzy function. This study was prepared using the methods used in [11]. The fuzzy conformable fractional derivative and integral have been redefined using an unknown kernel.

3. Generalized fuzzy conformable fractional derivative

In this section, we present a new definition for generalized fuzzy compatible fractional derivative using an unknown kernel.

Definition 3.1. *Let $k : [a,b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t-t_0)$, $k'(t-t_0) \neq 0$, whenever $t_0 \geq 0$. Let $f \in C(\mathbb{A},\mathscr{F}(\mathbb{R}))$ and $\alpha \in \mathbb{D}$. If $\exists D^{\alpha}f(t) \in \mathscr{F}(\mathbb{R})$, then, f is generalized fuzzy conformable fractional derivative at $t \in \mathbb{A}$, with respect to kernel k, if the limit*

$$\begin{aligned} D^{\alpha}f(t) &= \lim_{\varepsilon\to 0} \frac{f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right) \ominus f(t)}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{f(t) \ominus f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)}{\varepsilon}, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} D^{\alpha}f(t) &= \lim_{\varepsilon\to 0} \frac{f(t) \ominus f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)}{-\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right) \ominus f(t)}{-\varepsilon}. \end{aligned} \tag{3.2}$$

Theorem 3.2. *Let $f \in C(\mathbb{A},(\mathbb{R}))$ and $\alpha \in \mathbb{D}$, then f is fuzzy continuous at t.*

Proof. Let t, $t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)} \in (0,a)$ with $\varepsilon > 0$. Then, by properties of equation (2.10) and the triangle inequality, we have

$$\begin{aligned} d\left(f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right), f(t)\right) &= d\left(f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right) \ominus f(t), \overline{0}\right) \\ &\leq \varepsilon.d\left(\frac{\left(f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right) \ominus f(t)\right)}{\varepsilon}, f^{(\alpha)}(t)\right) + \varepsilon d\left(f^{(\alpha)}(t),\overline{0}\right) \end{aligned}$$

where ε is so small that the $H-$difference $f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right) \ominus f(t)$ exists. By the differentiability, the right-hand side goes to zero as $\varepsilon \longrightarrow 0^{+}$, and hence, f is right fuzzy continuous.
On the other side, let t, $t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)} \in (0,a)$ with $\varepsilon > 0$. Then, by properties of equation (2.10) and the triangle inequality, we have

$$\begin{aligned} d\left(f(t), f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)\right) &= d\left(f(t) \ominus f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right), \overline{0}\right) \\ &\leq \varepsilon d\left(f^{(\alpha)}(t),\overline{0}\right) + \varepsilon.d\left(f^{(\alpha)}(t), \frac{\left(f(t) \ominus f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)\right)}{\varepsilon}\right), \end{aligned}$$

where ε is so small that the $H-$difference $f(t)\ominus f\left(t-\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)$ exists. By the differentiability, the right-hand side goes to zero as $\varepsilon\longrightarrow 0^+$, and hence, f is left fuzzy continuous.
For the other part, similar proof can be made. □

Theorem 3.3. *Let $\alpha\in(0,1]$. If f is differentiable and f is α-differentiable, then*

$$D^{\alpha}f(t)=\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}f'_g(t). \tag{3.3}$$

Proof. Note that

$$D^{\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\right)\ominus f(t)}{\varepsilon}. \tag{3.4}$$

Set $h=\varepsilon\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}$, then one finds

$$\begin{aligned}
D^{\alpha}f(t) &= \lim_{h\to 0}\frac{f(t+h)\ominus f(t)}{h.k^{\alpha-1}(t-t_0)k'(t-t_0)} && (3.5)\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\lim_{h\to 0}\frac{f(t+h)\ominus f(t)}{h} && (3.6)\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}D^1f(t) && (3.7)\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}f'_g(t). && (3.8)
\end{aligned}$$

Thus the proof is complete. □

4. Generalized fuzzy conformable fractional integral

In this section, we present a new definition for generalized fuzzy conformable fractional integral using an unknown kernel.

Definition 4.1. *Let $t_0\geq 0$, $a\geq 0$ and $x\in(0,a)$. Also, let f be a function defined on $(a,x]$ and $\alpha\in\mathbb{R}$. Let $k:[a,b]\to\mathbb{R}$ be a continuous nonnegative map such that $k(t-t_0)$, $k'(t-t_0)\neq 0$. Then, the α-generalized fractional integral of f is defined by,*

$$I_{t_0}^{\alpha}(f)(x)=\int_{t_0}^{x}k^{\alpha-1}(t-t_0)k'(t-t_0)f(t)\,dt. \tag{4.1}$$

Theorem 4.2. *If $f\in C(\mathbb{A},\mathscr{F}(\mathbb{R}))$ with $\alpha\in\mathbb{D}$. then:*

$$\left[D^{\alpha}\left(I_{t_0}^{\alpha}f\right)(t)\right]=f(t). \tag{4.2}$$

Proof. Note that,

$$\begin{aligned}
\left[D^{\alpha}\left(I_{t_0}^{\alpha}f\right)(t)\right]=\frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\left[D^1\left(I_{t_0}^{\alpha}f\right)(t)\right] &= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(I_{t_0}^{\alpha}f\right)(t)\right]\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}k^{\alpha-1}(s-t_0)k'(s-t_0)f(s)\mathrm{d}s\right]\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}\left[k^{\alpha-1}(t-t_0)k'(t-t_0)f(t)\right]\\
&= \frac{k^{1-\alpha}(t-t_0)}{k'(t-t_0)}k^{\alpha-1}(t-t_0)k'(t-t_0)[f(t)]\\
&= f(t).
\end{aligned}$$

Thus the proof is complete. □

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