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Homothetic Motions and Dual Transformations

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Abstract: In this research, we produce a homothetic motion in E_1^n from a homothetic motion in E^n by using a dual transformation. Furthermore, we define a transition from Euclidean umbrella matrix to Lorentzian umbrella matrix. Then, we examine the invariance of the axis of the umbrella motion that is $\vec{x} = (1,1,..,1)$ in both spaces. We also provide examples to make our results clear. Moreover, we draw their figures to investigate visual representations. Finally, we study on homothetic motions in dual spaces.

Homotetik Hareketler ve Dual Dönüşümler

Anahtar Kelimeler
Homotetik Hareket
Şemsiye Hareketi
Dual Dönüşüm
Lorentz Uzayı
Dual Uzay
Kinematik.

Öz: Bu çalışmada, dual dönüşüm yardımıyla E_1^n deki homotetik hareketlerden E^n de homotetik hareketler elde ettik. Ayrıca, Öklidyen şemsiye matrisleri ile Lorentzian şemsiye matrisleri arasında bir geçiş sağladık. Daha sonra, şemsiye hareketinin ekseni olan $\vec{x} = (1,1,..,1)$ in iki uzayda da sabit kaldığını gösterdik. Elde edilen sonuçların pekiştirilmesi amacıyla örnekler vererek şekillerini çizdik. Son olarak, homotetik hareketleri dual uzaylarda çalıştık.

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1. Introduction

Kinematics is a subfield of physics, deals with the motions of points, bodies, and systems of bodies without considering the forces that cause them to move: mention frames, variables, and transformations. The study of kinematics is often referred to as the geometry of motion. Motion is the phenomenon of constant displacement of a rigid body relative to a certain reference point. Displacement of a rigid body is used to describe the motion of systems in mechanical engineering, robotics, biomechanics, astrophysics, and in other areas related. Homothetic motions of a rigid body in n-dimensional Euclidean space are generated by the homothetic transformations. In [1], the n-dimensional homothetic motion of a body in Euclidean space is generated by the transformation

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} h.A & a \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X \\ 1 \end{bmatrix} \tag{1}$$

where $h = h.I_n$ is a scalar matrix, $A \in \mathrm{SO(n)}$ and $a \in \mathbb{R}_1^n$. Here, if *h = 1* in (1) one-parameter motions are defined. If $A \in \mathrm{O}(n)$ that provides the property

$$AS = S,$$

where

$$S = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}_1^n,$$

then A is called an umbrella matrix. Umbrella motions and homothetic motions in Euclidean spaces are given in [2]. Also, homothetic motions are studied by several authors [3] - [6].

The relationship between Euclidean and Lorentzian rotational motion matrices is given by using dual transformations between $\mathrm{SO}(n+1)$ and $\mathrm{SO}(n,1)$ in [7]. In the light of this study, we examined dual transformations in dual spaces by investigating invariant axes in both spaces, see [8]. Additionally, we carry this research into Galilean spaces in [9]. Kinematics applications of dual transformations are also studied in [10]. In kinematics, there has been very important activities of an experimental nature concerning not only the study of models and the visualization of flows, but also that of objects like the human figure and the bodies of animals. Previous studies on kinematics can be used to obtain extensive information, cited as references [11] - [18]. Our paper is also expected to contribute to the existing literature on kinematics and its applications.

The main objective of this paper is to define a transition from Euclidean homothetic motion matrices to Lorentzian homothetic motion matrices by means of dual transformations. Even though many researchers were worked on affine kinematics in both spaces, the new and the most intriguing part of this study is to give the relationship between homothetic motions in different spaces. In other words, the dual transformation defined in this paper works as a handy tool for obtaining Lorentzian homothetic motions from Euclidean homothetic motions. Additionally, we acquire umbrella motion matrices in Lorentzian space with a similar method. We examine the invariance of the axis of the umbrella motion that is $\vec{x} = (1,1,..,1)$ in both spaces. Moreover, we provide some examples making effective our obtained results. Furthermore, we draw their figures to give visual representations. Considering the importance of dual space in kinematics, we also focus on homothetic motions in dual spaces.

2. Material and Method

This section includes two subsections to give a background for Lorentzian space and dual transformations. Since we present the concepts with their dual notions in the following subsections, it would be appropriate to give the preliminaries of dual space beforehand.

Definition 2.1 If a and a^* are real numbers and $\epsilon^2 = 0$, the combination $\hat{a} = a + \epsilon a^*$ is called a dual number, where ϵ is the dual unit.

Definition 2.2 The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$. The set $\mathbb{D}^3 = \{\vec{\hat{a}} = (\hat{a}_1, \hat{a}_2, \hat{a}_3) | \hat{a}_i \in \mathbb{D}, 1 \leq i \leq 3\}$ is called a $\mathbb{D}$-module or dual space.

Definition 2.3 The elements of $\mathbb{D}^3$ are called dual vectors. A dual vector $\vec{\hat{a}}$ can be written $\vec{\hat{a}} = \vec{a} + \epsilon\vec{a}^*$ where $\vec{a}$ and $\vec{a}^*$ are real vectors in $\mathbb{R}^3$.

Definition 2.4 The norm of a dual vector $\vec{\hat{a}}$ is defined by $|\vec{\hat{a}}| = |\vec{a}| + \epsilon\frac{\langle\vec{a},\vec{a}^*\rangle}{|\vec{a}|^2}$.

For more details about dual space see [19].

2.1 Background on Lorentzian space

We mention some fundamental definitions and properties in Lorentzian space that we use in this paper.

Definition 2.5 The Lorentzian metric $\langle,\rangle$ defined by

$$\langle u,v\rangle = u_1v_1 + u_2v_2 + \ldots + u_{n-1}v_{n-1} - u_nv_n \tag{2}$$

in E_1^n will be used in this study.

It is pointed out that $\langle,\rangle$ is a non-degenerate metric of index 1. It can also be written in the form:

$$\langle u,v\rangle = u^T \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{bmatrix} v = u^T G v. \tag{3}$$

After giving the Lorentzian metric, we recall that a vector $v \in E_1^n$ can have one of three casual characters as given below.

Definition 2.6 A vector $v \in E_1^n$ is called

- spacelike if $\langle v,v\rangle > 0$ or $v = 0$,
- timelike if $\langle v,v\rangle < 0$,
- lightlike if $\langle v,v\rangle = 0$ and $v \neq 0$.

Since we will be working with more matrices in this study, let us recall some properties of Lorentzian matrices, see [20].

Definition 2.7 An $n \times n$ matrix S is called

- semi symmetric if $S^T = GSG$ or $S = GS^TG$,
- semi skew-symmetric if $S^T = -GSG$ or $S = -GS^TG$,
- semi-orthogonal if $S^T = GS^{-1}G$ or $S^{-1} = GS^TG$, where G is the sign matrix of Lorentzian space, see [21].

We will use dual vectors in the sections concerning dual spaces, so we need the following definition.

Definition 2.8 The Lorentzian inner product of dual vectors $\vec{\hat{a}}$ and $\vec{\hat{b}}$ is defined by

$$\left\langle \vec{\hat{a}}, \vec{\hat{b}} \right\rangle = \langle\vec{a},\vec{b}\rangle + \epsilon(\langle\vec{a},\vec{b}^*\rangle + \langle\vec{a}^*,\vec{b}\rangle)$$

with $\vec{\hat{a}} = \vec{a} + \epsilon\vec{a}^*$ and $\vec{\hat{b}} = \vec{b} + \epsilon\vec{b}^*$. A dual vector $\vec{\hat{a}}$ is called *timelike* if $\langle\vec{\hat{a}},\vec{\hat{a}}\rangle < 0$, *spacelike* if $\langle\vec{\hat{a}},\vec{\hat{a}}\rangle > 0$ and *lightlike* (or *null*) if $\langle\vec{\hat{a}},\vec{\hat{a}}\rangle = 0$, where $\langle,\rangle$ is Lorentzian inner product. We call the dual space $\mathbb{D}^3$ together with this Lorentzian inner product as *dual Lorentzian space* and indicate it by $\mathbb{D}_1^3$.

Previous studies in Lorentzian space can be used to achieve more information [22] – [24]. Also, in Lorentzian space, rotational motions are studied by [25] and [26].

2.2 Dual transformations

The dual transformation between $\mathrm{SO}(n)\backslash\{a_{nn}=0\}$ and $\mathrm{SO}(n-1,1)$ which is defined below, will be used for obtaining semi-orthogonal matrices from orthogonal matrices. We acquire Lorentzian matrices from Euclidean matrices by using this dual transformation.

Definition 2.9 Dual transformation between $\mathrm{SO}(n)\backslash\{a_{nn}=0\}$ and $\mathrm{SO}(n-1,1)$ is defined in Dohi et al. (2010). Two sets can be given by

$$\mathrm{SO}(n)=\{A\in GL(n,\mathbb{R})|A^TA=AA^T=I_n,\det A=1\},$$

$$\mathrm{SO}(n-1,1)=\{A\in GL(n,\mathbb{R})|A^TGA=AGA^T=G,\det A=1\},$$

where $G=\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1\end{bmatrix}$ and I_n is $n\times n$ identity matrix.

Let $A\in SO(n)$, then it can be written in the block form as

$$A=\begin{bmatrix} B & C \\ D & a_{nn}\end{bmatrix},$$

where $a_{nn}\neq 0$. Here, B is an $(n-1)\times(n-1)$ square matrix, C is a column matrix and D is a row matrix. Since $a_{nn}\neq 0$, then we can use the following two sets given by

$$\mathfrak{S}_1=\{A\in\mathrm{SO}(n)|a_{nn}\neq 0\},$$

$$\mathfrak{S}_2=\{A\in\mathrm{SO}(n-1,1)|a_{nn}\neq 0\}.$$

Therefore, the dual transformation can be defined as

$$f:\mathfrak{S}_1\to\mathfrak{S}_2$$

$$f:A\mapsto f(A)=\frac{1}{a_{nn}}\begin{bmatrix} a_{nn}(B^{-1})^T & C \\ -D & 1\end{bmatrix}, \tag{4}$$

here T denotes transposition.

We now give the definition of dual transformation in dual spaces. We will use it for obtaining dual semi-orthogonal matrices from dual orthogonal matrices.

Definition 2.10 There is dual transformation between $\widehat{\mathrm{SO}}(n)\backslash\{\hat{a}_{nn}=0\}$ and $\widehat{\mathrm{SO}}(n-1,1)$. Firstly, we give the following sets:

$$\widehat{\mathrm{SO}}(n)=\{\hat{A}\in GL(n,\mathbb{D})|\hat{A}^T\hat{A}=\hat{A}\hat{A}^T=I_n,\det\hat{A}=1\},$$

$$\widehat{\mathrm{SO}}(n-1,1)=\{\hat{A}\in GL(n,\mathbb{D})|\hat{A}^TG\hat{A}=\hat{A}G\hat{A}^T=G,\det\hat{A}=1\},$$

where $G=\begin{bmatrix} I_{n-1} & 0 \\ 0 & -1\end{bmatrix}$ and I_n is $n\times n$ identity matrix.

We write the dual matrix $\hat{A}\in\widehat{\mathrm{SO}}(n)$ in the block form as

$$\hat{A}=\begin{bmatrix} \hat{B} & \hat{C} \\ \hat{D} & \hat{a}_{nn}\end{bmatrix},$$

where $\hat{a}_{nn}\neq 0$. Since $\hat{a}_{nn}\neq 0$, then two sets can be written as

$$\widehat{\mathfrak{S}_1}=\{\hat{A}\in\widehat{\mathrm{SO}}(n)|\hat{a}_{nn}\neq 0\},$$

$$\widehat{\mathfrak{S}_2}=\{\hat{A}\in\widehat{\mathrm{SO}}(n-1,1)|\hat{a}_{nn}\neq 0\}.$$

Now, f dual transformation can be defined as below

$$f: \widehat{\mathfrak{S}_1} \to \widehat{\mathfrak{S}_2}$$

$$f: \hat{A} \mapsto f(\hat{A}) = \frac{1}{\hat{a}_{nn}} \begin{bmatrix} \hat{a}_{nn}(\hat{B}^{-1})^T & \hat{C} \\ -\widehat{D} & 1 \end{bmatrix} \tag{5}$$

For more details about dual transformation in dual space see [8].

3. Results

This section includes five subsections to investigate homothetic motions and umbrella motions with the help of dual transformations. We also carry the results into the dual space.

3.1 Homothetic motions and dual transformations

In this section, we examine homothetic motions by means of dual transformations. We obtain a Lorentzian homothetic motion from a Euclidean homothetic motion.

Theorem 3.1 Let $H \in E^n$ given by

$$H = \begin{bmatrix} h.A & a \\ 0 & 1 \end{bmatrix} \tag{6}$$

where $h = h.I_n$ is a scalar matrix, $A \in \mathrm{SO}(n-1)$ and $a \in \mathbb{R}_1^{n-1}$.

f_h defines a dual transformation,

$$f_h: E^n \to E_1^n$$

$$H \mapsto f_h(H) = H_L = \begin{bmatrix} h.f(A) & a \\ 0 & 1 \end{bmatrix} \tag{7}$$

where f is the dual transformation given in (4), thus $f(A) \in \mathrm{SO}(n-2,1)$. The semi-orthogonal matrix $H_L \in E_1^n$ represents the homothetic motion in n-dimensional Lorentzian space.

Proof We show that

$$f_h^2(H) = f_h(f_h(H))$$

$$= f_h(H_L), \quad f^2 = id.$$

$$= H$$

$$f_h^2 = id.$$

Thus, f_h is a dual transformation.

3.2 Applications with one-parameter homothetic motions

After examining homothetic motions with dual transformations, we investigate one-parameter homothetic motions by means of dual transformations. One-parameter homothetic motion in n-dimensional Euclidean space is generated by the transformation

$$\begin{bmatrix} Y(t) \\ 1 \end{bmatrix} = \begin{bmatrix} h.A(t) & a(t) \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X(t) \\ 1 \end{bmatrix} \tag{8}$$

where $h = h.I_n$ is a scalar matrix, $A(t) \in \mathrm{SO}(n)$.

By using f_h dual transformation in (7), Lorentzian one-parameter homothetic motion matrix can be represented by

$$H_L(t) = \begin{bmatrix} h.f\big(A(t)\big) & a(t) \\ 0 & 1 \end{bmatrix}. \tag{9}$$

Example 1 Let $H \in E^4$ be a one-parameter homothetic motion matrix is given by

$$H(t) = \begin{bmatrix} -2t\cos^2(t) + t & 2t\sin(t) - 2t\cos(t) & 2t\sin(t)\cos(t) + 2t & t \\ 2t\sin(t) + 2t\cos(t) & t & 2t\cos(t) - 2t\sin(t) & t^2 \\ 2t\sin(t)\cos(t) - 2t & 2t\sin(t) + 2t\cos(t) & 2t\cos^2(t) - t & t^3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, Lorentzian one-parameter homothetic motion matrix $H_L(t)$ can be obtained by using f_h as follows

$$H_L(t) = \begin{bmatrix} \dfrac{t}{2t\cos^2(t) - t} & \dfrac{-2t\sin(t) - 2t\cos(t)}{2t\cos^2(t) - t} & \dfrac{2t\sin(t)\cos(t) + 2t}{2t\cos^2(t) - t} & t \\ \dfrac{-2t\sin(t) + 2t\cos(t)}{2t\cos^2(t) - t} & \dfrac{-2t\cos^2(t) + t}{2t\cos^2(t) - t} & \dfrac{2t\cos(t) - 2t\sin(t)}{2t\cos^2(t) - t} & t^2 \\ \dfrac{-2t\sin(t)\cos(t) + 2t}{2t\cos^2(t) - t} & \dfrac{-2t\sin(t) - 2t\cos(t)}{2t\cos^2(t) - t} & \dfrac{1}{2t\cos^2(t) - t} & t^3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 2 Let $A(t)$ be a homothetic matrix is given by

$$A(t) = \begin{bmatrix} \dfrac{2(t - t^3)}{1 + 3t^2} & \dfrac{4(t^3 - t^2)}{1 + 3t^2} & \dfrac{4(t^3 + t^2)}{1 + 3t^2} \\ \dfrac{4(t^3 + t^2)}{1 + 3t^2} & \dfrac{2(t - t^3)}{1 + 3t^2} & \dfrac{4(t^3 - t^2)}{1 + 3t^2} \\ \dfrac{4(t^3 - t^2)}{1 + 3t^2} & \dfrac{4(t^3 + t^2)}{1 + 3t^2} & \dfrac{2(t - t^3)}{1 + 3t^2} \end{bmatrix}.$$

If we multiply the matrix $A(t)$ with the curve $\phi(s) = (\sin(s), \cos(s), s^3)$, then we obtain the matrix $A(t).\phi(s)$. The elements of the matrix $A(t).\phi(s)$ can be represented by a surface $S_1 = \Psi(t,s) = (\frac{2\sin(s)(t-t^3)}{1+3t^2} + \frac{4\cos(s)(t^3-t^2)}{1+3t^2} + \frac{4s^3(t^3+t^2)}{1+3t^2}, \frac{4\sin(s)(t^3+t^2)}{1+3t^2} + \frac{2\cos(s)(t-t^3)}{1+3t^2} + \frac{4s^3(t^3-t^2)}{1+3t^2}, \frac{4\sin(s)(t^3-t^2)}{1+3t^2} + \frac{4\cos(t^3+t^2)}{1+3t^2} + \frac{2s^3(t-t^3)}{1+3t^2})$. See Fig. (1).

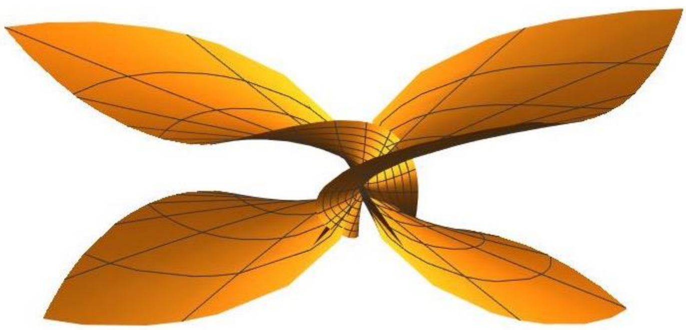

Figure 1. The surface $S_1 \in E^3$

By using f dual transformation, we obtain the Lorentzian homothetic matrix $A_L(t)$.

$$A_L(t) = \begin{bmatrix} 1 & \frac{-4(t^3+t^2)}{2(t-t^3)} & \frac{4(t^3+t^2)}{2(t-t^3)} \\ \frac{-4(t^3-t^2)}{2(t-t^3)} & 1 & \frac{4(t^3-t^2)}{2(t-t^3)} \\ \frac{-4(t^3-t^2)}{2(t-t^3)} & \frac{-4(t^3+t^2)}{2(t-t^3)} & 1 \end{bmatrix}.$$

By multiplying the matrix $A_L(t)$ with $\phi(s)$, we acquire the matrix $A_L(t).\phi(s)$. The elements of the matrix $A_L(t).\phi(s)$ can be expressed as a surface $S_2 = \Psi(t,s) = (\sin(s) + \frac{-4\cos(s)(t^3+t^2)}{2(t-t^3)} + \frac{4s^3(t^3+t^2)}{2(t-t^3)}, \frac{-4\sin(s)(t^3-t^2)}{2(t-t^3)} + \cos(s) + \frac{4s^3(t^3-t^2)}{2(t-t^3)}, \frac{-4\sin(s)(t^3-t^2)}{2(t-t^3)} + \frac{-4\cos(s)(t^3+t^2)}{2(t-t^3)} + s^3)$. See Fig. (2).

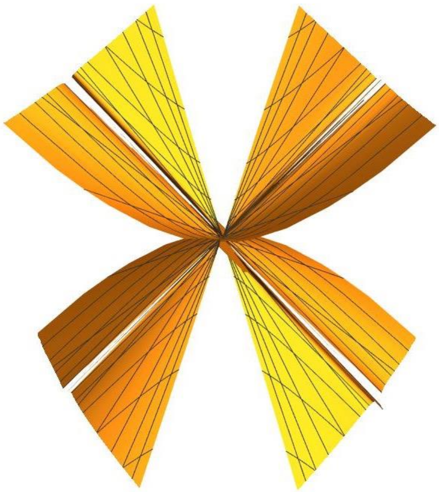

Figure 2. The surface $S_2 \in E_1^3$

3.3 Umbrella motions and dual transformations

In this section, we define a transition from a Euclidean umbrella motion to a Lorentzian umbrella motion. We obtain an umbrella matrix in E_1^n from an umbrella matrix in E^n with the help of a dual transformation.

Theorem 3.2 Let $U \in E^n$ given by

$$U = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \tag{10}$$

where A is an $(n-1) \times (n-1)$ umbrella matrix. f_u defines a dual transformation,

$$f_u: E^n \to E_1^n$$

$$U \mapsto f_u(U) = U_L = \begin{bmatrix} f(A) & C \\ 0 & 1 \end{bmatrix} \tag{11}$$

where f is the dual transformation given in (4), thus $f(A) \in \mathrm{SO}(n-2,1)$. The semi-orthogonal matrix $U_L \in E_1^n$ represents the umbrella motion in n-dimensional Lorentzian space.

Proof f_u is a dual transformation, since it holds

$$f_u^2(U) = f_u(f_u(U))$$

$$= f_u(U_L), \quad f^2 = id.$$

$$= U$$

$$f_u^2 = id.$$

Example 3 Let U represents the umbrella motion in E^4 given by

$$U=\begin{bmatrix} \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \cos\theta \\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \sin\theta \\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

By applying the f_u dual transformation to the umbrella motion matrix U, we obtain the Lorentzian matrix $f_u(U)=U_L$ as follows

$$U_L=\begin{bmatrix} 1 & -1-\sqrt{3} & 1+\sqrt{3} & \cos\theta \\ -1+\sqrt{3} & 1 & 1-\sqrt{3} & \sin\theta \\ -1+\sqrt{3} & -1-\sqrt{3} & 1 & \theta \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Theorem 3.3 f dual transformation leaves invariant the axis of the umbrella matrix in E^n and E_1^n, which is $\vec{x}=(1,1,..,1)$.

Proof The Euclidean umbrella matrix $A\in \mathrm{SO}(n)$ and the Lorentzian umbrella motion matrix $f(A)\in \mathrm{SO}(n-1,1)$ leave the same axis invariant, where f is the dual transformation given in (4). Thus, it holds that

$$A.\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},\quad f(A).\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Example 4 Let A be an umbrella matrix in E^3 given in *Ex. 3* as follows

$$A=\begin{bmatrix} \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} \\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}.$$

We acquire the Lorentzian umbrella matrix A_L under the f dual transformation as follows

$$A_L=\begin{bmatrix} 1 & -1-\sqrt{3} & 1+\sqrt{3} \\ -1+\sqrt{3} & 1 & 1-\sqrt{3} \\ -1+\sqrt{3} & -1-\sqrt{3} & 1 \end{bmatrix}.$$

Now, let us verify that f dual transformation leaves invariant the axis of the matrix A, which is $\vec{x}=(1,1,..,1)$, as given in Theorem 3.3.

$$\begin{bmatrix} \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} \\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}.\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1-\sqrt{3} & 1+\sqrt{3} \\ -1+\sqrt{3} & 1 & 1-\sqrt{3} \\ -1+\sqrt{3} & -1-\sqrt{3} & 1 \end{bmatrix}.\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Example 5 Let A be an umbrella matrix is given by

$$A=\begin{bmatrix}\frac{1-\theta^2}{1+3\theta^2} & \frac{2(\theta^2-\theta)}{1+3\theta^2} & \frac{2(\theta^2+\theta)}{1+3t^2}\\ \frac{2(\theta^2+\theta)}{1+3\theta^2} & \frac{1-\theta^2}{1+3\theta^2} & \frac{2(\theta^2-\theta)}{1+3\theta^2}\\ \frac{2(\theta^2-\theta)}{1+3\theta^2} & \frac{2(\theta^2+\theta)}{1+3\theta^2} & \frac{1-\theta^2}{1+3\theta^2}\end{bmatrix}.$$

If we multiply the matrix A with the curve $\phi(\alpha)=(\sin(\alpha),\cos(\alpha),\alpha^3)$, then we obtain the matrix $A.\phi(\alpha)$. The elements of the matrix $A.\phi(\alpha)$ can be represented by a surface $S_3=\Psi(\theta,\alpha)=(\frac{2\alpha^3(\theta^2+\theta)}{3\theta^2+1}+\frac{\sin(\alpha)(1-\theta^2)}{3\theta^2+1}+\frac{2\cos(\alpha)(\theta^2-\theta)}{3\theta^2+1},\frac{2\alpha^3(\theta^2-\theta)}{3\theta^2+1}+\frac{2\sin(\alpha)(\theta^2+\theta)}{3\theta^2+1}+\frac{\cos(\alpha)(1-\theta^2)}{3\theta^2+1},\frac{\alpha^3(1-\theta^2)}{3\theta^2+1}+\frac{2\sin(\alpha)(\theta^2-\theta)}{3\theta^2+1}+\frac{2\cos(\alpha)(\theta^2+\theta)}{3\theta^2+1})$. See Fig. (3).

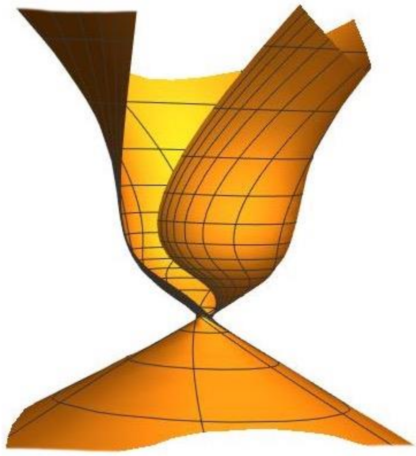

Figure 3. The surface $S_3 \in E^3$

With the help of f dual transformation, we can get the Lorentzian matrix A_L as below

$$A_L=\begin{bmatrix}1 & \frac{-2(\theta^2+\theta)}{1-\theta^2} & \frac{2(\theta^2+\theta)}{1-\theta^2}\\ \frac{-2(\theta^2-\theta)}{1-\theta^2} & 1 & \frac{2(\theta^2-\theta)}{1-\theta^2}\\ \frac{-2(\theta^2-\theta)}{1-\theta^2} & \frac{-2(\theta^2+\theta)}{1-\theta^2} & 1\end{bmatrix}.$$

By multiplying the matrix A_L with $\phi(\alpha)$, we acquire the matrix $A_L.\phi(\alpha)$. The elements of the matrix $A_L.\phi(\alpha)$ can be expressed as a surface $S_4=\Psi(\theta,\alpha)=(\frac{2\alpha^3(\theta^2+\theta)}{1-\theta^2}-\frac{2\cos(\alpha)(\theta^2+\theta)}{1-\theta^2}+\sin(\alpha),\frac{2\alpha^3(\theta^2-\theta)}{1-\theta^2}+\frac{-2\sin(\alpha)(\theta^2-\theta)}{1-\theta^2}+\cos(\alpha),\alpha^3+\frac{-2\sin(\alpha)(\theta^2-\theta)}{1-\theta^2}-\frac{2\cos(\alpha)(\theta^2+\theta)}{1-\theta^2})$. See Fig. (4).

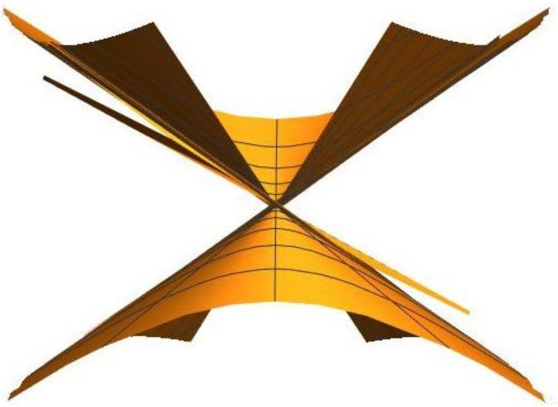

Figure 4. The surface $S_4 \in E_1^3$

3.4. Dual homothetic motions and dual transformations

In this section, we examine homothetic motions in n-dimensional dual space $\mathbb{D}^n$. We give a transition from a dual homothetic motion matrix in $\hat{E}^n$ to a dual homothetic motion matrix in $\hat{E}_1^n$ by means of dual transformations.

Theorem 3.4 Let $\widehat{H} \in \hat{E}^n$ given by

$$\widehat{H} = \begin{bmatrix} h.\hat{A} & \hat{a} \\ 0 & 1 \end{bmatrix} \tag{12}$$

where $h = h.I_n$ is a scalar matrix, $\hat{A} \in S\widehat{O}(n-1)$ and $\hat{a} \in \mathbb{D}_1^{n-1}$. f_h defines a dual transformation,

$$f_h: \hat{E}^n \to \hat{E}_1^n$$

$$\widehat{H} \mapsto f_h(\widehat{H}) = \widehat{H}_L = \begin{bmatrix} h.f(\hat{A}) & \hat{a} \\ 0 & 1 \end{bmatrix} \tag{13}$$

where f is the dual transformation given in (5), thus $f(\hat{A}) \in S\widehat{O}(n-2,1)$. The dual semi-orthogonal matrix $\widehat{H}_L \in \hat{E}_1^n$ represents the homothetic motion in n-dimensional dual Lorentzian space.

Proof We show that

$$f_h^2(\widehat{H}) = f_h(f_h(\widehat{H}))$$

$$= f_h(\widehat{H}_L), \quad f^2 = id.$$

$$= \widehat{H}$$

$$f_h^2 = id.$$

Therefore, f_h is a dual transformation.

Example 6 Let $\widehat{H} \in \hat{E}^4$ be a dual homothetic matrix for $\hat{\theta} = \theta + \epsilon$ is given by

$$\widehat{H} = \begin{bmatrix} \frac{\cos^2\hat{\theta} - \sin^2\hat{\theta}}{3} & \frac{2\cos\hat{\theta}\sin\hat{\theta} - 2}{3} & \frac{2\cos\hat{\theta} + 2\sin\hat{\theta}}{3} & \hat{\theta}^2 \\ \frac{2\cos\hat{\theta}\sin\hat{\theta} + 2}{3} & \frac{-\cos^2\hat{\theta} + \sin^2\hat{\theta}}{3} & \frac{2\sin\hat{\theta} - 2\cos\hat{\theta}}{3} & 2\cos\hat{\theta} \\ \frac{2\cos\hat{\theta} - 2\sin\hat{\theta}}{3} & \frac{2\sin\hat{\theta} + 2\cos\hat{\theta}}{3} & \frac{1}{3} & 2\sin\hat{\theta} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thus, we can obtain the dual homothetic matrix $\widehat{H_L}$ with the help of f_h dual transformation is given in (13).

$$\widehat{H_L} = \begin{bmatrix} \sin^2\hat{\theta} - \cos^2\hat{\theta} & -2 - 2\cos\hat{\theta}\sin\hat{\theta} & 2\cos\hat{\theta} + 2\sin\hat{\theta} & \hat{\theta}^2 \\ 2 - 2\sin\hat{\theta}\cos\hat{\theta} & -\sin^2\hat{\theta} + \cos^2\hat{\theta} & -2\cos\hat{\theta} + 2\sin\hat{\theta} & 2\cos\hat{\theta} \\ -2\cos\hat{\theta} + 2\sin\hat{\theta} & -2\cos\hat{\theta} - 2\sin\hat{\theta} & 3 & 2\sin\hat{\theta} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.5 Dual umbrella motions and dual transformations

In this section, we carry our work in umbrella motions from n-dimensional real space to dual space $\mathbb{D}^n$. We give a transition from a dual Euclidean umbrella motion to a dual Lorentzian umbrella motion by using the dual transformation. Thence, we acquire a dual umbrella matrix in $\hat{E}_1^n$ from a dual umbrella matrix in $\hat{E}^n$.

Theorem 3.5 Let $\widehat{U} \in \hat{E}^n$ given by

$$\widehat{U} = \begin{bmatrix} \hat{A} & \hat{C} \\ 0 & 1 \end{bmatrix} \tag{14}$$

where $\hat{A}$ is an $(n-1) \times (n-1)$ dual umbrella matrix. f_u defines a dual transformation,

$$f_u : \hat{E}^n \to \hat{E}_1^n$$

$$\widehat{U} \mapsto f_u(\widehat{U}) = \widehat{U}_L = \begin{bmatrix} f(\hat{A}) & \hat{C} \\ 0 & 1 \end{bmatrix} \tag{15}$$

where f is the dual transformation given in (5), thus $f(\hat{A}) \in \widehat{SO}(n-2,1)$. The dual semi-orthogonal matrix $\widehat{U}_L \in \hat{E}_1^n$ represents the umbrella motion in n-dimensional dual Lorentzian space.

Proof Since f_u is a dual transformation, it satisfies

$$\begin{aligned} f_u^2(\widehat{U}) &= f_u(f_u(\widehat{U})) \\ &= f_u(\widehat{U}_L), \quad f^2 = id. \\ &= \widehat{U} \\ f_u^2 &= id. \end{aligned}$$

4. Discussion and Conclusion

The geometry of the motion is important in the study of spatial mechanisms. It has a number of applications in geometric modeling of mechanical products or in the design of robotic motion. In this study, homothetic motions of a rigid body are examined. The new and the exciting part of this study is to define the dual transformation as a handy tool for obtaining one homothetic motion from another. We define a transition from Euclidean homothetic motion matrices to Lorentzian homothetic motion matrices. We give this transition by using dual transformations. With a similar method, we acquire umbrella motions in Lorentzian spaces. We also investigate the invariance of the axis of the umbrella motion that is $\vec{x} = (1,1,..,1)$ in both spaces. Additionally, related examples of matrices are provided. Furthermore, we draw their figures to investigate visual representations. Finally, because of the importance of the dual space in kinematics, robotics, and other areas related, we carry this work into dual spaces. Our paper is expected to contribute to the existing literature on kinematics.

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